ME 6590  Multibody Dynamics
Principal Moments of Inertia and Principal Directions

Properties of Real Symmetric Matrices

All real symmetric matrices have the following properties:

- The eigenvalues of a real symmetric matrix are real.
- The eigenvectors of a real symmetric matrix can always be chosen to be real.
- A real symmetric matrix is diagonalizable.
- Every real symmetric matrix possesses a complete set of orthonormal eigenvectors.
- For every real symmetric \( n \times n \) matrix \([A]\), there exists an \( n \times n \) real orthogonal matrix \([M]\) such that \([M]^T[A][M] = [D]\), where \([D]\) is a diagonal matrix.
- \([M]\) is called the modal matrix. Its columns are formed by the eigenvectors of \([A]\).
- \([D]\) is a diagonal matrix whose entries are the eigenvalues of \([A]\).
- Eigenvalues appear in the same columns of \([D]\) as their associated eigenvectors appear in \([M]\).

Principal Moments of Inertia and Principal Directions

- The inertia dyadic of the body for a set of axes passing through its mass-center and parallel to the unit vector set \( B : (e_1, e_2, e_3) \) is defined as
  \[
  I'_G = \sum_{i=1}^{3} \sum_{j=1}^{3} I'_{ij} e_i e_j
  \]

- The components of the dyadic form a real symmetric \( 3 \times 3 \) matrix \([I'_G]\). Consequently, this matrix has all the properties listed above.
- The eigenvectors of \([I'_G]\) define the principal directions of the body for the mass-center \( G \). Recall that the principal directions are those directions for which all products of inertia are zero.
- The eigenvalues of \([I'_G]\) are the principal moments of inertia of the body for \( G \), that is, they are the moments of inertia about the principal directions.
- Note that the principal directions and inertia vary from point to point in the body, but there exists only one set for any given point.
Calculation of Principal Moments of Inertia and Principal Directions

The principal moments of inertia of a body for a given point, say the mass-center \(G\), may be calculated (as you would to find the eigenvalues of any \(3\times3\) matrix) by setting

\[
\det \begin{bmatrix}
(I'_{xx} - \lambda) & -I'_{xy} & -I'_{xz} \\
-I'_{xy} & (I'_{yy} - \lambda) & -I'_{yz} \\
-I'_{xz} & -I'_{yz} & (I'_{zz} - \lambda)
\end{bmatrix} = 0. 
\]

(1)

By expanding the determinant, the resulting characteristic equation can be written as

\[
\lambda^3 + (-I'_{xx} - I'_{yy} - I'_{zz})\lambda^2 + (I'_{xx} I'_{yy} + I'_{xx} I'_{zz} + I'_{yy} I'_{zz} - I'_{xy} I'_{yz} - I'_{xz} I'_{yz} - I'_{xz} I'_{yz})\lambda + (-I'_{xx} I'_{xz} + I'_{xx} I'_{yz} + I'_{yy} I'_{xz} + I'_{yy} I'_{yz} + I'_{xx} I'_{yz} + I'_{xx} I'_{xz} + 2I'_{xy} I'_{yz} I'_{xz}) = 0
\]

(2)

The three roots to this equation are the three principal moments of inertia.

Letting \(I_i\) \((i = 1,2,3)\) represent the three principal moments of inertia, we can find the principal directions (as you would find the eigenvectors of any \(3\times3\) matrix) by setting

\[
\begin{bmatrix}
(I'_{xx} - I_i) & -I'_{xy} & -I'_{xz} \\
-I'_{xy} & (I'_{yy} - I_i) & -I'_{yz} \\
-I'_{xz} & -I'_{yz} & (I'_{zz} - I_i)
\end{bmatrix} \{a_{i1}, a_{i2}, a_{i3}\} = \{0\}
\]

(3)

Since the coefficient matrix is singular, these equations do not have a single solution. We can find an eigenvector to within a constant multiplier only.

However, if we impose the further condition that \(a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1\), then the eigenvector is unique. The components of the eigenvector are then the components of a unit vector pointing in the principal direction. Recall that the components of a unit vector are the direction cosines for that direction.

To solve Eq. (3), simply choose a value for one of the \(a_j\) \((j = 1,2,3)\), and then solve for the other two. Then, normalize the resulting vector.