ME 6590 Multibody Dynamics

Time Derivative of the (Coordinate) Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider a body \( B : (e_1, e_2, e_3) \) moving in a fixed reference frame \( R : (N_1, N_2, N_3) \). If \( r \) is a vector fixed in the body \( B \), then the derivative of \( r \) may be written as

\[
\frac{dR}{dt} = \dot{r} = R \omega_B \times r
\]

When performing the cross product, we have a choice of what reference frame to use to express each of the vectors. In the following paragraphs, "primes" will indicate vector components in \( B : (e_1, e_2, e_3) \), and no primes will indicate vector components in \( R : (N_1, N_2, N_3) \). Also, let \([C]\) be the transformation matrix that relates the two sets of unit vectors as defined by the equation

\[
[N] = [C]\{e\}.
\]

**Case 1**: \( \dot{r} \) expressed in \( R : (N_1, N_2, N_3) \), but \( R \omega_B \) and \( r \) expressed in \( B : (e_1, e_2, e_3) \)

In this case, we write

\[
\dot{r} = \sum_{i=1}^{3} \dot{r}_i N_i, \quad R \omega_B = \sum_{i=1}^{3} \omega'_i e_i, \quad \text{and} \quad r = \sum_{i=1}^{3} r'_i e_i.
\]

The three sets of vector components are related by the matrix form of Eq. (1).

\[
\{\dot{r}\} = [C]\{[\omega']\{r'\}\} = ([C][\omega']\{r\}\}
\]

**Case 2**: \( \dot{r} \) and \( R \omega_B \) expressed in \( R : (N_1, N_2, N_3) \), but \( r \) expressed in \( B : (e_1, e_2, e_3) \)

In this case, we write

\[
\dot{r} = \sum_{i=1}^{3} \dot{r}_i N_i, \quad \omega_B = \sum_{i=1}^{3} \omega_i N_i, \quad \text{and} \quad r = \sum_{i=1}^{3} r'_i e_i.
\]

The three sets of vector components are related by the matrix form of Eq. (1)

\[
\{\dot{r}\} = [\tilde{\omega}][C]\{r'\} = ([\tilde{\omega}][C]\{r\}\}
\]
**Time Derivative of the Transformation Matrices**

The above results can be used to determine *two different forms* of the time derivative of the transformation matrix \([C]\). To do this, we first note that the components of \(\mathbf{r}\) in the two different reference frames are related as follows

\[
\{r\} = [C]\{r'\}
\]

This matrix equation can be differentiated directly to give

\[
\{\dot{r}\} = [\dot{C}]\{r'\} + [C]\{\dot{r}'\} = [\dot{C}]\{r'\}
\]

Here, we have taken advantage of the fact that since \(\mathbf{r}\) is fixed in the body, \(\{r'\} = \{0\}\). Comparing this result with Eqs. (2) and (3) gives the two forms of \([\dot{C}]\).

\[
[\dot{C}] = [C][\dot{\omega}'] \quad \text{and} \quad [\dot{C}] = [\dot{\omega}][C]
\]