

Limit Law 1

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ then $\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$.

Proof:

Given $\varepsilon > 0$, consider $\frac{\varepsilon}{2}$. By definition of limit there are positive numbers δ_1, δ_2 such that $|f(x) - L| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_1$ and $|g(x) - M| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$\begin{aligned} | [f(x) + g(x)] - [L + M] | &= | [f(x) - L] + [g(x) - M] | \\ \text{(By Triangle Inequality)} &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

whenever $0 < |x - a| < \delta$. Thus $\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$.

Similar Proof:

Given $\varepsilon > 0$, consider $\frac{\varepsilon}{2}$. By definition of limit there are positive numbers δ_1, δ_2 such that $-\frac{\varepsilon}{2} < f(x) - L < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_1$ and $-\frac{\varepsilon}{2} < g(x) - M < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then adding the inequalities yields

$$-\varepsilon < f(x) + g(x) - L - M < \varepsilon \quad \text{whenever } 0 < |x - a| < \delta, \text{ in other words}$$

$$-\varepsilon < f(x) + g(x) - (L + M) < \varepsilon \quad \text{whenever } 0 < |x - a| < \delta. \text{ Thus}$$

$$| [f(x) + g(x)] - [L + M] | < \varepsilon \quad \text{whenever } 0 < |x - a| < \delta.$$