Limit Law 4

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}[f(x)g(x)] = LM$.

Proof:

Given $\varepsilon > 0$, chose $\varepsilon_1 > 0$ such that $(|L| + |M|)\varepsilon_1 + {\varepsilon_1}^2 = \varepsilon$. By definition of limit there are positive numbers δ_1, δ_2 such that $|\ f(x) - L\ | < \varepsilon_1$ whenever $0 < |x - a| < \delta_1$ and $|\ g(x) - M\ | < \varepsilon_1$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$
\begin{aligned}
|\ f(x)g(x) - LM\ | &= |\ f(x)g(x) - f(x)M + f(x)M - LM\ | \\
(\textit{By Triangle Inequality})\quad &\le |\ f(x)g(x) - f(x)M\ | + |\ f(x)M - LM\ | \\
&= |\ f(x)\ | \cdot |\ g(x) - M\ | + |\ f(x) - L\ | \cdot |\ M\ | \\
(\textit{By Triangle Inequality})\quad &\le (|L| + |f(x) - L|) \cdot |g(x) - M| + |f(x) - L| \cdot |M| \\
&< (|L| + \varepsilon_1)\varepsilon_1 + \varepsilon_1|M| \\
&= (|L| + |M|)\varepsilon_1 + {\varepsilon_1}^2 = \varepsilon,
\end{aligned}
$$

whenever $0 < |x - a| < \delta$. Thus $\lim_{x\to a}[f(x)g(x)] = LM$.