

Squeeze Theorem

If $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} h(x) = L$, and $f(x) \leq g(x) \leq h(x)$ near x then $\lim_{x \rightarrow a} g(x) = L$.

Proof:

Let $\varepsilon > 0$. By definition of limit there are positive numbers δ_1, δ_2 such that

$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta_1$ and $|h(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta_2$.

Since $f(x) \leq g(x) \leq h(x)$ near x , there is a positive number δ_3 such that

$f(x) \leq g(x) \leq h(x)$ whenever $0 < |x - a| < \delta_3$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then whenever $0 < |x - a| < \delta$ we have all three of the following: $f(x) - L \leq g(x) - L \leq h(x) - L$, $-\varepsilon < f(x) - L < \varepsilon$, and $-\varepsilon < h(x) - L < \varepsilon$.

Combining these yields that $-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon$ whenever $0 < |x - a| < \delta$. Thus $|g(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, which means that $\lim_{x \rightarrow a} g(x) = L$ since ε was arbitrary.