MATH 3740
Week 2: lecture #2

Bonus problem:

Consider RC circuit

\[ \frac{q}{C} \text{ charge on capacitor} \]

\[ \frac{1}{C} q = U_c \text{ in voltage} \]

Equation

\[ R \frac{dq}{dt} + \frac{1}{C} q = E_0 \]

\[ q(0) = 0 \]

Find \( q(t) \) and sketch it.

Solution.

\[ \frac{dq}{dt} = \frac{1}{RC} E_0 - \frac{1}{C} q \text{ in linear and separable equation} \]

\[ \frac{dq}{dt} = a - bq \]

\[ a = \frac{E_0}{R}, \quad b = \frac{1}{RC} \]

\[ \frac{dq}{a - bq} = dt \quad \int \frac{dq}{a - bq} = \int dt \]

\[ -\frac{1}{b} \ln(a - bq) = t - \frac{1}{b} \ln a \]

\[ \ln(a - bq) = -bt + \ln C \quad C \text{-constant} \]

\[ a - bq = C e^{-bt} \]

\[ q(0) = 0 \rightarrow C = a \]

\[ a - bq = ae^{-bt} \rightarrow \]

\[ -q(t) = \frac{1}{b} (a - ae^{-bt}) = \frac{a}{b} (1 - e^{-bt}) \]

\[ q(t) = E_0 C (1 - e^{-bt}) \]
\[ Q(t) = E_0 C \left( 1 - e^{-\frac{t}{RC}} \right) \]

S. 1. 6. Substitution method and exact equations.

Consider \( \frac{dy}{dx} = f(x, y) \)

we can find solutions for separable or linear differential equations. Power tool tool is a substitution (change of variables) method!

Example. \( \frac{dy}{dx} = (x + y + 2)^2 \)

\( y' = F(ax + by + c) \) type eq.-n.

Change of variables

\( v = x + y + 2 \rightarrow v' = 1 + y' \)

\[ dx = \frac{dv}{1 + v^2} \]

is separable

\[ \int \frac{dv}{1 + v^2} = \int dx \quad \text{arctan} v = x + C \]

\[ v = \tan(x + C) \]

\[ \therefore \quad \tan^{-1}(v + C) \]
\[ y = \tan(x + c) - x - 2 \]

In general,
\[ y' = F(ax + by + c), \ b \neq 0 \]

change of variables
\[ \nu = ax + by + c \]
\[ \nu' = a + by' + c \quad \text{but} \quad y' = F(\nu) \]

Thus, new differential eqn
\[ \frac{d\nu}{dx} = a + b \cdot F(\nu) \quad \text{is separable diff. equation!} \]

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Homogeneous equation

\[ y' = F\left(\frac{y}{x}\right) \quad \text{for some function} \]

Such equation appears in many applications.

\[ \begin{align*}
\text{Ex:} \quad \frac{y'}{x+y} &= \frac{1}{1+\frac{y}{x}} \quad F\left(\frac{y}{x}\right) = \frac{1}{1+\frac{y}{x}} \\
\text{Substitution:} \quad \nu = \frac{y}{x} \quad \text{or} \quad y = \nu x \\
\nu x = y \quad \left(\nu \frac{d\nu}{dx}\right)' = y' \quad (\text{but} \quad y' = F(\nu)) \\
\nu' x + \nu = F(\nu) \quad \text{or} \quad \frac{x \, d\nu}{dx} = F(\nu) - \nu \quad \text{separate equaion} \end{align*} \]
Example

\[ \frac{d^2 y}{dx^2} = y + y^2 + x^2 \quad y(1) = 0 \]

\[ y' = \frac{y}{x} + \frac{y^2}{x^2} + 1 \]

\[ V = \frac{y}{x} \quad \text{or} \quad y = Vx \quad (Vx)' = y' \]

\[ V'x + V = V + V^2 + 1 \]

\[ x \frac{dv}{dx} = 1 + V^2 \quad \Rightarrow \quad \frac{dv}{1 + V^2} = \frac{dx}{x} \]

\[ \arctan V = \ln x + C \]

\[ V(1) = \frac{y(1)}{x} = 0 \quad \Rightarrow \quad \arctan 0 = \ln 1 + C \quad \Rightarrow \quad C = 0 \]

\[ \arctan V = \ln x \]

\[ V = \tan(\ln x) \quad \text{or} \quad \frac{y}{x} = \tan(\ln x) \]

\[ y = x \tan(\ln x) \]

\[ \# 7 \quad xy^2 \quad y' = x^3 + y^3 \]

\[ y' = \frac{2cx^2}{y^2} + \frac{y}{x} \quad \leftarrow \text{homogeneous DE} \]

Substitution \( V = \frac{y}{x} \) \( \quad VX = y \quad \Rightarrow \quad V'x + V = y' \)

\[ xV' + V = \frac{1}{V^2} + V \]

\[ y \quad x \frac{dv}{dx} = \frac{1}{V^2} \quad \Rightarrow \quad V^2 dv = \frac{1}{x} \quad dx \]

\[ \frac{1}{3}V^3 = \ln |x| + \frac{4}{3} \]
\[ v^3 = 3 \ln |x| + C, \quad v = \frac{y}{x} \]

\[ y^3 = x^3 \left( 3 \ln |x| + C \right) \]
\[ y = x \left( 3 \ln |x| + C \right)^{\frac{1}{3}} \]

**Exact differential equations**

We consider differential equation which can be written in the form

\[ M(x,y) \, dx + N(x,y) \, dy = 0 \]

or

\[ N(x,y) \frac{dy}{dx} + M(x,y) = 0 \]

**Question:** When it is possible to write solution of this problem in \( F(x, y(x)) = 0 \) for some function.

We have

\[ F(x, y) \]

\[ \frac{d}{dx} F(x, y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \]

\[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \]

Compare \( M + N \frac{dy}{dx} = 0 \)

Then if \[ \begin{align*}
M &= \frac{\partial F}{\partial x} \\
N &= \frac{\partial F}{\partial y}
\end{align*} \]

the solution \( F = 0 \)
\[ M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y} \]

When such \( F \) exists for given \( M(x,y) \) and \( N(x,y) \)

We know that if such \( F(x,y) \) exists then

\[ \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x} \]

are equal. Inverse is also true.

If \[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]
then solution of the equation is given by \( F(x,y) = 0 \)

Such differential equations are called **exact**.

Def. DE \( Mdx + Ndy = 0 \) is called exact if

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

Then its general solution is given by some equation

\[ F(x,y(x)) = 0 \]
How to find $F$?

Use relation

$$\frac{\partial F}{\partial x} = M(x,y) \Rightarrow F(x,y) = \int M(x,y) \, dx + g(y)$$

$g(y)$ is unknown.

To find $g$, differentiate

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left( \int M(x,y) \, dx + g(y) \right) = N(x,y)$$

differential equations for $g(y)$

Example

$$(3x^2y^2 + e^x \sin y) \, dx + (2x^3y + e^x \cos y) \, dy = 0$$

is this equation exact?

$M = 3x^2y^2 + e^x \sin y$, $N = 2x^3y + e^x \cos y$

Check

$$\frac{\partial M}{\partial y} = 6x^2y + e^x \cos y$$

$$\frac{\partial N}{\partial x} = 6x^2y + e^x \cos y$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn is Exact!

How to find $F$

$$\frac{\partial F}{\partial x} = \overbrace{M}^{3x^2y^2 + e^x \sin y} \Rightarrow F = \int (3x^2y^2 + e^x \sin y) \, dx =$$

$$x^3y^2 + e^x \sin y + g(y)$$
\[ F = x^3 y^2 + e^x \sin y + g(y) \]

\( g(y) \) is unknown. To find it, use condition \( \frac{\partial F}{\partial y} = N \)

\[ \frac{\partial F}{\partial y} = 2x^3 y + e^x \cos y + g'(y) = 2x^3 y + e^x \cos y \]

or \( g'(y) = 0 \) \( \rightarrow \) \( g(y) = C \) constant

General solution of the exact eqn

\[ F(x,y) = C \]

\[ x^3 y^2 + e^x \sin y + C = 0 \]

Problems from Sect. 1.6

#34 \( (2xy^2 + 3x^2) \, dx + (2x^2 y + 4y^3) \, dy = 0 \)

\[ M = 2xy^2 + 3x^2 \quad N = 2x^2 y + 4y^3 \]

\[ \frac{\partial M}{\partial y} = 4xy \quad \frac{\partial N}{\partial x} = 4xy \]

\( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \), eq'n is exact

Find \( F \) such that \( \frac{\partial F}{\partial x} = M \), \( \frac{\partial F}{\partial y} = N \)

\[ \frac{\partial F}{\partial x} = 2xy^2 + 3x^2 \quad \int (2xy^2 + 3x^2) \, dx = x^2 y^2 + \frac{3}{2} x^3 + g(y) \]

To find \( g(y) \) use

\[ \frac{\partial F}{\partial y} = 2x^2 y + g'(y) = N = 2x^2 y + 4y^3 \]

\( \rightarrow \quad g'(y) = 4y^3 \)
\[ g'(y) = 4y^3 \rightarrow g(y) = \int 4y^3 dy = y^4 + C \]

Then \[ F = \frac{x^2 y^2}{y^2} + 2 + g'(y) = \]

\[ = x^2 y^2 + x^3 + y^4 + C \]

general solution of the DE

\[ x^2 y^2 + x^3 + y^4 + C = 0 \]

#38

\[(x + \arctan y) dx + \frac{x + y}{1 + y^2} dy = 0\]

\[ M = x + \arctan y \quad N = \frac{x + y}{1 + y^2} \]

\[ \frac{\partial M}{\partial y} = 1 + y^2 \quad \frac{\partial N}{\partial x} = 1 + y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

the DE is exact

Find \( F \) such that \( \frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N \)

\[ F = \int M \, dx = \int (x + \arctan y) \, dx = \frac{1}{2} x^2 + x \arctan x + \]

+ \( g(y) \)  

To find \( g(y) \)

\[ \frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left( \frac{x}{1 + y^2} + g(y) \right) = \frac{x}{1 + y^2} + g'(y) = N \]

\[ \frac{x}{1 + y^2} + g'(y) = \frac{x + y}{1 + y^2} \]

\[ g'(y) = \frac{y}{1 + y^2} \quad g(y) = \int \frac{y}{1 + y^2} \, dy = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u + C = \]

\[ = \frac{1}{2} \ln (1 + y^2) + C \quad u = 1 + y^2 \]
\[ F(x, y) = \frac{1}{2} x^2 + x \arctan y + \frac{1}{2} \ln(1 + y^2) + C \]

Solution of the exact DE is given
\[ \frac{1}{2} x^2 + x \arctan y + \frac{1}{2} \ln(1 + y^2) + C = 0 \]

Bernoulli equation
\[ \frac{dy}{dx} + P(x) y = Q(x) y^n \quad n \neq 0 \]
\[ \frac{dy}{dx} + P(x) y = Q(x) y \quad n = 1 \]

Find substitution such that this equation will be transformed into linear differential equation.

We try to use substitution
\[ V = y^\alpha \quad \alpha \text{ is a number} \]
Then
\[ V' = \alpha y^{\alpha-1} y' = \alpha y^{\alpha-1} \left( -P(x)y + Q(x)y^n \right) \]
or
\[ V' = -\alpha P(x) y^\alpha + \alpha Q(x) y^{\alpha-1+n} \]
Recall \( y^\alpha = V \) and choose \( \alpha \) such that
\[ \alpha - 1 + n = 0 \text{ or } \alpha = 1 - n \]
Then
\[ V' = -\alpha P(x) V + \alpha Q(x) V \]
linear diff.
with respect to \( V \) eqn.
For Bernoulli equation use

Substitution \( V = y^{1-h} \)

Example

#22

\[ x^2 y' + 2xy = 5y^4 \]

Or Bernoulli equation

\[ y' + \frac{2}{x} y = \frac{5}{x^2} y^4 \]

\( h = 4 \) substitution \( V = y^{1-4} = y^{-3} \)

\[ V' = (y^{-3})' = -3y^{-4}(y') = -3y^{-4}\left(-\frac{2}{x} y + \frac{5}{x^2} y^4\right) \]

or \[ V' = -3\left(-\frac{2}{x}\right)y^{-3} - \frac{15}{x^2} \]

\[ V' = +6\frac{1}{x} V = -\frac{15}{x^2} \]

\[ V' - 6\frac{1}{x} V = -\frac{15}{x^2} \leq \text{linear DE} \]

\[ p(x) = -\frac{6}{x} \quad \int p(x) \, dx = -6 \ln x = 6 \ln x^{-6} \]

\[ \varphi(x) = e^{\ln x^{-6}} = x^{-6} = \frac{1}{x^6} \]

\[ \frac{1}{x^6} V' - 6\frac{1}{x^7} V = -\frac{15}{x^8} \]

\[ (\frac{1}{x^6} V)' = -\frac{15}{x^8} \]

\[ \frac{1}{x^6} V = -\frac{15}{x^7} + C \]

\[ V = \frac{15}{x} + C x^6 \]
Part \( v = y^{-3} = \frac{1}{y^3} \)

\[
\frac{1}{y^3} = \frac{15}{7x} + Cx^6
\]

\[
y = (\frac{15}{7x} + Cx^6)^{-\frac{1}{3}}
\]

More problems in your homework.

To check problems for Review Section (page 74), you should be able to do any of them.

Page 74:

#1 \( x^3 + 3y - xy' = 0 \)

Solve equation
Chapter 1 First-Order Differential Equations

- Is it exact? That is, when the equation is written in the form $M \, dx + N \, dy = 0$, is $\partial M/\partial y = \partial N/\partial x$ (Section 1.6)?
- If the equation as it stands is not separable, linear, or exact, is there a plausible substitution that will make it so? For instance, is it homogeneous (Section 1.6)?

Many first-order differential equations succumb to the line of attack outlined here. Nevertheless, many more do not. Because of the wide availability of computers, numerical techniques are commonly used to approximate the solutions of differential equations that cannot be solved readily or explicitly by the methods of this chapter. Indeed, most of the solution curves shown in figures in this chapter were plotted using numerical approximations rather than exact solutions. Several numerical methods for the appropriate solution of differential equations will be discussed in Chapter 2.

Chapter 1 Review Problems

Find general solutions of the differential equations in Problems 1 through 30. Primes denote derivatives with respect to $x$.

1. $x^3 + 3y = xy' = 0$
2. $xy^2 + 3y^2 - x^2 y' = 0$
3. $xy + y^2 - x^2 y' = 0$
4. $2xy^3 + e^x + (3x^2 y^2 + \sin y)y' = 0$
5. $3y + x^4 y' = 2xy$
6. $2xy^2 + x^2 y' = y^2$
7. $2x^2 y + x^3 y' = 1$
8. $2xy + x^2 y' = y^2$
9. $xy' + 2y = 6x^2 \sqrt{y}$
10. $y' = 1 + x^2 + y^2 + x^2 y^2$
11. $x^2 y' = xy + 3y^2$
12. $6xy^3 + 2y^4 + (9x^2 y^2 + 8xy^3)y' = 0$
13. $4xy^2 + y' = 5x^4 y^2$
14. $x^3 y' = x^2 y - y^3$
15. $y' + 3y = 3x^2 e^{-3x}$
16. $y' = x^2 - 2xy + y^2$
17. $e^x + ye^{xy} + (e^y + x e^{xy})y' = 0$
18. $2x^2 y - x^3 y' = y^3$
19. $3x^5 y^2 + x^3 y' = 2y^2$
20. $xy' + 3y = 3x^{-3/2}$
21. $(x^2 - 1)y'' + (x - 1)y = 1$
22. $xy' = 6y + 12x^4 y^{2/3}$
23. $e^y + y \cos x + (xe^y + \sin x)y' = 0$
24. $9x^2 y^2 + x^{3/2} y' = y^2$
25. $2y + (x + 1)y' = 3x + 3$
26. $9x^{1/2} y^4/3 - 12x^{1/3} y^{3/2} + (8x^{3/2} y^{1/3} - 15x^{6/5} y^{1/2}) y' = 0$
27. $y + x y' = 2e^{2x}$
28. $(2x + 1)y'' + y = (2x + 1)^{3/2}$
29. $y' = \sqrt{x + y}$
30. $y' = \sqrt{x + y}$

Each of the differential equations in Problems 31 through 36 is of two different types considered in this chapter—separable, linear, homogeneous, Bernoulli, exact, etc. Hence, derive general solutions for each of these equations in two different ways; then reconcile your results.

31. $\frac{dy}{dx} = (3y + 7)x^2$
32. $\frac{dy}{dx} = xy^3 - xy$
33. $\frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy}$
34. $\frac{dy}{dx} = x + 3y$
35. $\frac{dy}{dx} = \frac{2xy + 2x}{x^2 + 1}$
36. $\frac{dy}{dx} = \sqrt{x^2 + y}$
\[ x^3 + 3y - xy' = 0 \rightarrow \text{linear DE} \]

\[ xy' - 3y = x^3 \rightarrow y' - \frac{3}{x} y = x^2 \ (\star) \]

integrating factor \( P(x) = -\frac{3}{x} \)

\[ \int P(x) \, dx = \int -\frac{3}{x} \, dx = -3 \ln x = \ln x^{-3} \]

\[ y = e^{\int P(x) \, dx} = e^{\ln x^{-3}} = \frac{1}{x^3} \]

multiply \((\star)\) by \( P = \frac{1}{x^3} \)

\[ \frac{1}{x^3} y' - \frac{3}{x^4} y = \frac{1}{x} \]

\[ \left( \frac{1}{x^3} y \right)' = \frac{1}{x} \rightarrow \frac{1}{x^3} y = \ln |x| + C \]

or solution

\[ y = x^3 (\ln |x| + C) \]

\# 23 \[ e^y + y \cos x + (xe^y + \sin x)y' = 0 \]

Is it exact DE?

\[ (e^y + y \cos x) \, dx + (xe^y + \sin x) \, dy = 0 \]

\[ M = e^y + y \cos x, \quad N = xe^y + \sin x \]

Check \( \frac{\partial M}{\partial y} = e^y + \cos x \), \( \frac{\partial N}{\partial x} = e^y + \cos x \)

yes, equation is exact, its solution can be found in the form

\[ F(x, y(x)) = 0 \]
To find $F$ recall that
\[ \frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N. \]

Then
\[ F = \int M(x,y) \, dx = \int (e^y + y \sin x) \, dx = xe^y + y \sin x + g(y). \]

To find $g$ use the second relation $\frac{\partial F}{\partial y} = N$.
\[ \frac{\partial F}{\partial y} = xe^y + y \sin x + g'(y) = \overbrace{ xe^y + y \sin x }^{N(x,y) \quad \text{from (**)}}. \]

Then
\[ g'(y) = 0 \rightarrow g(y) = c \]
\[ F(x,y) = xe^y + y \sin x + C = \]

Solution
\[ xe^y + y \sin x + C = 0 \]

Bonus problem. #26
\[ \frac{1}{2} y^{4/3} - 12 x^{1/5} y^{3/2} + (8x^{3/2} y^{1/3} - 15 x^{6/5} y^{1/2}) y' = 0 \]