This is a problem about changing amount of salt \( x(t) \) in tank

We derived eqn for \( x(t) \) in class

\[
\frac{dx}{dt} = r_i c_i - r_o c_0
\]

where \( r_i \) incoming flow rate
\( r_o \) outgoing flow rate

in \#33 \( r_i = r_o = 5 \)

\( c_i \) concentration of salt in incoming flow
\( c_0 \) concentration of salt in tank

\[
c_0 = \frac{x_0}{V} = \frac{x}{1000}
\]

Then from equation (\#)

\[
\frac{dx}{dt} = 0 - \frac{1}{1000} x \cdot 5
\]

\[
\frac{dz}{dx} = -\frac{1}{200} x, \quad x(10) = 100
\]

Find \( x(t) \), separate variables \( x(t) = (e^{-\frac{t}{200}}) \)

use initial cond. \( x(t) = 100 \cdot 100 e^{-\frac{t}{200}} \)

Find \( t \) when \( x(t) = 10 \), \( 100 e^{-\frac{t}{200}} = 10 \)

\[
e^{\frac{t}{200}} = 10 \quad \frac{t}{200} t = \ln 10
\]

\[
\boxed{t = (\ln 10) 200}
\]
\[ yy' + x = \sqrt{x^2 + y^2} \]

This equation is homogeneous one

\[ y' = F\left(\frac{y}{x}\right) \]

To see it we consider

\[ y' + \frac{x}{y} = \frac{1}{y} \sqrt{x^2 + y^2} \]

or

\[ y' = -\frac{x}{y} + \frac{\sqrt{\left(\frac{x}{y}\right)^2 + 1}}{\sqrt{\left(\frac{x}{y}\right)^2 + 1}} \]

We use substitution \( y = v x \) \( (v = \frac{y}{x}) \)

Then

\[ v' x + v = y' = -\frac{1}{v} + \sqrt{\frac{1}{v^2} + 1} \]

or

\[ x \frac{dv}{dx} = -\frac{1}{v} - v + \sqrt{\frac{1 + v^2}{v^2}} \]

we rewrite it

\[ x \frac{dv}{dx} = -\frac{1 - v^2 + \sqrt{1 + v^2}}{v} \]

and separate variables

\[ \int \frac{v \, dv}{-1 - v^2 + \sqrt{1 + v^2}} = \int \frac{dx}{x} \]

To find first integral we use substitution

\[ u = 1 + v^2, \quad 2u \, du = 2v \, dv \]

then

\[ \int \frac{v \, dv}{-1 - v^2 + \sqrt{1 + v^2}} = \int \frac{u \, du}{-u^2 + u} = \int \frac{du}{1 - u} = \ln|1 - u| \]
Thus

\[-\ln \left| 1 - \sqrt{1 + v^2} \right| = \ln |x| - \ln C\]

\[\ln \left| 1 - \sqrt{1 + v^2} \right| = -\ln |x| + \ln C\]

after exponentiation

\[1 - \sqrt{1 + v^2} = \frac{C}{x}\]

\[\sqrt{1 + v^2} = 1 - \frac{C}{x} \rightarrow 1 + \left(\frac{y}{x}\right)^2 = \left(1 - \frac{C}{x}\right)^2\]

Finally

\[\left(\frac{y}{x}\right)^2 = \left(1 - \frac{C}{x}\right)^2 - 1\]

\[y = \sqrt{\left((x - C)^2 - x^2\right)}\]
S.2.1 Mathematical models of Population Dynamics.

Example of mathematical modelling

Let $P(t)$ be some population (people, or animals or bacteria).

How to describe change of population $P(t)$.

Assume that we know

birth rate (number of births per unit of population per unit of time)

$\beta(t) = \beta(t, P)$ it can depend on size of population

death rate (number of deaths per unit of population per unit of time)

$\delta(t, P)$

Then consider balance relation: for small time interval $\Delta t$ the change of population is

\[ \Delta P = P(t + \Delta t) - P(t) \approx \beta(t) P(t) \Delta t - \delta(t) P(t) \Delta t \]

or

\[ \frac{\Delta P}{\Delta t} = \beta(t) P - \delta(t) P \]
By taking limit as \( \Delta t \to 0 \)

\[
\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = \beta(t) P - \delta(t) P
\]

or equation of population dynamics

\[
\frac{dP}{dt} = \beta(t) P - \delta(t) P
\]

In general rates \( \beta(t) = \beta(t, P) \)
\( \delta(t) = \delta(t, P) \)

We consider a case where

\( \beta(t) = \beta_0 - \beta_1 P \) - birth rate is decreasing with growth of population

\( \delta(t) = \delta_0 \)

Then

\[
\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0) P
\]

or

\[
\frac{dP}{dt} = (\beta_0 - \delta_0 - \beta_1 P) P
\]

\[
\frac{dP}{dt} = k(M - P) P
\]

constants \( k = \beta_1 \), \( M = \frac{\beta_0 - \delta_0}{\beta_1} \)
Equation
\[ \frac{dP}{dt} = kP(M-P), \quad P(0) = P_0 \]

is called \underline{logistic equation}.

We are interested in finding its solution and study their properties.

Separable equation

\[ \int \frac{dP}{P(M-P)} = \int k \, dt \]

How to find integral of rational function

\[ \int \frac{1}{P(M-P)} \, dP = ? \]

Present \[ \frac{1}{P(M-P)} \] as a sum of proper fractions

\[ \frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}, \quad A, B \text{ undetermined} \]

\[ \frac{A}{P} + \frac{B}{M-P} = \frac{A(M-P) + BP}{P(M-P)} = \frac{1}{P(M-P)} \]

Then \[ A(M-P) + BP = 1 \] for all \( P \)

\[ (B-A)P + AM = 1 \rightarrow \text{linear system} \]

\[ B - A = 0, \quad AM = 1 \rightarrow A = B = \frac{1}{M} \]
Thus,

\[ \int \frac{1}{P(M-P)} \, dp = \int \left( \frac{1}{M-P} + \frac{1}{M(M-P)} \right) \, dp = \frac{1}{M} \left( \ln P - \ln (M-P) \right) = \frac{1}{M} \ln \frac{P}{M-P} \]

(since \( \ln a - \ln b = \ln \frac{a}{b} \))

But \( \int \frac{dp}{P(M-P)} = \int k \, dt \)

So we obtain

\[ \ln \frac{P}{M-P} = kMt + \ln c \]

After exponentiating

\[ \frac{P}{M-P} = Ce^{kMt} \]

but \( P(0) = P_0 \)

we get \( C = \frac{P_0}{M-P_0} \)

\[ \frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt} \]

\( \frac{P}{M-P} = D, \quad D = \frac{P_0}{M-P_0} e^{kMt}, \quad \Rightarrow \quad P = D(M-P) \)

\( \Rightarrow (1+D)P = DM \quad \Rightarrow \quad P = \frac{DM}{1+D} \)

\[ P(t) = \frac{MP_0 e^{kMt}}{(M-P_0) \left( 1 + \frac{P_0}{M-P_0} e^{kMt} \right)} \]
or \[ P(t) = \frac{M P_0 e^{kHt}}{(M-P_0)e^{kHt} + P_0} \]

\[ P(t) = \frac{M P_0}{(M-P_0)e^{kHt} + P_0} \] is convenient for analysis.

1. \[ \frac{dP}{dt} = kP(M-P), \quad P(0) = P_0 \]

\[ P(t) = \frac{M P_0}{P_0 + (M-P_0)e^{-kHt}} \]

Case \( P_0 = M \rightarrow P(t) \equiv M \) for all \( t \)

equilibrium solution

2. Case \( 0 < P_0 < M \) \( (M-P_0)e^{-kHt} \rightarrow 0 \)

\[ P(t) = \frac{M P_0}{(M-P_0)e^{-kHt} + P_0} \rightarrow M \] and decreasing

\[ \text{is increasing} \]

3. Case \( P_0 > M \) \( (M-P_0)e^{-kHt} \rightarrow 0 \)

\[ P(t) \uparrow M \] and decreasing

\[ P^{\rightarrow} \rightarrow M \] \( M \)-equilibrium solution!
Logistic equation describes very well population dynamics in real world (look at textbook's example)

Problems.

#10 fish population in lake, attached by death rate which is \( \beta = 0 \), \( \delta \) death rate is proportional to \( \frac{1}{V_o} \). Let \( P(0) = 900 \), \( P(6) = 441 \) when \( P(t) = 0 \)?

\[
\frac{dP}{dt} = \beta P - \delta P \quad \beta = 0 \quad \delta = \frac{k}{V_o}
\]

Then \( \frac{dP}{\sqrt{P}} = -k \sqrt{P} \) separable eq - u

\[
\int \frac{dP}{\sqrt{P}} = -k \int \sqrt{P} \, dt = -k t + C
\]

\[
2 \sqrt{900} = -kt + C \quad C = ? \quad P(0) = 900
\]

\[
2 \sqrt{900} = C \rightarrow C = 60. \quad \text{How to find } k
\]

\[
t = 6 \quad P(6) = 441
\]

\[
2 \sqrt{441} = -k \cdot 6 + 60 \rightarrow 42 - 60 = -6k
\]

\[
t = 3 \rightarrow P(t) = (60 - \frac{3}{2} t)^2
\]

\[
P(t) = 0 \rightarrow t = 20 \text{ weeks!}
\]
Problems

(Use partial fractions to find integrals)

#4 Solve

\[ \frac{dx}{dt} = 9 - 4x^2, \quad x(0) = 0 \]

and sketch solution

\[ \frac{dx}{dt} = (3 - 2x)(3 + 2x) \]

\[ \int \frac{dx}{(3 - 2x)(3 + 2x)} = dt \]

\[ \frac{1}{(3 - 2x)(3 + 2x)} = \frac{A}{3 - 2x} + \frac{B}{3 + 2x} \]

\[ = \frac{A(3 + 2x) + B(3 - 2x)}{(3 - 2x)(3 + 2x)} = \frac{-2x(A - B) + 3A + 3B}{(3 - 2x)(3 + 2x)} \]

Then

\[ -2x(A - B) + 3A + 3B = 1 \quad \text{for all} \quad x \]

\[ A - B = 0 \quad 3A + 3B = 1 \quad A = B = \frac{1}{6} \]

Then

\[ \int \frac{1}{(3 - 2x)(3 + 2x)} \, dx = \int \left( \frac{1}{6} \left( \frac{1}{3 - 2x} + \frac{1}{3 + 2x} \right) \right) \, dx = \]

\[ = -\frac{1}{12} \ln |3 - 2x| + \frac{1}{12} \ln |3 + 2x| = t + \frac{1}{12} \ln \left( \frac{3 + 2x}{3 - 2x} \right) \]

\[ \ln \left| \frac{3 + 2x}{3 - 2x} \right| = 12t + \ln C \]
\[ \ln \frac{3 + 2x}{3 - 2x} = 12t + \ln C \]

Then
\[ \frac{3 + 2x}{3 - 2x} = C e^{12t} \]

Solve with respect to \( x(t) \)

in general case \( x(0) = x_0 \)

\[ \frac{3 + 2x}{3 - 2x} = \frac{3 + 2x_0}{3 - 2x_0} e^{12t} \]

\[ \frac{3 + 2x}{3 - 2x} = D \quad \text{for} \quad D = \frac{3 + 2x_0}{3 - 2x_0} e^{12t} \]

\[ 3 + 2x = 3D - 2Dx \]

\[ 2(1 + D)x = 3D - 3 \quad \Rightarrow \quad x = \frac{3D - 3}{2(1 + D)} \]

\[ x(t) = \frac{3}{2} \left( \frac{3 + 2x_0}{3 - 2x_0} e^{12t} - 1 \right) \]

\[ x(t) = \frac{3}{2} \frac{(3 + 2x_0)e^{12t} - (3 - 2x_0)}{(3 + 2x_0)e^{12t} + (3 - 2x_0)} \]

Check that \( x(0) = x_0 \)

If \( x_0 = 0 \) then

\[ x(t) = \frac{3}{2} \frac{3 e^{12t} - 3}{3 e^{12t} + 3} \]

\[ x(t) = \frac{3}{2} \frac{e^{12t} - 1}{e^{12t} + 1} \rightarrow \frac{3}{2} \]

\[ x = \frac{3}{2} \quad \Rightarrow \quad x(t) = \frac{3}{2} \text{ in equilibrium} \]
\[ \frac{dx}{dt} = (3+2x)(3-2x), \quad x(0) = x_0 \]

\[ x = \frac{3}{2} \text{ equilibrium solution } x(t) \rightarrow \frac{3}{2} \]

**Question:** Can we produce a general picture?

### S. 2.2. Equilibrium solutions and their stability

**Consider DE**

\[ \frac{dx}{dt} = f(x), \quad f \text{ is given} \]

**Definition:** Solution \( x(t) = a \) for all \( t \) is called equilibrium solution ("steady-state" solution).

\[ x(t) = a \text{ for } t \] (equilibrium)

**How to find equilibrium solutions**

Let \( x(t) = a \) then \( \frac{dx}{dt} = \frac{df}{dx} a = 0 = f(a) \)

Equilibrium solutions are roots \( f(a) = 0 \) of algebraic equation.

**Ex**

\[ \frac{dx}{dt} = x^2 - 2x - 3 \]

\[ f(x) = x^2 - 2x - 3 \]
\[
\frac{dx}{dt} = x^2 - 2x - 3, \quad f(x) = x^2 - 2x - 3
\]

Find equilibrium solutions:

\[x^2 - 2x - 3 = 0\]

\[c^2 + 6c + 9 = 0\]

\[x_{1,2} = \frac{-6 \pm \sqrt{36 - 36}}{2} = \frac{-6 \pm 0}{2} = -3\]

Two equilibrium solutions:

\[x_1 = 3, \quad x_2 = -2\]

\[
\begin{align*}
\frac{dx}{dt} = f(x) &= (x - 3)(x + 2) \\
\end{align*}
\]

\[f(x) = \begin{cases} 
> 0 & x > 3 \\
< 0 & -2 < x < 3 \\
> 0 & x < -2
\end{cases}\]

Thus:

\[
\frac{dx}{dt} = \begin{cases} 
> 0 & x > 3 \\
< 0 & 2 < x < 3 \\
> 0 & x < -2
\end{cases}\]

\[x(t)\] is increasing if \(x(t) > 3\)

\[x(t)\] is decreasing if \(2 < x(t) < 3\)

\[x(t)\] is increasing if \(x(t) < -2\)
Thus, we can sketch pictures of solutions between roots of $f(x) = 0$.

In this picture for solutions $x(t)$ such that $x(0)$ is close to equilibrium $x_{eq} = -2$ we have that

$$\lim_{t \to +\infty} x(t) = -2 \Rightarrow \text{equilibrium is asymptotically stable.}$$

In many control engineering applications ("cruise controller") design feedback control with asymptotically stable equilibrium.
Consider
\[ \frac{dx}{dt} = f(x) \]

let \( x(t) = a \) u equilibrium solution
\( f(a) = 0 \)

**Definition** Equilibrium solution \( x(t) = a \)
of (1) u called asymptotically stable if

(i) \( x(t) = a \) is Lyapunov stable
(for any \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that
for any solution \( x(t) \) such that \( |x(t) - a| < \delta \)
we have
\[ |x(t) - a| < \varepsilon \text{ for any } t \geq 0 \]

(ii) any such solution attracts to \( a \)
\[ \lim_{t \to +\infty} x(t) = a \]

\[ x(t) \]
\[ x = a \]
\[ x \in (-\varepsilon, \varepsilon + \varepsilon) \]
Problem #14 (p. 93)

Find all equilibrium solutions (critical points) and indicate stability or instability of equilibrium solutions. Sketch solutions.

\[ \frac{dx}{dt} = x(x^2 - 4) \]

\[ f(x) = x(x^2 - 4) = x(x - 2)(x + 2) \]

\[ f(x) = 0 \quad x_1 = 0 \quad x_2 = 2 \quad x_3 = -2 \quad \text{equilibria} \]

\[ f(x) = \begin{cases} 
> 0 & x > 2 \\
< 0 & 0 < x < 2 \\
> 0 & -2 < x < 0 \\
< 0 & x < -2
\end{cases} \]

- \( x = 2 \) ← unstable
- \( x = 0 \) ← stable (asymptotically)
- \( x = -2 \) ← unstable
\[
\frac{dx}{dt} = x^2(x^2 - 4)
\]

\[f(x) = x^2(x - 2)(x + 2)\]

Three equilibrium solutions:

\[x_1 = 0, \quad x_2 = 2, \quad x_3 = -2\]

\[f'(x) = \begin{cases} 
> 0 & x > 2 \\
< 0 & 0 < x < 2 \\
< 0 & -2 < x < 0 \\
> 0 & x < -2
\end{cases}\]

- \(x = 2\) unstable
- \(x = 0\) unstable
- \(x = -2\) asymptotically stable