S.5.3 General solutions of linear differential equations with constant coefficients

Consider $n$-th order linear DE

$$(A) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

**Theorem.** Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linear independent solutions of $(A)$ then any solution $y(x)$ is represented as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

**How to find $y_1(x), \ldots, y_n(x)$**

Use characteristic equation $y_k(x) = e^{rk}$

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0$$

If you have a repeated root then you have solutions in the form $y_1 = e^{rx}$, $y_2 = xe^{rx}$,

**Ex.**

$$y^{(6)} + 2y^{(5)} + 4y^{(4)} = 0$$

$h = 6$ we need 6 linear independent solutions

Characteristic $a_6 r^6 + a_5 r^5 + a_4 r^4 = 0$

$$r^6 (r^2 + 4r + 4) = 0$$
\[ r^4 (r^2 + 2)^2 = 0 \]

Six roots:
\[ r_1 = r_2 = r_3 = r_4 = 0 \quad y_1 = 1, \quad y_2 = x, \quad y_3 = x^2, \quad y_4 = x^3 \]

\[ r_5 = r_6 = -2 \quad y_5 = e^{-2x}, \quad y_6 = xe^{-2x} \]

General solution:
\[ y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-2x} + c_6 xe^{-2x} \]

Complex characteristic numbers.

Consider \[ y'' + 4y = 0 \]

Characteristic equation \[ r^2 + 4 = 0 \] or \[ r^2 = -4 \]

What are roots? Roots are complex numbers!

Solution of eqn. \[ r^2 = -1, \text{ root } r = \pm i \]

\( i \) - imaginary unit \( (i)^2 = -1 \), \( i = \sqrt{-1} \)

Complex number
\[ z = x + yi = x + iy, \quad x, y \text{ are real, } i^2 = -1 \]

Ex. \( z = 3 + 2i \)

For \( z = x + iy \) \( x \) u called Real part
\( x = \text{Re} z \)
\( y \) u called Imaginary part
\( y = \text{Im} z \)
Operations with complex numbers

\[ Z_1 = x_1 + iy_1, \quad z = x_2 + iy_2 \]

By definition, sum \[ Z_1 + Z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \]

\[ Z_1 = 3 - 2i, \quad Z_2 = 2 + 7i \]

\[ Z_1 + Z_2 = 5 + 5i \]

Product \[ Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) \]

\[ Z_1 = (3 + 2i), \quad Z_2 = (-2 + 4i) \]

\[ Z_1Z_2 = (3 + 2i)(-2 + 4i) = -6 + 8i + 4i - 8i^2 = -6 + 8i + 4(1) = -2 + 8i \]

Easy to check \[ Z_1 + Z_2 = Z_2 + Z_1, \quad Z_1Z_2 = Z_2Z_1 \]

\[ Z_3(Z_1 + Z_2) = Z_3Z_1 + Z_3Z_2 \]

Conjugate number

\[ Z = x + iy \]

Conjugate \[ \bar{Z} = x - iy \]

\[ Z\bar{Z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 \geq 0 \]
Application of conjugate:

\[
\frac{3+2i}{1-4i} = \frac{(3+2i)(1+4i)}{(1-4i)(1+4i)} = \frac{3-8+i(8+2)}{1+4^2} = -5 + 10i
\]

\[
\frac{3-8+i(8+2)}{1+4^2} = -5 + \frac{10}{17}i
\]

Properties of conjugate numbers

- \[\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\]
- \[\overline{z_1 + z_2} = (x_1 + x_2, y_1 + y_2)\]
- \[\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} = \overline{x_1 + x_2 - i(y_1 + y_2)} = \overline{z_1} + \overline{z_2}\]
- \[\overline{z_1 z_2} = \overline{z_1} \overline{z_2}\]
- \[\overline{z_1 z_2} = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\]
- \[\overline{z_1 z_2} = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)\]
- \[\overline{z_1 z_2} = (x_1 + iy_1)(x_2 - iy_2) = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)\]. Thus, \[\overline{z_1 z_2} = \overline{z_1} \overline{z_2}\]
Corollary \[ \overline{z^2} = \overline{z} \cdot \overline{z} = (\overline{z})^2 \]
\[ \overline{z^3} = \overline{z^2} \cdot \overline{z} = (\overline{z})^3 \]
\[ \overline{z^n} = (\overline{z})^n \]

Fundamental Theorem of Algebra
For any polynomial of degree \( n \):
\[ z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 \]
there exists exactly \( n \) complex roots

Example
\[ z^2 - 3z + 4 = 0 \]
\[ z_1 = \frac{3 + \sqrt{9 - 16}}{2} = \frac{3 + \sqrt{-7}}{2} = \frac{3}{2} + \frac{\sqrt{7}}{2}i \]
\[ z_2 = \frac{3 + \sqrt{9 - 16}}{2} = \frac{3 - \sqrt{-7}}{2} = \frac{3}{2} - \frac{\sqrt{7}}{2}i \]

\( z_1, z_2 \) are conjugate

\[ z^2 + 1 = 0 \quad z_1 = \sqrt{-1} = i \]
\[ z_2 = -\sqrt{-1} = -i \]

two complex roots
they are conjugate \( z_1 = \overline{z_2} \)

General Fact: for polynomial equation with real coefficients
for any root \( z \) there exists conjugate root \( \overline{z} \).
Proposition. Let polynomial equation

\[ z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 \]

have real coefficients

\[ \overline{a_{n-1}} = a_{n-1}, \quad \overline{a_{n-2}} = a_{n-2}, \ldots, \overline{a_0} = a_0 \]

Then if \( z \) is a root of it, then the conjugate \( \overline{z} \) is also a root.

Note \( z \) is a real number

\[ z = x + i0 \quad \text{if and only if} \quad \overline{z} = x + i0 = z \]

\[ z = \overline{z} \quad \text{for real numbers}. \]

Proof. Let \( z \) be a root

\[ z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 \]

Take

\[ \overline{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = 0 \]

\[ \overline{z^n} + \overline{a_{n-1} z^{n-1}} + \ldots + \overline{a_1 z} + \overline{a_0} = 0 \]

\[ \overline{(z)}^n + a_{n-1} \overline{(z)}^{n-1} + \ldots + a_1 \overline{(z)} + a_0 = 0 \]

The same. \( \overline{z} \) is also a root. 

Ex.

\[ z^2 + 2z + 1 = 0 \]

\[ z_1 = \frac{-2 + \sqrt{4 - 8}}{2} = \frac{-2 + \sqrt{-4}}{2} = -1 + i \quad z_1 = \overline{z}_2 \]

\[ z_2 = \frac{-2 - \sqrt{-4}}{2} = -1 - i \]
If coefficients are not real numbers. 

6x. \( z^2 + 2iz + 2 = 0 \)

\[ z_1 = \frac{-2i + \sqrt{-4 - 8}}{2} = \frac{-2i + \sqrt{12}i}{2} = (1 + \sqrt{3})i \]

\[ z_2 = \frac{-2i - \sqrt{-12}}{2} = (-1 - \sqrt{3})i \]

\[ z_1 \neq z_2 \]

You need to master computations with complex numbers.

**Euler formula:**

\[ e^{x+iy} = e^x (\cos y + i \sin y) \]

**Special case:** \( e^{iy} = \cos y + i \sin y \)

Proof by using power Taylor series.

**Important fact:** for complex \( z = a + ib \)

\[ (e^{zx})' = r e^{zx} \]
It means that if $r$ is a characteristic root
\[ r^n + a_{n-1}r^{n-1} + \ldots + a_1r + a_0 = 0 \]

Then function
\[ e^{rx} \]
\[ e^{rx} \]
is a solution of the differential equation
\[ \lambda y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \]

But \[ e^{rx} = e^{ax + ibx} = e^{ax}(\cos bx + i \sin bx) \]

a complex-valued solution of \( \lambda \)

We need Real-Valued solution

Note $r = a + ib$ is characteristic root then \( \bar{r} = a - ib \) is also characteristic root.

Then we have two complex-valued solutions
\[ Z_1(x) = e^{ax}(\cos bx + i \sin bx) = e^{rx} \]
\[ Z_2(x) = e^{ax}(\cos bx - i \sin bx) = e^{\bar{r}x} \]
\[ y_1 = \frac{Z_1 + Z_2}{2} = e^{ax} \cos bx \] it real-valued solution
\[ y_2 = \frac{Z_1 - Z_2}{2i} = e^{ax} \sin bx \] it real-valued solution
Important Fact. If we have a characteristic root \( r = \alpha + i\beta \) (and conjugate \( \bar{r} = \alpha - i\beta \)) then we have two linear independent solutions

\[
y_1 = e^{\alpha x} \cos \beta x, \quad y_2 = e^{\alpha x} \sin \beta x
\]

Ex. \( y'' - 2y' + 2y = 0 \)

Find general solution

\[
r^2 - 2r + 2 = 0 \quad r = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i
\]

\[
r_1 = \frac{2 + \sqrt{4 - 4}}{2} = 1 + i
\]

\[
r_2 = \frac{2 - \sqrt{4 - 4}}{2} = 1 - i
\]

Real linear independent solutions

\[
y_1 = e^x \cos x, \quad y_2 = e^x \sin x
\]

General solution

\[
y = C_1 e^x \cos x + C_2 e^x \sin x
\]

Problem. Find solution \( y(x) \) of

\[
y'' - 2y' + 2y = 0
\]

Satisfy \( y(0) = 1 \quad y'(0) = 3 \)

\[
y = C_1 e^x \cos x + C_2 e^x \sin x, \quad y = C_1 (e^x \cos x - e^x \sin x) + C_2 (e^x \cos x + e^x \sin x)
\]

\[
y(0) = 1 \quad C_1 + C_2 = 3 \quad C_2 = 2
\]

\[
y'(0) = 3 \quad C_1 = 1 \quad C_2 = 2
\]
Solution: \[ y = e^x \cos x + 2e^x \sin x \]

\[ e^x y'' + 4y = 0 \]

\[ r^2 + 4 = 0 \quad r^2 = -4 \]

\[ r_1 = 2i \quad r_2 = -2i \]

Two linear independent solutions

\[ y_1 = e^x \cos 2x \quad y_2 = e^x \sin 2x \]

general solution

\[ y = C_1 \cos 2x + C_2 \sin 2x \]

Example

\[ y^{(4)} + 6y'' + 9y = 0 \]

\[ r^4 + 6r^2 + 9 = 0 \]

\[ (r^2 + 3)^2 = 0 \]

four roots 1 repeated

\[ r_1 = \sqrt{3}i \quad r_2 = -\sqrt{3}i \quad r_3 = \sqrt{3}i \quad r_4 = -\sqrt{3}i \]

We need for solution

\[ y_1 = \cos \sqrt{3}x \quad y_2 = \sin \sqrt{3}x \quad y_3 = x \cos \sqrt{3}x \quad y_4 = x \sin \sqrt{3}x \]

\[ y = C_1 \cos (\sqrt{3}x) + C_2 \sin (\sqrt{3}x) + C_3 x \cos (\sqrt{3}x) + C_4 x \sin (\sqrt{3}x) \]
S.5.4 Mechanical Vibrations

"Springs - mass" mechanical system

Newton Law

\[ m \dddot{x} = F \]

\( x(t) \) in displacement

\[ F = -kx \quad (\text{Hooke's law}) \]

deceleration

Then \( x(t) \) satisfies the differential equation

\[ m \dddot{x} = -kx \]

\[ m \dddot{x} + kx = 0 \]

or

\[ \dddot{x} + \omega^2 x = 0 \quad \text{where} \quad \omega^2 = \frac{k}{m} \]

General solution:

\[ r^2 + \omega^2 = 0 \quad r_1 = i\omega, \quad r_2 = -i\omega \]

General solution

\[ x(t) = A \cos \omega t + B \sin \omega t \]

\( A, B \) are constants, instead of \( C_1, C_2 \)

\( x(t) \) is a periodic motion.
\[ x(t) = A \cos \omega t + B \sin \omega t, \quad \omega = \sqrt{\frac{k}{m}} \]

\[ x(t) = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \cos \omega t + \frac{B}{\sqrt{A^2 + B^2}} \sin \omega t \right) \]

Note
\[ \left( \frac{A}{\sqrt{A^2 + B^2}} \right)^2 + \left( \frac{B}{\sqrt{A^2 + B^2}} \right)^2 = 1 \]

There exist angle \( \alpha \)
\[ \cos \alpha = \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \alpha = \frac{B}{\sqrt{A^2 + B^2}} \]

\( \alpha \) is called the phase angle

\[ x(t) = C \left( \cos \alpha \cos \omega t + \sin \alpha \sin \omega t \right) = C \cos (\omega t - \alpha) \]

where \( C = \sqrt{A^2 + B^2} \)

\( x(t) \) is a periodic motion (vibration)

\( C \) is amplitude

Period of vibration
\[ \omega T = 2\pi \quad \Rightarrow \quad T = \frac{2\pi}{\omega} \]
Oscillations in electric circuits
Vibrations in mechanical systems

\[ x(t) = C \cos(\omega t - \alpha) \]

- \( C \) - amplitude, \( \omega \) - circular frequency
- \( \alpha \) - phase angle

\[ T = \frac{2\pi}{\omega} \quad \text{period}, \quad \nu = \frac{1}{T} \quad \text{frequency} \]

(number of cycles per sec)

Problem 3 (5.5.4) Mass \( m = 3 \text{ kg} \) is attached to the end of spring which is stretched 20 cm by the force \( F = 15 \text{ N} \). It is set in motion \( \dot{x}(0) = x_0 \) and \( v(0) = -10 \text{ m/s} \).

Find amplitude, frequency, period and phase angle.

\[ m x'' + k x = 0 \]

\( m = 3 \text{ kg}, \quad k = \frac{F}{x} = \frac{15}{0.2} = 75 \)

\( 3x'' + 75x = 0 \quad x'' + 25x = 0 \)

\( 1^2 + 25 = 0 \quad \omega = 5 \)

\( x(t) = A \cos 5t + B \sin 5t \quad x' = -5A \sin 5t + 5B \cos 5t \)

\( x(0) = A = 0 \quad x'(0) = 5B = -10 \quad B = -2 \quad c = \sqrt{A^2 + B^2} = 0 \)

\( \cos \alpha = 0 \quad \sin \alpha = -1 \)

\( \omega = \frac{3}{\sqrt{11}} \)
\[ x(t) = -2 \sin \sqrt{5} t = 2 \cos \left(5t - \frac{3\pi}{2}\right) \]

amplitude \( C = 5 \), circular frequency \( \omega = 5 \)

\[ T = \frac{2\pi}{5}, \quad \text{frequency} \quad \nu = \frac{5}{2\pi}, \quad \alpha = \frac{3}{2}\pi \]

Little Trigonometry.

\[ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \]

\[ \alpha = \pi \]

Example

\[ x(t) = 2 \cos 3t - 2 \sin 3t \]

find amplitude \( C \), \( T \) and \( \alpha \)

\[ A = 2, \quad B = -2, \quad C = \sqrt{A^2 + B^2} = \sqrt{4 + 4} = 2\sqrt{2} \]

\[ x(t) = 2\sqrt{2} \left( \frac{1}{\sqrt{2}} \cos 3t - \frac{1}{\sqrt{2}} \sin 3t \right) = 2\sqrt{2} \cos \left(3t - \frac{7}{4}\pi \right) \]

\[ \cos \alpha = \frac{1}{\sqrt{2}} \]

\[ \sin \alpha = -\frac{1}{\sqrt{2}} \]

\[ \alpha' = 2\pi - \pi = \frac{7}{4}\pi \]