MATH 3740
Week 6, lecture #1
S. 5.4 Mechanical vibrations

"mass - spring system"

\[ mx'' + kx = 0 \]

or \[ x'' + \omega^2 x = 0, \quad \omega = \sqrt{\frac{k}{m}} \]

\[ x(t) = C \cos(\omega t - \alpha) \]

- amplitude
- phase angle
- \( \omega \) - circular frequency
- \( T = \frac{2\pi}{\omega} \) - period

Physical Pendulum

\[ s = l \theta \]

acceleration \( s'' = (l\theta)'' = l\theta'' \)

Newton Law

\[ m l \theta'' = -mg \sin \theta \]

or \[ l \theta'' + g \sin \theta = 0 \]

\[ \theta'' + \omega^2 \sin \theta = 0, \quad \omega^2 = \frac{g}{l}, \quad \omega = \sqrt{\frac{g}{l}} \]

\[ T = 2\pi \sqrt{\frac{l}{g}} \]

nonlinear equation!

describes physical pendulum
\[ \theta'' + \omega^2 \sin \theta = 0 \]

Mathematical pendulum:

Linearization of the nonlinear equation near equilibrium solutions

When \( \theta \approx 0 \) (\( \theta \) is small)
then \( \sin \theta \approx \theta \)

We consider equation

\[ \theta'' + \omega^2 \theta = 0 \]

\[ \theta(t) = C \cos(\omega t - \phi) \] periodic motion

"granddaddy clocks"

\[ T = 2\pi \sqrt{\frac{l}{g}} \]

acceleration of gravity

\[ g = \frac{GM}{R^2} \] \( R \) is the distance to the center of Earth

\[ T = 2\pi \sqrt{\frac{l}{g} \frac{R^2}{\delta m n}} \] Earth is not spheric but oblate

\[ T_1 = 2\pi \sqrt{\frac{l_1 R_1^2}{\delta M_1 n}} \quad T_2 = 2\pi \sqrt{\frac{l_2 R_2^2}{\delta M_2 n}} \]

\[ \frac{T_1}{T_2} = \frac{R_1}{R_2} \] different periods different designs.
Free damped motion

"mass - spring - dashpot" system

\[ mx'' = -kx - cx' \]

Equation of motion

\[ F_r = -c x' \text{ resistance / fluid friction force} \]

\[ mx'' + cx' + kx = 0 \]

Introduce notation \( \frac{c}{m} = 2p, \frac{k}{m} = \omega_0^2 \)

Then

\[ x'' + 2px' + \omega_0^2 x = 0 \]

Find solution

Characteristic equation

\[ r^2 + 2pr + \omega_0^2 = 0 \]

\[ r_1 = -p + \sqrt{p^2 - \omega_0^2}, \quad r_2 = -p - \sqrt{p^2 - \omega_0^2} \]

Let \( p > \omega_0 \) then \( r_1, r_2 \) are real number

(over-damped case)

\( r_1 < 0, r_2 < 0 \)

General solution

\[ x(t) = Ae^{r_1 t} + Be^{r_2 t} \]

\( \lim_{t \to \infty} x(t) = 0 \)

Stable system
Case \( p < \omega_0 \) (underdamped case)

\[
\begin{align*}
 r_1 &= -p + i \sqrt{p^2 - \omega_0^2} \\
r_2 &= -p - i \sqrt{p^2 - \omega_0^2}
\end{align*}
\]

Denote \( \omega = \sqrt{p^2 - \omega_0^2} \) \( \lim \)

\[
x(t) = A e^{-pt} \cos \omega t + B e^{-pt} \sin \omega t
\]

we have

\[
\lim_{t \to +\infty} x(t) = 0
\]

**Problem 15.5.4**

\[
m x'' + c x' + k x = 0
\]

\[
m = \frac{1}{2}, \quad c = 3, \quad k = 4, \quad x_0 = 2, \quad x_0 = x'(0) = 0
\]

Find \( x(t) \)

**Equation of motion**

\[
\frac{1}{2} x'' + 3 x' + 4 x = 0
\]

\[
x'' + 6 x' + 8 x = 0
\]

**overdamped**

\[
r^2 + 6 r + 8 = 0
\]

\[
r_1 = \frac{-6 + \sqrt{36 - 32}}{2} = \frac{-6 + 2}{2} = -2
\]

\[
r_2 = \frac{-6 - 2}{2} = -4
\]

\[
x(t) = A e^{-2t} + B e^{-4t}
\]

\[
x(0) = A + B = 2 \quad x'(0) = -2A - 4B = 0
\]

\[
-A - 2B = 0
\]

\[
B = 2A - 4 \quad (x = 4e^{-2t} - 2e^{-4t})
\]
S.7.5. Nonhomogeneous Equation
and undetermined coefficients.

Consider nonhomogeneous equation
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x) \]
in terms of linear operator
\[ L(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 \]
we have
\[ L(D)y = f(x) \]
nonhomogeneous equation
\[ L(D)y = 0 \]
homogeneous eqn

and we know general solution of it
\[ y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \]

\[ L(D)y_c = 0 \]

How to find general solution of nonhomogeneous equation?

(1) \( L(D)y = f(x) \)

Assume we know particular solution \( y_p(x) \) of \( (1) \) nonhomogeneous equation
Assume we know particular solution $y_p$ of

$$L(D)y_p = f(x)$$

Then the general solution of nonhomogeneous solution $y(x)$ is represented as

$$y(x) = y_p(x) + y_c(x) = y_p(x) + C_1y_1(x) + C_2y_2(x) + \ldots + C_ny_n(x)$$

The proof is easy: let $y$ is a general solution of $(\star)$ ($L(D)y = f(x)$)

Then

$$L(D)y - L(D)y_p = f(x) - f(x) = 0$$

Thus $y - y_p$ is a solution of homogeneous equation

$$y - y_p = y_c = C_1y_1 + C_2y_2 + \ldots + C_ny_n$$

Then, to find solution of $L(D)y = f$

we need to find a particular solution $y_p$ of $(\star)$

$$L(D)y_p = f(x)$$
How to find a particular solution?

L(D)y = f(x)

for important classes of f(x)

for example, \( f(x) = P_m(x) \) - polynomial of degree m. Then try to find

\[ y_p(x) = A_m x^m + A_{m-1} x^{m-1} + \ldots + A_1 x + A_0 \]

\( A_m, A_{m-1}, \ldots, A_0 \) are undetermined coefficients.

\[ y^2 - 4y = x^3 + 2x \]

find a particular solution

\[ y_p = A_3 x^3 + A_2 x^2 + A_1 x + A_0 \]

\[ y'_p = 3A_3 x^2 + 2A_2 x + A_1 \]

\[ y''_p = 6A_3 x + 2A_2 \]

\[ y''_p - 4y = 6A_3 x + 2A_2 - 4(A_3 x^3 + A_2 x^2 + A_1 x + A_0) = \]

\[ = (-4A_3) x^3 - 4A_2 x^2 + (6A_3 - 4A_1) x + (2A_2 - 4A_0) \]

\[ = f(x) = x^3 + 2x = 1 \cdot x^3 + 0 \cdot x^2 + 2 \cdot x + 0 \]

Thus \(-4A_3 = 1, -4A_2 = 0, 6A_3 - 4A_1 = 2, 2A_2 - 4A_0 = 0\)
Then \( A_3 = -\frac{1}{4}, \) \( A_2 = 0 \)
\[ A_1 = \frac{(6A_3 - 2)}{4} = -\frac{7}{8} \quad A_0 = 0 \]

\[ y = x^3 - \frac{7}{8}x \]

**General solution**

\( (**) y'' - 4y = x^3 + 2x \)

Consider homogeneous equation

\[ y'' - 4y = 0 \quad r^2 - 4 = 0 \]
\[ r_1 = 2 \quad r_2 = -2 \]

\[ y_c = C_1 e^{2x} + C_2 e^{-2x} \]

**General solution of non homogeneous equation**

\[
\boxed{y = y_p + y_c = -\frac{1}{4}x^3 - \frac{7}{8}x + C_1 e^{2x} + C_2 e^{-2x}}
\]

**Example** \( f(x) \) in trigonometric function

\( f(x) = \cos(\omega x) \) or \( \sin(\omega x) \)

Then \( y_p = A \cos(\omega) \)

**undetermined coefficients**

\[ y'' + 2y' - 3y = \cos(2x) \]

Find \( y_p = A \cos(2x) + B \sin(2x) \)
\[ y = A \cos 2x + B \tan 2x \]
\[ y' = -2A \sin 2x + 2B \sec 2x \]
\[ y'' = -4A \cos 2x - 4B \tan 2x \]
\[ y'' + 2y' - 3y = (-4A \cos 2x - 4B \tan 2x) + 
+ 2(-2A \sin 2x + 2B \cos 2x) - 3(A \cos 2x + B \tan 2x) = 
= (-4A - 4B - 3A) \cos 2x + (-4B - 4A - 3B) \tan 2x = 
= \cos 2x \]

Then find \( A, B \):

\[ -7A + 4B = 1 \quad \Rightarrow \quad -7A + B = 1 \]
\[ -4A - 7B = 0 \quad \Rightarrow \quad \left( -\frac{16}{7} - 7 \right) B = -\frac{4}{7} \]

\[ B = \frac{4}{65} \quad A = \frac{-7}{65} \]

\[ y_p = -\frac{7}{65} \cos 2x + \frac{4}{65} \tan 2x \]

Find \( y_c \), \( y' + 2y' - 3y = 0 \)

\[ r^2 + 2r - 3 = 0 \]
\[ r_1 = -3 \quad r_2 = +1 \]

General \( y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x} \)

\[ y = -\frac{7}{65} \cos 2x + \frac{4}{65} \tan 2x + c_1 e^{-3x} + c_2 e^x \]
General Rule:

Let \( f(x) = e^{ax} \cos \beta x \cdot P_m(x) \)

\( P_m(x) \) is a polynomial of degree \( m \).

Find a general solution of \( L(D)y = f(x) \).

Step 1. Look at homogeneous equation

\( L(D)y_c = 0 \)

Find its characteristic roots \( r_1, r_2, \ldots, r_n \)

and consider \( r = \alpha + i\beta \)

**Non-Resonance case** (\( r_0 \) is not a characteristic root)

Then

\( y_p(x) \) can be found in the form

\[ y_p(x) = R_m(x) e^{\alpha x} \cos \beta x + S_m(x) e^{\alpha x} \sin \beta x \]

\( R_m(x), S_m(x) \) are polynomials with undetermined coefficients!

**Resonance case** (\( r_0 \) is one of the characteristic roots)

\[ y_p = x^s \left( R_m(x) e^{\alpha x} \cos \beta x + S_m(x) e^{\alpha x} \sin \beta x \right) \]

\( s \geq 1 \) \( if \) root \( r_0 \) is repeated
Ex. Problem #1 S 5.5

\[ y'' + 16y = e^{3x} \]

Step 1 Homogeneous equation

\[ y'' + 16y = 0 \quad r^2 + 16 = 0 \]

\[ r_1 = 4i, \quad r_2 = -4i, \quad y_c = c_1 \cos 4x + c_2 \sin 4x \]

\[ f(x) = e^{3x} = P_n(x)e^{\alpha x} \cos \beta x \]

\[ m = 0, \quad \alpha = 3, \beta = 0 \]

\[ \text{number } r_0 = \alpha + i\beta = 3 \quad \text{and it is not} \]
\[ \text{one of } r_1, \text{or } r_2. \quad \text{Non Resonance case!} \]

\[ y_p = Ae^{3x}, \quad A \text{ undetermined coefficient} \]

\[ y_p' = 3Ae^{3x} \]

\[ y_p'' = 9Ae^{3x} \quad y_p'' + 16y_p = 9Ae^{3x} + 16Ae^{3x} = \]

\[ = 25Ae^{3x} = f(x) = e^{3x} \]

\[ A = \frac{1}{25} \]

\[ \text{Thus } y_p = \frac{1}{25} e^{3x} \]

General solution

\[ y = \frac{1}{25} e^{3x} + c_1 \cos 4x + c_2 \sin 4x \]
Problem 4.21 (only set an approximate form of a particular solution $y_p$)

$$y'' - 2y' + 2y = e^x \sin x$$

Homogeneous solution

$$y'' - 2y' + 2y = 0 \quad r^2 - 2r + 2 = 0$$

Characteristic roots $r_1 = 1 + i$, $r_2 = 1 - i$

$$y_h = c_1 e^x \cos x + c_2 e^x \sin x$$

Consider $f(x) = P_n(x) e^{\alpha x} \sin \beta x = e^{\alpha x} \sin x$

Then $m = 0$, $\alpha = 1$, $\beta = 1$

$r_0 = \alpha + i\beta = 1 + i$ is one of the characteristic roots! RESONANCE!

$$y_p(x) = x(A e^x \cos x + B e^x \sin x),$$

find $A, B$.

Problem

$$y^{(5)} - y^{(3)} = 3x^2 - x$$

Homogeneous equation

$$y^{(5)} - y^{(3)} = 0 \quad r^5 - r^3 = r^3(r^2 - 1) = 0$$

$r_1 = 0, r_2 = 0, r_3 = 0, r_4 = 1, r_5 = -1$

$$y_h = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$$

$$f(x) = 3x^2 - x = P_n(x) e^{\alpha x} \sin \beta x$$

$m = 2$, $\alpha = 0$, $\beta = 0$
\( r_0 = 0 + i \alpha \) is a characteristic vector which is repeated three times.

\[ y_p = x^3 (Ax^2 + Bx + C) \]

\[ d \text{ a particular solution if } f(x) = f_1(x) + f_2(x) \]

\[ L(D) y = f_1(x) + f_2(x) \]

Then \( y_p = y_{p_1} + y_{p_2} \) where \( L(D) y_{p_1} = f_1(x) \), \( L(D) y_{p_2} = f_2(x) \).

\[ \text{Proof: } L(D) y = L(D)(y_{p_1} + y_{p_2}) = L(D)y_{p_1} + L(D)y_{p_2} = f_1(x) + f_2(x) \]

\[ E \]

\[ y'' + y = 3 \cos 2x + (x+2) e^x \]

\[ f_1 = 3 \cos 2x \quad f_2 = (x+2) e^x \]

\[ L(D)y = 0 \quad y'' + y = 0 \quad r^2 + 1 = 0 \quad \nu_1 = i, \nu_2 = -i \]

\[ y_c = c_1 \cos x + c_2 \sin x \]

\[ y_{p_1} = 3x^2 \cos 2x \quad m = 0, \alpha = 0, \beta = 2 \quad r_{o_1} = 2 \cos \pm 1 \quad \text{Non Res} \]

\[ y_{p_1} = A \cos x + B \sin x \]

\[ y_{p_2} = (x+2) e^x \quad m = 1, \alpha = 1, \beta = 0 \quad r_{o_2} = 1 \pm i \quad \text{Non Res} \]

\[ y_p = (A_1 x + A_2) e^x \]