22–25 TANGENT

Given a curve C: $\mathbf{r}(t)$, find a tangent vector $\mathbf{r}'(t)$, a unit tangent vector $\mathbf{u}'(t)$, and the tangent of C at P. Sketch the curve and the tangent.

22. $\mathbf{r}(t) = [t,\ \ t^2,\ \ 0]$, P: $(2, 4, 0)$

23. $\mathbf{r}(t) = [5 \cos t,\ \ 5 \sin t,\ \ 0]$, P: $(4, 3, 0)$

24. $\mathbf{r}(t) = [3 \cos t,\ \ 3 \sin t,\ \ 4t]$, P: $(3, 0, 8\pi)$

25. $\mathbf{r}(t) = [\cosh t,\ \ \sinh t]$, P: $(\frac{5}{3}, \frac{4}{3})$

26–28 LENGTH

Find the length and sketch the curve.

26. Circular helix $\mathbf{r}(t) = [2 \cos t,\ \ 2 \sin t,\ \ 6t]$ from $(2, 0, 0)$ to $(2, 0, 24\pi)$

27. Catenary $\mathbf{r}(t) = [t,\ \ \cosh t]$ from $t = 0$ to $t = 1$

28. Hypocycloid $\mathbf{r}(t) = [a \cos^3 t,\ \ a \sin^3 t]$, total length

29. Show that (10) implies $\ell = \int_a^b \sqrt{1 + y'^2}\, dx$ for the length of a plane curve C: $y = f(x)$, $z = 0$, $a \leqq x \leqq b$.

30. Polar coordinates $\rho = \sqrt{x^2 + y^2}$, $\theta = \arctan (y/x)$ give $\ell = \int_\alpha^\beta \sqrt{\rho^2 + \rho'^2}\, d\theta$, where $\rho' = d\rho/d\theta$. Derive this. Use it to find the total length of the **cardioid** $\rho = a(1 - \cos \theta)$. Sketch this curve. *Hint.* Use (10) in App. 3.1.

31. CAS PROJECT. Polar Representations. Use your CAS to graph the following famous curves[4] and investigate their form depending on parameters a and b.

$\rho = a\theta$ *Spiral of Archimedes*

$\rho = ae^{b\theta}$ *Logarithmic spiral*

$\rho = \dfrac{2a \sin^2 \theta}{\cos \theta}$ *Cissoid of Diocles*

$\rho = \dfrac{a}{\cos \theta} + b$ *Conchoid of Nicomedes*

$\rho = a/\theta$ *Hyperbolic spiral*

$\rho = \dfrac{3a \sin 2\theta}{\cos^3 \theta + \sin^3 \theta}$ *Folium of Descartes*

$\rho = 2a \dfrac{\sin 3\theta}{\sin 2\theta}$ *Maclaurin's trisectrix*

$\rho = 2a \cos \theta + b$ *Pascal's snail*

32–34 CURVES IN MECHANICS

Velocity and Acceleration. Forces on moving objects (cars, airplanes, etc.) require that the engineer knows corresponding **tangential** and **normal accelerations.** Find them, along with the **velocity** and **speed,** for the following motions. Sketch the path.

32. $\mathbf{r}(t) = [4t,\ \ -3t,\ \ 0]$

33. $\mathbf{r}(t) = [t,\ \ t^2,\ \ 0]$

34. $\mathbf{r}(t) = [\cos t,\ \ 2 \sin t,\ \ 0]$

35. (**Cycloid**) Given

$$\mathbf{r}(t) = (R \sin \omega t + \omega R t)\mathbf{i} + (R \cos \omega t + R)\mathbf{j}.$$

This cycloid is the path of a point on the rim of a wheel of radius R that rolls without slipping along the x-axis. Find $\mathbf{v}$ and $\mathbf{a}$ at the maximum y-values of the curve.

36. CAS PROJECT. Paths of Motions. Gear transmissions and other engineering constructions often involve complicated paths whose study is greatly facilitated by the use of a CAS. To grasp the idea, graph the following paths and find the velocity, the speed, and the tangential and normal accelerations.

(**a**) $\mathbf{r}(t) = [2 \cos t + \cos 2t,\ \ 2 \sin t - \sin 2t]$ (*Steiner's hypocycloid*)

(**b**) $\mathbf{r}(t) = [\cos t + \cos 2t,\ \ \sin t - \sin 2t]$

(**c**) $\mathbf{r}(t) = [\cos t,\ \ \sin 2t,\ \ \cos 2t]$

(**d**) $\mathbf{r}(t) = [ct \cos t,\ \ ct \sin t,\ \ ct]$ $(c \neq 0)$

37. (**Sun and earth**) Find the acceleration of the earth toward the sun from (19) and the fact that the earth revolves about the sun in a nearly circular orbit with an almost constant speed of 30 km/sec.

38. (**Earth and moon**) Find the centripetal acceleration of the moon toward the earth, assuming that the orbit of the moon is a circle of radius 239,000 miles $= 3.85 \cdot 10^8$ m, and the time for one complete revolution is 27.3 days $= 2.36 \cdot 10^6$ sec.

39. (**Satellite**) Find the speed of an artificial earth satellite traveling at an altitude of 80 miles above the earth's surface, where $g = 31$ ft/sec^2. (The radius of the earth is 3960 miles.)

40. (**Satellite**) A satellite moves in a circular orbit 450 miles above the earth's surface and completes 1 revolution in 100 min. Find the acceleration of gravity at the orbit from these data and from the radius of the earth (3960 miles).

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[4]Named after ARCHIMEDES (c. 287–212 B.C.), DESCARTES (Sec. 9.1), DIOCLES (200 B.C.), MACLAURIN (Sec. 15.4), NICOMEDES (250? B.C.) ÉTIENNE PASCAL (1588–1651), father of BLAISE PASCAL (1623–1662).

THEOREM 2

Path Dependence

The line integral (3) generally depends not only on **F** *and on the endpoints A and B of the path, but also on the path itself along which the integral is taken.*

PROOF Almost any example will show this. Take, for instance, the straight segment C_1: $\mathbf{r}_1(t) = [t, t, 0]$ and the parabola C_2: $\mathbf{r}_2(t) = [t, t^2, 0]$ with $0 \leqq t \leqq 1$ (Fig. 221) and integrate $\mathbf{F} = [0, xy, 0]$. Then $\mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) = t^2$, $\mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) = 2t^4$, so that integration gives 1/3 and 2/5, respectively. ■

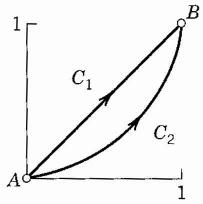

Fig. 221. Proof of Theorem 2

PROBLEM SET 10.1

1–12 LINE INTEGRAL. WORK DONE BY A FORCE

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the following data. If **F** is a force, this gives the work done in the displacement along C. (Show the details.)

1. $\mathbf{F} = [y^3, \ x^3]$, C the parabola $y = 5x^2$ from A: (0, 0) to B: (2, 20)
2. **F** as in Prob. 1, C the shortest path from A to B. Is the integral smaller? Give reason.
3. **F** as in Prob. 1, C from A straight to (2, 0), then vertically up to B
4. $\mathbf{F} = [x^2, \ y^2, \ 0]$, C the semicircle from (2, 0) to (−2, 0), $y \geqq 0$
5. $\mathbf{F} = [xy^2, \ x^2y]$, C: $\mathbf{r} = [\cosh t, \ \sinh t, \ 0]$, $0 \leqq t \leqq 2$. Sketch C.
6. $\mathbf{F} = [e^x, \ e^y]$ clockwise along the circle with center (0, 0) from (1, 0) to (0, −1)
7. $\mathbf{F} = [z, \ x, \ y]$, C: $\mathbf{r} = [\cos t, \ \sin t, \ t]$ from (1, 0, 0) to $(1, 0, 4\pi)$
8. $\mathbf{F} = [\cosh x, \ \sinh y, \ e^z]$, C: $\mathbf{r} = [t, \ t^2, \ t^3]$ from (0, 0, 0) to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$
9. **F** as in Prob. 8, C the straight segment from (0, 0, 0) to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$
10. $\mathbf{F} = [x, \ -z, \ 2y]$ from (0, 0, 0) straight to (1, 1, 0), then to (1, 1, 1), back to (0, 0, 0)
11. $\mathbf{F} = [e^x, \ e^y, \ e^z]$, $\mathbf{r} = [t, \ t^2, \ t^2]$ from (0, 0, 0) to (2, 4, 4). Sketch C.
12. $\mathbf{F} = [y^2, \ x^2, \ \cos^2 z]$, C as in Prob. 7. Sketch C.
13. **WRITING PROJECT. From Definite Integrals to Line Integrals.** Write a short report (1–2 pages) with examples on line integrals as generalizations of definite integrals. The latter give the area under a curve. Explain the corresponding geometric interpretation of a line integral.
14. **PROJECT. Independence of Representation.**

    Dependence on Path. Consider the integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, where $\mathbf{F} = [xy, -y^2]$.

    (a) One path, several representations. Find the value of the integral when $\mathbf{r} = [\cos t, \sin t]$, $0 \leqq t \leqq \pi/2$. Show that the value remains the same if you set $t = -p$ or $t = p^2$ or apply two other parametric transformations of your own choice.

    (b) Several paths. Evaluate the integral when C: $y = x^n$, thus $\mathbf{r} = [t, t^n]$, $0 \leqq t \leqq 1$, where $n = 1, 2, 3, \cdots$. Note that these infinitely many paths have the same endpoints.

    (c) Limit. What is the limit in (b) as $n \to \infty$? Can you confirm your result by direct integration without referring to (b)?

    (d) Show path dependence with a simple example of your choice involving two paths.

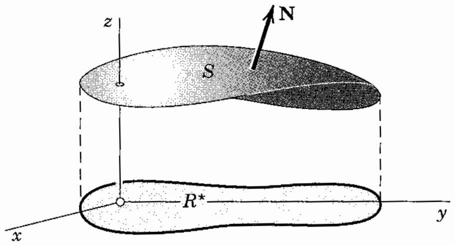

Fig. 248. Formula (11)

Here R^* is the projection of S into the xy-plane (Fig. 248) and the normal vecto points *up*. If it points *down*, the integral on the right is preceded by a minus sig

From (11) with $G = 1$ we obtain for the **area** $A(S)$ of $S\colon z = f(x, y)$ the form

$$A(S) = \iint_{R^*} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy \tag{12}$$

where R^* is the projection of S into the xy-plane, as before.

PROBLEM SET 10.6

1–12 FLUX INTEGRALS (3) $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$

Evaluate these integrals for the following data. Indicate the kind of surface. (Show the details of your work.)

1. $\mathbf{F} = [2x,\ 5y,\ 0]$, $S\colon \mathbf{r} = [u,\ v,\ 4u + 3v]$, $0 \leqq u \leqq 1$, $-8 \leqq v \leqq 8$
2. $\mathbf{F} = [x^2,\ y^2,\ z^2]$, $S\colon x + y + z = 4$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$
3. $\mathbf{F} = [x - z,\ y - x,\ z - y]$, $S\colon \mathbf{r} = [u \cos v,\ u \sin v,\ u]$, $0 \leqq u \leqq 3$, $0 \leqq v \leqq \pi$
4. $\mathbf{F} = [e^y,\ -e^z,\ e^x]$, $S\colon x^2 + y^2 = 9$, $x \geqq 0$, $y \geqq 0$, $0 \leqq z \leqq 2$
5. $\mathbf{F} = [x,\ y,\ z]$, $S\colon \mathbf{r} = [u \cos v,\ u \sin v,\ u^2]$, $0 \leqq u \leqq 4$, $-\pi \leqq v \leqq \pi$
6. $\mathbf{F} = [\cosh yz,\ 0,\ y^4]$, $S\colon y^2 + z^2 = 1$, $0 \leqq x \leqq 20$, $z \geqq 0$
7. $\mathbf{F} = [1,\ 1,\ 1]$, S the sphere of radius 1 and center 0
8. $\mathbf{F} = [\tan xy,\ x^2 y,\ -z]$, $S\colon y^2 + \frac{1}{9}z^2 = 1$, $1 \leqq x \leqq 4$
9. $\mathbf{F} = [0,\ x,\ 0]$, $S\colon x^2 + y^2 + z^2 = a^2$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$
10. $\mathbf{F} = [y^2,\ x^2,\ z^4]$, $S\colon z = 4\sqrt{x^2 + y^2}$, $0 \leqq z \leqq 8$, $y \geqq 0$
11. $\mathbf{F} = [y^3,\ x^3,\ z^3]$, $S\colon x^2 + 4y^2 = 4$, $x \geqq 0$, $y \geqq 0$, $0 \leqq z \leqq h$
12. $\mathbf{F} = [\cosh y,\ 0,\ \sinh x]$, $S\colon z = x + y^2$, $0 \leqq y \leqq x$, $0 \leqq x \leqq 1$
13. **CAS EXPERIMENT.** Write a program for surface integrals (3) that prints intermedi (**F**, **F**•**N**, the integral over one of the two Can you experimentally obtain rules on fun surfaces giving integrals that can be evalua usual methods of calculus? Make a list of p negative results.

14–20 SURFACE INTEGRALS (6) $\iint_S G(\mathbf{r})$

Evaluate these integrals for the following data. In kind of surface. (Show the details.)

14. $G = \cos y + \sin x$, $S\colon x + y + z = 2$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$
15. $G = 5(x + y + z)$, $S\colon z = x + 2y$, $0 \leqq y \leqq x$, $0 \leqq x \leqq 2$
16. $G = ye^x + xe^y + e^z$, $S\colon x^2 + y^2 = 16$, $y \geqq 0$, $0 \leqq z \leqq 4$
17. $G = (x^2 + y^2 + z^2)^2$, $S\colon z = \sqrt{x^2 + y^2}$, y $0 \leqq z \leqq 2$
18. $G = ax + by + cz$, $S\colon x^2 + y^2 + z^2 = 1$, y
19. $G = \arctan(y/x)$, $S\colon z = x^2 + y^2$, $1 \leqq z \leqq 9$, $x \geqq 0$, $y \geqq 0$
20. $G = 3xy$, $S\colon z = xy$, $0 \leqq x \leqq 1$, $0 \leqq y \leqq$
21. **(Fun with Möbius)** Make Möbius strips slim rectangles R of grid paper (graph paper) the short sides together after giving the pa twist. In each case count the number of par

for proper direction on C and normal vector $\mathbf{n}$ on S. Since curl $\mathbf{F} = \mathbf{0}$ in D, the surface integral and hence the line integral are zero. This and Theorem 2 of Sec. 10.2 imply that the integral (9) is path independent in D. This completes the proof. ■

PROBLEM SET 10.9

1–8 DIRECT INTEGRATION OF THE SURFACE INTEGRALS

Evaluate the integral $\iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dA$ directly for the given $\mathbf{F}$ and S.

1. $\mathbf{F} = [4z^2, \;\; 16x, \;\; 0]$, S: $z = y$ $(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$
2. $\mathbf{F} = [0, \;\; 0, \;\; 5x \cos z]$, S: $x^2 + y^2 = 4, y \geqq 0, 0 \leqq z \leqq \frac{1}{2}\pi$
3. $\mathbf{F} = [-e^y, \;\; e^z, \;\; e^x]$, S: $z = x + y$ $(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$
4. $\mathbf{F} = [3 \cos y, \;\; \cosh z, \;\; x]$, S the square $0 \leqq x \leqq 2, 0 \leqq y \leqq 2, z = 4$
5. $\mathbf{F} = [e^{2z}, \;\; e^z \sin y, \;\; e^z \cos y]$, S: $z = y^2$ $(0 \leqq x \leqq 4, 0 \leqq y \leqq 1)$
6. $\mathbf{F} = [z^2, \;\; x^2, \;\; y^2]$, S: $z^2 = x^2 + y^2, y \geqq 0, 0 \leqq z \leqq 2$
7. $\mathbf{F} = [z^2, \;\; \frac{3}{2}x, \;\; 0]$, S the square $0 \leqq x \leqq a, 0 \leqq y \leqq a, z = 1$
8. $\mathbf{F} = [y^3, \;\; -x^3, \;\; 0]$, S: $x^2 + y^2 \leqq 1, z = 0$

9. Verify Stokes's theorem for $\mathbf{F}$ and S in Prob. 7.
10. Verify Stokes's theorem for $\mathbf{F}$ and S in Prob. 8.

11–18 EVALUATION OF $\oint_C \mathbf{F} \bullet \mathbf{r}'\, ds$

Calculate this line integral by Stokes's theorem for the following $\mathbf{F}$ and C. Assume the Cartesian coordinates to be right-handed and the z-component of the surface normal used to be positive. (Show the details.)

11. $\mathbf{F} = [-3y, \;\; 3x, \;\; z]$, C the circle $x^2 + y^2 = 4, z = 1$
12. $\mathbf{F} = [4z, \;\; -2x, \;\; 2x]$, C the intersection of $x^2 + y^2 = 1$ and $z = y + 1$
13. $\mathbf{F} = [y^2, \;\; x^2, \;\; -x + z]$, around the triangle with vertices (0, 0, 1), (1, 0, 1), (1, 1, 1)
14. $\mathbf{F} = [y, \;\; xy^3, \;\; -zy^3]$, C the circle $x^2 + y^2 = a^2, z = b\ (> 0)$
15. $\mathbf{F} = [y, \;\; z^2, \;\; x^3]$, C as in Prob. 12
16. $\mathbf{F} = [x^2, \;\; y^2, \;\; z^2]$, C the intersection of $x^2 + y^2 + z^2 = 4$ and $z = y^2$
17. $\mathbf{F} = [\cos \pi y, \;\; \sin \pi x, \;\; 0]$, around the rectangle with vertices (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 0)
18. $\mathbf{F} = [z, \;\; x, \;\; y]$, C as in Prob. 13

19. **(Stokes's theorem not applicable)** Evaluate $\oint_C \mathbf{F} \bullet \mathbf{r}'\, ds$, $\mathbf{F} = (x^2 + y^2)^{-1}[-y, x]$, C: $x^2 + y^2 = 1, z = 0$, oriented clockwise. Why can Stokes's theorem not be applied? What (false) result would it give?
20. **WRITING PROJECT. Grad, Div, Curl in Connection with Integrals.** Make a list of ideas and results on this topic in this chapter. See whether you can rearrange or combine parts of your material. Then subdivide the material into 3–5 portions and work out the details of each portion. Include no proofs but simple typical examples of your own that lead to a better understanding of the material.

CHAPTER 10 REVIEW QUESTIONS AND PROBLEMS

1. List the kinds of integrals in this chapter and how the integral theorems relate some of them.
2. How can work of a variable force be expressed by an integral?
3. State from memory how you can evaluate a line integral. A double integral.
4. What do you remember about path independence? Why is it important?
5. How did we use Stokes's theorem in connection with path independence?
6. State the definition of curl. Why is it important in this chapter?
7. How can you transform a double integral or a surface integral into a line integral?
8. What is orientation of a surface? What is its role in connection with surface integrals?
9. State the divergence theorem and its applications from memory.
10. State Laplace's equation. Where in physics is it important? What properties of its solutions did we discuss?

11–20 LINE INTEGRALS $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r}$ (WORK INTEGRALS)

Evaluate, with $\mathbf{F}$ and C as given, by the method that seems most suitable. Recall that if $\mathbf{F}$ is a force, the integral gives the work done in a displacement along C. (Show the details.)

11. $\mathbf{F} = [x^2, \;\; y^2, \;\; z^2]$, C the straight-line segment from (4, 1, 8) to (0, 2, 3)