ME5390
Part of Advanced Thermal Design

Fin Design with Thermal Radiation

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1.10  Fin Design with Thermal Radiation

1.10.1  Review of Thermal Radiation

Radiation is the electromagnetic waves (or photons), propagating through a transparent medium or even in a vacuum effectively as one of the heat transfer mechanisms. Radiation is a spectrum having a wide range of the wavelengths from microwaves to gamma rays. The thermal radiation encompasses a range from infrared to ultraviolet, which includes the light. Rainbow from red to violet exhibits that the light is indeed a spectrum. The surface of any matter emits the electromagnetic radiation if the temperature of the surface is greater than absolute zero, and also absorbs the radiation from the surroundings, which is called irradiation. The emissivity and absorptivity of real bodies vary depending on the finishes of the surfaces and the nature of the irradiation, lying between 0 and 1. This feature provides engineers with potential for improvement in the control of heat transfer. It is essential to define a perfect emitter, absorber, and also diffuser, which is called a blackbody. The nature of radiation involves spectral and directionality. The spectral emissive powers of a blackbody and a real surface are illustrated in Figure 1.12, where the emissive power of the real surface shows always less than that of the blackbody.

![Figure 1.12 Spectral distributions of a blackbody and a real surface.](image)

The second nature of thermal radiation relates to its directionality, as shown in Figure 1.13. Blackbody has a diffuse nature.
A blackbody is a perfect emitter and absorber. A blackbody emits radiation energy uniformly in all directions, which is called a diffusive emitter. Plank’s law provides the spectral distribution of blackbody emission as:

$$E_{λ, b}(λ, T) = \frac{C_1}{\lambda^5} \left( \exp\left(\frac{C_2}{\lambda T}\right) - 1 \right)$$  \hspace{1cm} (1.147)

where $C_1=3.472 \times 10^8 \text{W·μm}^4/\text{m}^2$ and $C_2=1.439 \times 10^4 \text{μm·K}$. Equation (1.147) is a function of both wavelength and temperature, which is plotted in Figure 1.14, showing that the spectral emissive power distribution depends on the temperature and wavelength.
If we integrate the spectral distribution, Equation (1.147), for a temperature of a blackbody with respect to wavelength $\lambda$ over the entire range, we have the total emissive power of the blackbody, which is known the Stefan-Boltzmann law. The total emissive power is

$$E_b = \sigma T^4$$

(1.148)

where $E_b$ is the total emissive power of a blackbody and $T$ is the surface temperature of the body and $\sigma = 5.67 \times 10^{-8}$ W/m$^2 \cdot$K$^4$, which is termed the Stefan-Boltzmann constant.

Therefore, the emissive power of a real surface is

$$E = \varepsilon \sigma T^4$$

(1.148a)

where $\varepsilon$ is the emissivity.

**Irradiation G**

Irradiation G is an incident radiation, which may originate from other surfaces or surroundings, being independent of the finishes of the incepting surface, which is shown in Figure 1.15. Irradiation will have spectral and directional properties. Irradiation could be partially absorbed, reflected, or transmitted as shown.

$$\rho + \alpha + \gamma = 1$$

(1.149)

where $\rho$ is the reflectivity, $\alpha$ is the absorptivity, and $\gamma$ is the transmissivity.

![Figure 1.15](image)

Figure 1.15 Surface radiosity J includes emission $\varepsilon E_b$ and reflection $\rho G$; irradiation G includes absorption $\alpha G$, reflection $\rho G$, and transmission $\gamma G$.

For instance, if the surface is opaque or perfectly diffuse, the transmissibility $\gamma$ is zero.
\[ \rho + \alpha = 1 \]  

(1.150)

**Radiosity J**
Radiosity \( J \) is all the radiant energy leaving a surface, consisting of the direct emission \( \varepsilon E_b \) from the surface and the reflected portion \( \rho G \) of the irradiation, which is shown in Figure 1.15. The radiosity is

\[
J = \varepsilon E_b + \rho G
\]  

(1.151)

**Kirchhoff’s Law**
Consider a large, isothermal enclosure of surface temperature \( T_s \) within which several small bodies are confined as shown in Figure 1.16. Under steady-state conditions, thermal equilibrium must exist between the bodies and the enclosure. Regardless of the orientation and the surface properties of the enclosure, the irradiation experienced by anybody in the enclosure must be diffuse and equal to emission from a blackbody at \( T_s \).

![Figure 1.16 Radiation exchange in an isothermal enclosure.](image)

The irradiation in the *enclosure* behaves like a blackbody.

\[
G = E_b(T_s)
\]  

(1.152)

The absorbed irradiation is

\[
G_{abs} = \alpha G = \alpha E_b(T_s)
\]  

(1.152a)

At thermal equilibrium, the energy absorbed by the body must be equal to the energy emitted. Otherwise, there would be an energy flow into or out the body, which would raise or lower its temperature. For a body denoted by 2, the energy balance gives
\[ A_2 \alpha_2 G - A_2 E_2 = 0 \] (1.153)

Or equivalently

\[ \alpha_2 E_b(T_s) - \varepsilon_2 E_b(T_s) = 0 \] (1.154)

Eventually,

\[ \alpha_2 = \varepsilon_2 \] (1.155)

The same thing happens in other bodies. Therefore, in an enclosure, we have

\[ \alpha = \varepsilon \] (1.156)

The total emissivity of the surface is equal to its total absorptivity, which is known as the Kirchhoff’s law. Since we know that they have inherently the spectral and directional properties as \( \alpha_{\lambda, \theta} = \varepsilon_{\lambda, \theta} \), we conclude that they are independent of the spectral and directional distributions of the emitted and incident radiation. Note that this relation is derived under the condition that the surface temperature is equal to the temperature of the source of the irradiation. However, since \( \varepsilon \) and \( \alpha \) vary weakly with temperature, the relation may be assumed and greatly simplifies the radiation calculation.

**Gray Surface**

Gray surface is defined as one for which \( \alpha_\lambda \) and \( \varepsilon_\lambda \) are independent of \( \lambda \) over the spectral regions of the irradiation and the surface emission.

**Diffuse-Gray Surface**

A common assumption in enclosure calculations is that surfaces are diffuse-gray. The diffuse-gray surface emits and absorbs a fraction of radiation for all direction and all wavelengths. For diffuse-gray surface, total emissivity and absorptivity are all equal. The total absorptivity is independent of the nature of the incident radiation.

**Radiation Exchange for a Single Surface in an Enclosure**

Consider a diffuse-gray surface at \( T_s \) in an enclosure (surroundings) at \( T_{\text{sur}} \). Equation (1.156) can be applied with an assumption of the diffuse-gray surface in the enclosure. The net radiation exchange of the surface is obtained

\[ q = A \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4) \] (1.157)

### 1.10.2 View Factor

View factor \( F_{ij} \) is defined as the fraction of the radiation leaving surface \( i \) that is intercepted by surface \( j \), which is shown in Figure 1.17. We consider radiation heat transfer between two or
more surfaces, which is often the primary quantity of interest in fin design. Two important relations are suggested:

Reciprocity relation:

\[ A_i F_{ij} = A_j F_{ji} \]  \hspace{1cm} (1.158)

Summation rule:

\[ \sum_{j=1}^{N} F_{ij} = 1 \]  \hspace{1cm} (1.159)

Figure 1.17 View factor associated with radiation exchange between two finite surfaces.

Table 1.2 View factors for two- and three dimensional geometries.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>View Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Small Object in a Large Cavity</td>
<td>( \frac{A_1}{A_2} = 0 )  \hspace{1cm} ( F_{12} = 1 ) \hspace{1cm} (1.160)</td>
</tr>
<tr>
<td>2. Concentric Cylinder (infinite length)</td>
<td>( \frac{A_1}{A_2} = \frac{r_1}{r_2} )  \hspace{1cm} ( F_{12} = 1 )  \hspace{1cm} ( F_{21} = \frac{A_1}{A_2} F_{12} )</td>
</tr>
</tbody>
</table>
3. Large (infinite) Parallel Plate

\[ A_1 = A_2 = A \quad F_{12} = 1 \]  
(1.162)

4. Perpendicular plates with a common edge

\[ F_y = \frac{1+w_j/w_i{\sqrt{1+\left(\frac{w_j}{w_i}\right)^2}}}{2} \]  
(1.163)

5. Hemispherical concentric cylinder

\[ F_{13} = \frac{1}{2} + \frac{1}{\pi} \left( \frac{r_2}{r_1} \right) \left[ \sqrt{1 - \left( \frac{r_2}{r_1} \right)^2} \right] + \frac{1}{\pi} \sin^{-1} \left( \frac{r_1}{r_2} \right) - \frac{1}{\pi} \frac{r_2 - r_1}{r_1} \]  
(1.164)

6. Concentric cylinder (infinite length) with variable angle \( \alpha \)

If \( \alpha \geq \cos^{-1} \left( \frac{r_1}{r_2} \right) \)

\[ F_{13} = 1 - \frac{\pi}{2\alpha} + \frac{r_2}{ar_1} \left[ 1 - \left( \frac{r_1}{r_2} \right)^2 \right] \sqrt{1 - \left( \frac{r_1}{r_2} \right)^2} + \frac{1}{\alpha} \sin^{-1} \left( \frac{r_1}{r_2} \right) - \frac{1}{\alpha} \frac{r_2 - r_1}{r_1} \]  
\[ F_{22} = \frac{2\sqrt{r_2^2 - r_1^2} + 2r_1 \left[ \alpha - \frac{\pi}{2} + \sin^{-1} \left( \frac{r_1}{r_2} \right) \right] - 4 \alpha (r_2 + r_1)}{4(r_2 - r_1)} \]

Otherwise

\[ F_{13} = \frac{\sqrt{r_2^2 + r_1^2 - 2r_2r_1 \cos(\alpha)} - (r_2 - r_1)}{ar_1} \]
\[ F_{22} = \frac{2\sqrt{r_2^2 + r_1^2 - 2r_2r_1 \cos(\alpha)} - \alpha (r_2 + r_1)}{4(r_2 - r_1)} \]
1.10.3 Radiation Exchange between Diffuse-Gray Surfaces

The net rate of heat transfer at the surface is obtained by applying a heat balance on the control volume at the surface.

\[ J_i = \alpha_i E_i \]

The diffuse-gray surface in an enclosure yields

\[ \varepsilon_i = \alpha_i \]  

(1.174)

The heat balance at the surface gives

\[ q_i = A_i (\varepsilon_i E_{bi} - \alpha_i G_i) \]  

(1.175)

The net rate of heat transfer is also equal to the difference between the surface radiosity and the irradiation.

\[ q_i = A_i (J_i - G_i) \]  

(1.176)

Combining the above three equations and eliminating the irradiation \( G_i \) gives

\[ q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i} \]  

(1.177)

The net radiation exchange between the radiosities in the enclosure is expressed

\[ q_{ij} = \frac{J_i - J_j}{(A_i F_j)^{-1}} \]  

(1.177a)

The net rate of heat transfer at surface \( i \) is connected to the number of radiosities in the enclosure and forms a radiation network (Figure 1.19).
Combining Equations (1.177) and (1.178), we obtain

\[
\sum_{j=1}^{N} \left( J_i - J_j \right) = \sum_{j=1}^{N} \left( J_i - J_j \right) \frac{E_{bi} - J_i}{(1 - \varepsilon_i) \varepsilon_i A_i (A_{ij} F_{ij})^{-1}}
\]

(1.179)

Figure 1.19 Radiation network between surface \(i\) and the remaining surfaces of the enclosure.

### 1.10.4 Single Longitudinal Fin with Radiation

Consider that a longitudinal fin is exposed to an environment at a temperature of zero Kelvin in space, where no air for the convection heat transfer exists, which is shown in Figure 1.20. Furthermore, radiation from the surroundings, such as the sun, earth, or a planet, is incident on the plate surfaces and the fluxes absorbed on the top and bottom sides are \(q_{\text{top}}^*\) and \(q_{\text{bot}}^*\), respectively. This problem can be solved numerically. However, we wish to develop some analytical equations to optimize the fin design with the optimum fin thickness and profile length that should maximize the heat dissipation. A small differential element for the purpose is constructed in the plate to apply the energy balance on the element (or the control volume) as shown in Figure 1.20. In this analysis, the fin plate length \(L\) is regarded as unity of 1 meter for simple calculations.
We assume that the base temperature is constant and the material properties do not change with temperature. The environment in orbit is at zero Kelvin, so the irradiation from the space is zero. However, the radiation fluxes from the sun and earth are not considered as an enclosure because they do not have the blackbody properties, so they are treated as the boundary conditions.

Applying the heat balance on the control volume yields

\[
q_x - \left( q_x + \frac{dq_x}{dx} \right) - dq_{rad} + \left( q_{top} + q_{bot} \right) dx
\]

(1.180)

where \( q_x \) is a conduction heat transfer as

\[
q_{rad} = -kt \frac{dT}{dx}
\]

(1.181)

and \( q_{rad} \) is the emissive power (radiation) from both sides of the plate surface

\[
dq_{rad} = 2dx \sigma \varepsilon T^4
\]

(1.182)

Equation (1.180) reduces to

\[
-kT \frac{d^2T}{dx^2} + 2\sigma \varepsilon T^4 = q_{top} + q_{bot}
\]

(1.183)

Equation (1.183) ends up with a nonlinear differential equation, which usually appears difficult to solve because of the nonlinearity, of which the solution would vary depending on the
boundary conditions. Numerical methods using computational software may be applied, but its optimization usually requires a great amount of work and time. We here try to obtain an optimum design with an analytical approach rather than a numerical approach toward the minimal weight or volume of fin for a given heat load.

The boundary conditions for the thin plate are $T=T_{\text{base}}$ specified at $x=0$ and, from an assumption of adiabatic tip, $dT/dx=0$ at $x=b$ [9]. To find $T(x)$, multiply Equation (1.167) by $dT/dx$ and integrate from $x$ to $b$.

\[
\int_{a}^{b} \left( -kt \frac{d^2 T}{dx^2} + 2\sigma e T^4 \right) \frac{dT}{dx} \, dx = \int_{a}^{b} (q_{\text{top}}^* + q_{\text{bot}}^*) \frac{dT}{dx} \, dx
\]  

(1.184)

Let $dv = \frac{d^2 T}{dx^2} \, dx$, then $v = \frac{dT}{dx}$ after integration. Note that $v=0$ at $x=b$ and $v=v$ at $x=x$. From the first term, we conduct a special integration by the substitution.

\[
\int_{a}^{b} \left( \frac{d^2 T}{dx} \right)^2 \, dx = \int_{0}^{v} \frac{1}{2} \left( \frac{dT}{dx} \right)^2 \, dx
\]  

(1.185)

Finally, we have

\[
-k t \left( \frac{dT}{dx} \right)^2 + \frac{2}{5} \varepsilon \sigma \left[ T_{\text{top}}^5 - T_{\text{bot}}^5 \right] = (q_{\text{top}}^* + q_{\text{bot}}^*) \left[ T - T(b) \right]
\]  

(1.186)

where $T=T(b)$ at $x=b$. Solving for $dT/dx$ yields

\[
\frac{dT}{dx} = - \left( \frac{4 \varepsilon \alpha}{5 k t} \right)^{1/2} \left\{ T_{\text{top}}^5 - T(b)^5 - \frac{5}{2 \varepsilon \sigma} (q_{\text{top}}^* + q_{\text{bot}}^*) \left[ T - T(b) \right] \right\}^{1/2}
\]  

(1.187)

The minus sign was chosen for the square root because $T(x)$ must be decreasing with $x$. Knowing the heat transfer rate at the entrance of the radiating fin using the Fourier’s law,

\[
q_f = -k t \left( \frac{dT}{dx} \right)_{x=0}
\]  

(1.188)

Using the temperature gradient $dT/dx$ at the entrance of the plate $x=0$ with Equation (1.187), where $T=T_{\text{base}}$, we obtain

\[
q_f = k t \left( \frac{4 \varepsilon \alpha}{5 k t} \right)^{1/2} \left\{ T_{\text{base}}^5 - T(b)^5 - \frac{5}{2 \varepsilon \sigma} (q_{\text{top}}^* + q_{\text{bot}}^*) \left[ T_{\text{base}} - T(b) \right] \right\}^{1/2}
\]  

(1.189)
Equation (1.189) contains four unknowns, \( t, b, q_f \) and \( T(b) \).

From Equation (1.187), separate the variables and integrate again to obtain

\[
x = \left( \frac{5kt}{4\varepsilon\sigma} \right)^{1/2} \int_{T}^{T_{base}} \frac{dT}{T^5 - T(b)^5 - (5/2\varepsilon\sigma)(q_{top} + q_{tot})(T - T(b))^{1/2}}
\]

which satisfies \( T=T_{base} \) at \( x=0 \) and \( T=T(b) \) at \( x=b \). For \( x=b \), we have

\[
b = \left( \frac{5kt}{4\varepsilon\sigma} \right)^{1/2} \int_{T(b)}^{T} \frac{dT}{T^5 - T(b)^5 - (5/2\varepsilon\sigma)(q_{top} + q_{tot})(T - T(b))^{1/2}}
\]

Equation (1.191) also contains four unknowns, \( t, b, q_f \) and \( T(b) \). Therefore, we have two

Equations (1.189) and (1.191) with four unknowns. There are obviously an infinite number of

the solutions. An idea is to find a minimal profile area (or volume of fin) for the optimum design.

See Example 1.5 for the solution. The temperature distribution with the obtained optimum

thickness \( t \) and profile length \( b \) is then found by evaluating the integral in Equation (1.190)

analytically to find \( x \) for various \( T \) values (in the lower limit of the integral) between \( T_{base} \) and \( T(b) \). Equations (1.190) and (1.191) was originally addressed in [9].
**Example 1.5 Single Rectangular Fin with Radiation**

A single rectangular fin in a radiator is used to dissipate energy in orbit as shown in Figure E1.5.1. Both sides of the fin are exposed to an environment at $T_e \approx 0K$ and also receives radiation from the sun at top and the earth at bottom, which are $q_{top}'' = 80W/m^2$ and $q_{bottom}'' = 30W/m^2$, respectively. The fin is diffuse-gray with emissivity $\varepsilon = 0.8$ on both sides and has a constant thermal conductivity of 28.5 W/m·K. The analysis can be done for a unity width of $L = 1m$. Design the fin for the optimum (minimum) volumes of the fin by providing the fin thicknesses, profile lengths, volumes of the fin, fin temperatures at the tip, and fin efficiencies for three heat dissipations per length: 150 W/m, 200 W/m and 250 W/m. And also provide a temperature distribution along the fin for the median dissipation of 200 W/m.

![Rectangular fin in orbit](image)

**Design Concept:**

The formulation of the governing equation, Equation (1.183), ends up with a nonlinear differential equation, which usually appears difficult to solve analytically because of the nonlinearity, of which the solutions would vary depending on the boundary conditions. Numerical solutions using computational software is always available these days, but its optimization requires a great amount of work and time. We here try to have an optimum design with an analytical approach rather than a numerical approach toward the minimal weight or the volume of fin for a given heat load.

We recall two Equations (1.189) and (1.191) with four unknowns, $t$, $b$, $q_f$ and $T(b)$. We denote $T(b)$ in Equation (1.189) as $T_b(t, q_f)$, because the unknown $T(b)$ is a function of both $t$ and $q_f$. In the same way, $b(t, q_f)$ in Equation (1.191) is eventually a function of both $t$ and $q_f$. MathCAD allow us to functionally solve $T_b(t, q_f)$ and $b(t, q_f)$ with the two equations, where $t$ and $q_f$ are variables.
**MathCAD format solution:**

We assume that the base temperature is constant and the material properties do not change with temperature. The environment in orbit is at zero Kelvin, so the irradiation from the space is zero. However, the radiation fluxes from the sun and earth are not considered as an enclosure, so treated as the boundary conditions.

Properties and information given:

\[
T_{\text{base}} := 400 \text{K} \quad k := 28.5 \frac{\text{W}}{\text{mK}} \quad \varepsilon := 0.8 \quad \sigma := 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^{2}\cdot\text{K}^{4}}
\]

\[
q_t := 80 \frac{\text{W}}{\text{m}^{2}} \quad q_b := 30 \frac{\text{W}}{\text{m}^{2}}
\]

Initial guesses (if possible, closest to the exact values, especially in such a nonlinear problem) are required for a ‘Find’ function, which is a built-in function, where ‘Given’ is also a required command function in connection with the ‘Find’ function.

Initial guess values:

\[
T_b := 300 \text{K} \quad b := 14 \text{cm}
\]

A ‘Given’ function is written as a command. Using Equations (1.189) and (1.191), we obtain

Given

\[
q_f = (k \cdot t) \left( \frac{4 \cdot \varepsilon \cdot \sigma}{5 \cdot k \cdot t} \right)^{\frac{1}{2}} \left[ T_{\text{base}}^{\frac{5}{2}} - T_b^{\frac{5}{2}} - \frac{5}{2 \cdot \varepsilon \cdot \sigma} \left( q_t + q_b \right) \left( T_{\text{base}} - T_b \right) \right] \left( T_{\text{base}} - T_b \right)^{\frac{1}{2}}
\]

\[
b = \left( \frac{5 \cdot k \cdot t}{4 \cdot \varepsilon \cdot \sigma} \right)^{\frac{1}{2}} \int_{T_b}^{T_{\text{base}}} \left[ T^{\frac{5}{2}} - T_b^{\frac{5}{2}} - \frac{5}{2 \cdot \varepsilon \cdot \sigma} \left( q_t + q_b \right) \left( T - T_b \right) \right]^{\frac{1}{2}} \text{d}T
\]

Obtain the functional solution as:

\[
\left( T_b(t, q_f), b(t, q_f) \right) := \text{Find}(T_b, b)
\]

We express the profile area, which is a product of the fin thickness and profile length.
We define three required heat loads per unit length as:

\[
q_{f1} := \frac{150}{m} \quad q_{f2} := \frac{200}{m} \quad q_{f3} := \frac{250}{m}
\]  

(E.1.5.5)

We try to plot the profile area along the fin thickness to find the minimal area.

The profile area curves show very steep slopes near the minimum values, which indicate the existence of singularities (values explode). The region beyond the singularity seems no practically interested. However, after several graphical explorations due to the nonlinearity of the governing equation, we can eventually figure out the optimum fin thicknesses with the minimal profile areas for the three heat loads: 150 W/m, 200 W/m, and 250 W/m, which are approximately, 3 mm, 6 mm and 9 mm, respectively. It is reminded that each curve represents the designated heat load. Finding the optimum thicknesses lead to find the corresponding optimum profile lengths, and fin temperatures at the tips. The profile length is plotted in Figure E1.5.3 for curiosity. The profile length appears not changing much in the regions after the optimum fin thicknesses.

Figure E1.5.2 Profile area (or volume of fin) vs. fin thickness
Now we try to plot the temperature distribution specific for a heat load of 200 W/m. The specific optimum thickness for the load is approximately:

\[ t := 6 \text{mm} \quad \text{(E1.5.6)} \]

In Equation (1.190), \( x \) is obviously a function of the fin temperature. However, it can be thought reversely, so we try to find the fin temperature as a function of \( x \) and also \( q_f \) using a ‘root’ function, which is also a built-in function in MathCAD.

Initial guess for the following ‘root’ function appears very sensitive to the convergence of the solution (Usually any numeric values would be accepted as a guess). An attempt was made to have the possibly closest value for the solution, which was found to be the fin temperature at the tip due to the nonlinearity.

Initial guess value:

\[ T_w = T_b(t, q_f) \quad \text{(E1.5.7)} \]
We plotted the above solution (the temperature distribution along the profile length), which took several minutes in computing by a regular PC. The fin temperature monotonically decreases as expected. Note that the fin temperature at the tip show approximately 320K for the given load of 200 W/m.

\[
T(x, q_f) := \sqrt{\left(\frac{5k t}{4\varepsilon \sigma}\right)^2 - \frac{1}{T_{base}}} \left[ T^5 - T_b(t,q_f)^5 - \frac{5}{2\varepsilon \sigma} (q_t + q_b)(T - T_b(t,q_f)) \right] dT - x, T
\]  

(E1.5.8)

We plotted the above solution (the temperature distribution along the profile length), which took several minutes in computing by a regular PC. The fin temperature monotonically decreases as expected. Note that the fin temperature at the tip show approximately 320K for the given load of 200 W/m.

\[
T(x, q_f^2) := \sqrt{\left(\frac{5k t}{4\varepsilon \sigma}\right)^2 - \frac{1}{T_{base}}} \left[ T^5 - T_b(t,q_f)^5 - \frac{5}{2\varepsilon \sigma} (q_t + q_b)(T - T_b(t,q_f)) \right] dT - x, T
\]  

(E1.5.8)

Figure E1.5.4 Temperature distribution for \(q_f=200\) W/m.

The corresponding values with the optimum fin thicknesses are sought for the summary results. Define the environment temperature for readers and the three optimum fin thicknesses.

\[
T_e := 0K
\]  

(E1.5.9)

\[
t_1 := 3mm \quad t_2 := 6mm \quad t_3 := 9mm
\]  

(E1.5.10)

Fin effectiveness is the ratio of the active fin heat transfer to the heat transfer without the fin at the base, where the fin was attached:
These values of effectiveness indicate that adding the fins increases the heat loads as shown in the values of the effectiveness. Fin efficiency is the ratio of the active fin heat transfer to the maximum possible heat transfer, which occurs when the fin temperature is constant at the base temperature:

\[
e_{f}(t, q_f) := \frac{q_f}{t \cdot e \cdot \sigma \left( T_{\text{base}}^4 - T_e^4 \right)} \tag{E1.5.11}
\]

\[
e_{f}(t_1, q_{f1}) = 43.058 \quad e_{f}(t_2, q_{f2}) = 28.706 \quad e_{f}(t_3, q_{f3}) = 23.921 \tag{E1.5.12}
\]

Now we calculate the optimum values with the optimum fin thicknesses.

\[
\eta_{f}(t, q_f) := \frac{q_f}{2 \cdot \left( b(t, q_f) + t \right) \cdot e \cdot \sigma \left( T_{\text{base}}^4 - T_e^4 \right)} \tag{E1.5.13}
\]

\[
\eta_{f}(t_1, q_{f1}) = 0.43 \quad \eta_{f}(t_2, q_{f2}) = 0.504 \quad \eta_{f}(t_3, q_{f3}) = 0.476 \tag{E1.5.14}
\]

Table E1.5.1 Summary of the optimum design of the rectangular fin for the minimal volume of the fin.

<table>
<thead>
<tr>
<th></th>
<th>Optimum I</th>
<th>Optimum II</th>
<th>Optimum III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_f) (W/m)</td>
<td>150</td>
<td>200</td>
<td>250</td>
</tr>
<tr>
<td>(t) (mm)</td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>(b) (cm)</td>
<td>14.7</td>
<td>16.5</td>
<td>21.7</td>
</tr>
<tr>
<td>(A_p) (cm²)</td>
<td>4.4</td>
<td>9.9</td>
<td>19.5</td>
</tr>
<tr>
<td>(T_b) (K)</td>
<td>298.6</td>
<td>319.0</td>
<td>312.8</td>
</tr>
<tr>
<td>(e_f)</td>
<td>43.0</td>
<td>28.7</td>
<td>23.9</td>
</tr>
<tr>
<td>(\eta_f)</td>
<td>0.43</td>
<td>0.504</td>
<td>0.476</td>
</tr>
</tbody>
</table>

Properties and information given:

\[
k = 28.5 \frac{W}{m \cdot K} \quad \varepsilon = 0.8 \quad T_{\text{base}} = 400K \quad q_l = 80 \frac{W}{m^2} \quad q_b = 30 \frac{W}{m^2}
\]
The summary of the optimum design of the rectangular fin is tabulated in Table E1.5.1. This would suggest that the mathematical methods in this problem save enormous work and time compared to the numerical methods. Note that this problem was optimized for the minimum weight of fins, not for the maximal fin efficiency.
Example 1.6 Axial Fins Array on a Cylindrical Pipe

A long axial fins array on a cylinder pipe is designed to dissipate energy from the pipe, in which a number of radioactive thermoelectric generators (RTG) produce electricity for the power of a spacecraft, which is illustrated in Figure E1.6.1. It is widely known that adding fins on a plane base does not increase the radiative dissipation, which means that adding fins on a plane provide additional weight and complexity with no gain on the heat load. A question arises whether adding fins on a cylindrical pipe will provide additional heat load. If it provides additional heat load, what is the optimal number of fins? We wish to study this problem mathematically. The radiative dissipation as a heat load by the 30-cm diameter pipe with fins, if justified, should be 1kW per meter at least. The base temperature in the cylindrical pipe is maintained at 400 K, while the fin temperatures are expected to be at the three different temperatures of 280 K, 290 K, and 300 K. The axial fins array is diffuse-gray with $\varepsilon=0.8$ and exposed to an environment at $T_e=3$ K. Design the fins array for its minimal weight.

![Figure E1.6.1 Axial fin array on a cylindrical pipe.](image)

**Design Concept:**

The exact solution for this problem seems very complex requiring significant specialty on the numerical computations. However, we here attempt to approach the problem mathematically with an acute assumption that the surface temperatures of the fins are constant and diffuse-gray, which makes the problem greatly simple. Two-dimensional analysis is applied because the axial fins are long. The concept of view factor accounts the radiation exchange between the surfaces of the fins and pipe. The radiation network allows easily to link the surfaces. One of the difficulties in this problem is to find the optimal angle between the fins, which determines the optimal number of fins. The view factors in the circular sector with variable angle were developed for the present purpose by author and listed in Table 1.2. The circular sector with a variable angle is
shown in Figure E1.6.2, where the system may be viewed as a three-surface enclosure, with the third surface being the cold environment denoted as a dotted line of surface area \(A_3\). Surfaces \(A_1\) and \(A_2\) of the cylinder pipe are real surfaces, while Surface \(A_3\) of the environment is considered as a blackbody according to the Kirchhoff’s law.

Figure E1.6.2 Circular sector

The radiation network is constructed by first identifying nodes associated with the radiosities of each surface, as shown in Figure E1.6.3. The relation between a real surface and radiosity is found in Equation (1.177) and the relation between radiosities is found in Equation (1.177a). Since the environment is regarded as a blackbody in an enclosure, we have \(J_3 = E_{b3}\) with \(\varepsilon = 1\), which is one of the three equations. Two more relations are obtained using Equation (1.179). Three equations with three unknowns provide the solutions for \(J_1\), \(J_2\), and \(J_3\), leading to calculate \(q_1\) and \(q_2\) and \(q_3 = q_1 + q_2\). From the radiation network of Figure E1.6.3, we need to obtain three view factors: \(F_{12}\), \(F_{13}\), and \(F_{23}\).

Figure E1.6.3 Radiation network for a circular sector.
Viewing from surface 1 in Figure 1.6.2 and using the summation rule of Equation (1.159), we have

\[ F_{12} + F_{13} = 1 \text{ or } F_{12} = 1 - F_{13} \]  
(E.1.6.1)

Note that surface 1 in this case cannot see its own surface 1. Using the reciprocity relation of Equation (1.158) between surfaces 1 and 2, we have

\[ A_1 F_{12} = A_2 F_{21} \text{ or } F_{21} = \frac{A_1}{A_2} F_{12} = \frac{A_1}{A_2} (1 - F_{13}) \]  
(E.1.6.2)

Viewing from surface 2 in Figure 1.6.2 and using the summation rule of Equation (1.159), we have

\[ F_{21} + F_{22} + F_{23} = 1 \text{ or } F_{23} = 1 - F_{21} - F_{22} \]  
(E.1.6.3)

where \( F_{22} \) means that surface 2 sees its own surface 2 only when the angle is small enough so that the curvature between the fins no longer hide other side’s surface 2.

Substituting Equation (E1.6.2) into (E1.6.3) yields

\[ F_{23} = 1 - \frac{A_1}{A_2} (1 - F_{13}) - F_{22} \]  
(E.1.6.4)

From this equation, we need to figure out only two view factors: \( F_{13} \) and \( F_{23} \), which are found in Table 1.2.

**MathCAD format solution:**

Properties and information given:

\[ \sigma := 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4} \quad \epsilon_1 := 0.8 \quad \epsilon_2 := 0.8 \]  
(E.1.6.5)

\[ T_1 := 400\text{K} \quad T_2 := 270\text{K} \quad T_e := 3\text{K} \]  
(E.1.6.6)

The fin dimensions are imported from Example 1.5 for the heat load of 200 W/m, which are obtained for the minimal volume of the fin. Considering the geometry in Figure E1.6.2, we have

\[ b := 16.5\text{cm} \quad t := 6\text{mm} \quad L := 1\text{m} \]  
(E.1.6.7)

\[ r_1 := 15\text{cm} \quad r_2 := r_1 + b \]  
(E.1.6.8)

Define the circular angle in terms of the number of fins: \( n \)
\[ \alpha(n) := \frac{2\pi}{n} \]  

The blackbody radiations for surfaces 1 and 2 are defined

\[ E_{b1} := \sigma \cdot T_1^4 \quad E_{b2}(T_2) := \sigma \cdot T_2^4 \]  

where we want to treat \( T_2 \) as a function too.

The areas of each surface are defined

\[ A_1(n) := \frac{2\pi \cdot r_1 \cdot L}{n} \quad A_2 := 2b \cdot L \quad A_3(n) := \frac{2\pi \cdot r_2 \cdot L}{n} \]  

View factors according to Table 1.2:

\[ F_{13}(n) := 1 - \frac{\pi}{2\alpha(n)} + \frac{r_2^2}{\alpha(n) \cdot r_1} \left[ 1 - \left( \frac{r_1}{r_2} \right)^2 \right]^{0.5} + \frac{1}{\alpha(n)} \cdot \sin \left( \frac{r_1}{r_2} \right) - \left( \frac{r_2 - r_1}{\alpha(n) \cdot r_1} \right) \]  

\[ F_{13}(n) := \begin{cases} F_{13}(n) & \text{if } \alpha(n) > \cos \left( \frac{r_1}{r_2} \right) \\ \frac{2\sqrt{r_2^2 - r_1^2 + 2 \cdot r_1 \left( \alpha(n) - \frac{\pi}{2} + \sin \left( \frac{r_1}{r_2} \right) \right) - \alpha(n) \cdot (r_2 + r_1)}}{4(r_2 - r_1)} & \text{otherwise} \end{cases} \]  

\[ F_{22}(n) := \begin{cases} F_{22}(n) & \text{if } \alpha(n) > \cos \left( \frac{r_1}{r_2} \right) \\ \frac{2\sqrt{r_2^2 + r_1^2 - 2 \cdot r_2 \cdot r_1 \cdot \cos(\alpha(n)) - \alpha(n) \cdot (r_2 + r_1)}}{4(r_2 - r_1)} & \text{otherwise} \end{cases} \]  

\[ F_{22}(n) := \begin{cases} F_{22}(n) & \text{if } F_{22}(n) > 0 \\ 0 & \text{otherwise} \end{cases} \]  

\[ F_{13}(n) := 1 - F_{13}(n) \]
Using Equation (1.179), we want to construct three equations from the radiation network of Figure E1.6.3 for $J_1$, $J_2$, and $J_3$.

Initial guesses

\[
J_1 := 100 \frac{W}{m^2} \quad J_2 := 100 \frac{W}{m^2} \quad J_3 := 100 \frac{W}{m^2}
\]  

(E1.6.19)

Given

\[
\frac{E_{b1} - J_1}{\left( \frac{1 - \varepsilon_1}{\varepsilon_1 A_1(n)} \right)} = \frac{J_1 - J_3}{(A_1(n) \cdot F_{13}(n))^{-1}} + \frac{J_1 - J_2}{(A_1(n) \cdot F_{12}(n))^{-1}}
\]  

(E1.6.20)

\[
\frac{E_{b2}(T_2) - J_2}{\left( \frac{1 - \varepsilon_2}{\varepsilon_2 A_2} \right)} = \frac{J_2 - J_1}{(A_1(n) \cdot F_{12}(n))^{-1}} + \frac{J_2 - J_3}{(A_2 \cdot F_{23}(n))^{-1}}
\]  

(E1.6.21)

\[
J_3 = \sigma \cdot T_e^4
\]  

(E1.6.22)

Solve for $J_1$, $J_2$, and $J_3$ as a function of $n$ and $T_2$.

\[
\begin{pmatrix}
J_1(n,T_2) \\
J_2(n,T_2) \\
J_3(n,T_2)
\end{pmatrix}
:= \text{Find}(J_1,J_2,J_3)
\]  

(E1.6.23)

Define the total heat dissipation from both the cylinder pipe and two fins in the circular sector.

\[
q_{\text{total}}(n,T_2) := n \left[ \frac{E_{b1} - J_1(n,T_2)}{\left( \frac{1 - \varepsilon_1}{\varepsilon_1 A_1(n)} \right)} + \frac{E_{b2}(T_2) - J_2(n,T_2)}{\left( \frac{1 - \varepsilon_2}{\varepsilon_2 A_2} \right)} \right]
\]  

(E1.6.24)
First of all, we want to see how the view factors vary along the number of fins (or the circular angle).

\[ n := 1, 1.1..10 \]

Figure 1.6.4 View factors vs. number of fins.

Remind that surface 1 = cylinder pipe, surface 2 = two fins each side, and surface 3 = blackbody environment. It is interesting to note that view factor \( F_{22} \) starts jumping up at \( n = 4.5 \), which implies that surface 2 does not see its other side surface until the number of fins reaches 4.5, which directly affects \( F_{23} \) that start decreasing (which causing the total heat dissipation to drop) as in Figure E1.6.4. Practically the number of fins must be integer, not fraction.

Now we inspect the total heat dissipation for three fin surface temperatures of surface 2 as:

\[
\begin{align*}
T_{2a} & := 280K \\
T_{2b} & := 290K \\
T_{2c} & := 300K
\end{align*}
\]

(E1.6.25)
Definitely we recognize the effect of the jumping-up of $F_{22}$ in the total heat dissipation of Figure E1.6.5, which becomes significant as lowering the fin temperatures of surface 2. When the base temperature of cylinder pipe is at 400 K, if the fins temperature is lower than 300 K, there is not much gain in the heat load with increasing the number of fins.

The number of fins seems dependent on the average fin temperature. If the fin temperature is not high enough, the number of fin is restricted by five (5). If the fin temperature is designed high enough, the number of fins may be increased more than five (5). The heat dissipations for all the three fin temperatures actually satisfy the least design requirement of 1 kW.
References

Problems

1.7 A thin plate of thickness \( t \) and length \( 2b \) is between two tubes in a radiator used to dissipate energy in orbit as shown in Figure P1.7. The dimension is long in the direction normal to the cross section shown. Both sides of the plate have the same emissivity and radiate to the environment at \( T \approx 0 \) K. Radiation from the surroundings, such as from the sun, earth, or a planet, is incident on the plate surfaces and the fluxes absorbed on the top and bottom sides are \( q_{\text{top}}^* = 190 \) W/m\(^2\) and \( q_{\text{bot}}^* = 50 \) W/m\(^2\). The plate is diffuse-gray with emissivity of \( \varepsilon = 0.8 \) on both sides, and has constant thermal conductivity \( k = 45.6 \) W/m·K. The base tube temperature \( T_{\text{tube}} \) is maintained at 400 K. Design the plate for a heat load of 200 W/m.

![Figure P1.7 Space radiator](image)

1.8 A long axial fins array on a cylinder pipe is designed to dissipate energy from the pipe, in which a number of radioactive thermoelectric generators (RTG) produce electricity for the power of a spacecraft, which is illustrated in Figure P1.8. It is widely known that adding fins on a plane base does not increase the radiative dissipation, which means that adding fins on a plane provide additional weight and complexity with no gain on the heat load. A question arises whether adding fins on a cylindrical pipe will provide additional heat load. If it provides additional heat load, what is the optimal number of fins? We wish to study this problem mathematically. The radiative dissipation as a heat load by the 60-cm diameter pipe with fins, if justified, should be 2.2kW per meter at least. The axial fins array is diffuse-gray with \( \varepsilon = 0.8 \) and exposed to an environment at \( T_e = 4 \) K. The base temperature in the cylindrical pipe is maintained at 400 K, while the fin temperatures are expected to be at 320 K. Design the fins array for its minimal weight.
Figure P1.8 Axial fin array on a cylindrical pipe.