§3.1, p. 116, #3 Let $\mathbb{C}$ be the set of complex numbers. Define addition in $\mathbb{C}$ by

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}, \quad \text{for all } a, b, c, d \in \mathbb{R} \tag{1}$$

and define scalar multiplication by

$$\alpha(a + b\mathrm{i}) = \alpha a + \alpha b\mathrm{i} \quad \text{for all scalars } \alpha \in \mathbb{R}, \quad \text{for all } a, b \in \mathbb{R}. \tag{2}$$

Show that $\mathbb{C}$ is a vector space with these operations.

Note: The Student Study Guide has an outline of the solution. We present here a model of a *complete* solution. It is important to study this example thoroughly, and read the discussion with care.

Discussion: Recall that $\mathbb{C}$ is defined to be the set of all numbers of the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. In symbols, the set of complex numbers is written as $\mathbb{C} = \{a + b\mathrm{i} \ : \ a, b \in \mathbb{R}\}$. The number a is called the *real part* and the number b is called the *imaginary part* of the complex number $a + b\mathrm{i}$. Note that the plus sign is to be treated as a *formal* symbol, in the sense that $a + b\mathrm{i}$ cannot be simplified any further.

We should examine the definition of addition and scalar multiplication in $\mathbb{C}$ carefully. Let's rewrite (1) to bring out an important point: not all the plus signs that appear in (1) mean the same thing. Let us write

$+_{\mathbb{C}}$ to denote addition of two quantities in $\mathbb{C}$,
$+_{\mathbb{R}}$ to denote addition of two quantities in $\mathbb{R}$,
$+$ to denote the *formal* plus between the real and imaginary parts of a complex number.

Here is (1) rewritten to bring out the difference between the meaning of the plus symbols that appear in it:

$$(a + b\mathrm{i}) +_{\mathbb{C}} (c + d\mathrm{i}) = (a +_{\mathbb{R}} c) + (b +_{\mathbb{R}} d)\mathrm{i}, \quad \text{for all } a, b, c, d \in \mathbb{R} \tag{3}$$

This equation tells us how to add two complex numbers (left hand side) to get a new complex number (right hand side). It says we have to add the real parts (which we know how to do, since these are two real numbers), and add the imaginary parts (which we also can do, since the imaginary parts are also real numbers). For example, (1) says that the sum of the complex numbers $2 - \mathrm{i}$ and $-7 + 5\mathrm{i}$ is the complex number $-5 - 4\mathrm{i}$, because

$$\begin{aligned} (2 - \mathrm{i}) + (-7 + 5\mathrm{i}) &= (2 - 7) + (-1 + 5)\mathrm{i} \qquad \text{by (1)} \\ &= -5 - 4\mathrm{i}. \end{aligned}$$

Similarly, (2) contains different kinds of multiplication on the left hand side and on the right hand side. Let us write

$\cdot_{\mathbb{C}}$ to denote multiplication of a complex number by a real number,
$\cdot_{\mathbb{R}}$ to denote multiplication between two real numbers

Then (2) becomes

$$\alpha \cdot_{\mathbb{C}} (a + b\mathrm{i}) = \alpha \cdot_{\mathbb{R}} a + \alpha \cdot_{\mathbb{R}} b\mathrm{i} \quad \text{for all scalars } \alpha \in \mathbb{R}, \text{ and for all } a, b \in \mathbb{R}, \tag{4}$$

which says that multiplication of a complex number by a scalar is to be done by multiplying both the real part and the complex part by the scalar. So for example,

$$\begin{aligned} -2\left(7 - \frac{2}{3}\mathrm{i}\right) &= (-2)(7) + (-2)\frac{-2}{3}\mathrm{i} \\ &= -14 + \frac{4}{3}\mathrm{i} \end{aligned}$$

Solution of Problem #3: To show that $\mathbb{C}$ is a vector space we must show that all eight axioms are satisfied.

A1. We must show that addition, as defined by (1), is commutative.
Let $x = x_1 + x_2\mathrm{i}$, and $y = y_1 + y_2\mathrm{i}$ be an arbitrary pair of complex numbers. Here x_1, x_2, y_1, y_2 are arbitrary real numbers. We need to show $x + y = y + x$. We have

$$\begin{aligned} x + y &= (x_1 + x_2\mathrm{i}) + (y_1 + y_2\mathrm{i}) && \text{substituting for } x,\ y \\ &= (x_1 + y_1) + (x_2 + y_2)\mathrm{i} && \text{by (1), which defines addition in } \mathbb{C} \\ &= (y_1 + x_1) + (y_2 + x_2)\mathrm{i} && \text{since } x_1, x_2, y_1, y_2 \in \mathbb{R}, \text{ and addition of real numbers} \\ & && \text{is commutative} \\ &= (y_1 + y_2\mathrm{i}) + (x_1 + x_2\mathrm{i}) && \text{by (1), which defines addition in } \mathbb{C} \\ &= y + x && \text{substituting for } x,\ y \end{aligned}$$

Since this equality holds for all $x,\ y \in \mathbb{C}$, we have established that addition, as defined by (1), is commutative.

A2. We must show that addition, as defined by (1), is associative.
Let $x = x_1 + x_2\mathrm{i}$, $y = y_1 + y_2\mathrm{i}$, and $z = z_1 + z_2\mathrm{i}$ be arbitrary complex numbers. We need to show $x + (y + z) = (x + y) + z$. We have

$$\begin{aligned} x + (y + z) &= x + ((y_1 + y_2\mathrm{i}) + (z_1 + z_2\mathrm{i})) && \text{substituting for } y,\ z \\ &= x + ((y_1 + z_1) + (y_2 + z_2)\mathrm{i}) && \text{by (1), which defines addition in } \mathbb{C} \\ &= (x_1 + x_2\mathrm{i}) + ((y_1 + z_1) + (y_2 + z_2)\mathrm{i}) && \text{substituting for } x \\ &= (x_1 + (y_1 + z_1)) + (x_2 + (y_2 + z_2))\mathrm{i} && \text{by (1), which defines addition in } \mathbb{C} \\ &= ((x_1 + y_1) + z_1) + ((x_2 + y_2) + z_2)\mathrm{i} && \text{since } x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}, \text{ and} \\ & && \text{addition of real numbers is associative} \end{aligned}$$

On the other hand,

$$\begin{aligned} (x + y) + z &= ((x_1 + x_2\mathrm{i}) + (y_1 + y_2\mathrm{i})) + z && \text{substituting for } x,\ y \\ &= ((x_1 + y_1) + (x_2 + y_2)\mathrm{i}) + z && \text{by (1), which defines addition in } \mathbb{C} \\ &= ((x_1 + y_1) + (x_2 + y_2)\mathrm{i}) + (z_1 + z_2)\mathrm{i} && \text{substituting for } z \\ &= ((x_1 + y_1) + z_1) + ((x_2 + y_2) + z_2)\mathrm{i} && \text{by (1), which defines addition in } \mathbb{C} \end{aligned}$$

Thus we have shown that $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{C}$. Thus addition, as defined by (1), is associative.

A3. We have to show that there exists an element in $\mathbb{C}$, to be called $\mathbf{0}$, such that $x + \mathbf{0} = x$ for every $x \in \mathbb{C}$. Consider the element $0 + 0\mathrm{i}$. Clearly this element belongs to $\mathbb{C}$ since it is of the form $a + b\mathrm{i}$; here a and b have both been chosen to be $0 \in \mathbb{R}$. Then

$$\begin{aligned} x + (0 + 0\mathrm{i}) &= (x_1 + x_2\mathrm{i}) + (0 + 0\mathrm{i}) && \text{substituting for } x \\ &= (x_1 + 0) + (x_2 + 0)\mathrm{i} && \text{by definition of addition in } \mathbb{C}\text{, see (1)} \\ &= x_1 + x_2\mathrm{i} && \text{since } x_1,\, x_2 \in \mathbb{R} \text{ and } a + 0 = a \text{ for all } a \in \mathbb{R} \\ &= x \end{aligned}$$

So, if we designate $\mathbf{0}$ as the element $0 + 0\mathrm{i}$ in $\mathbb{C}$, we see that axiom A3 is satisfied.

A4. We have to show that for each $x \in \mathbb{C}$, there exists another element, designated by $-x$, such that $x + (-x) = \mathbf{0}$. Note that $\mathbf{0}$ is the element identified in the proof of the previous axiom.

So we start with $x = x_1 + x_2\mathrm{i}$, an arbitrary element in $\mathbb{C}$. Consider the element in $\mathbb{C}$ defined by $-x_1 + (-x_2)\mathrm{i}$. We will represent this element by $-x$. We claim that $x + (-x) = \mathbf{0}$. We have

$$\begin{aligned} x + (-x) &= (x_1 + x_2\mathrm{i}) + (-x_1 + (-x_2)\mathrm{i}) && \text{substituting for } x \text{ and } -x \\ &= (x_1 + (-x_1)) + (x_2 + (-x_2))\mathrm{i} && \text{by definition of addition in } \mathbb{C}\text{, see (1)} \\ &= (x_1 - x_1) + (x_2 - x_2)\mathrm{i} && \text{properties of addition in } \mathbb{R} \\ &= 0 + 0\mathrm{i} && \text{properties of addition in } \mathbb{R} \\ &= \mathbf{0} && \text{definition of } \mathbf{0} \end{aligned}$$

This establishes our claim, and proves that A4 holds in $\mathbb{C}$.

A5 Let $\alpha \in \mathbb{R}$ be an arbitrary scalar, and $x = x_1 + x_2\mathrm{i}$, $y = y_1 + y_2\mathrm{i}$ be arbitrary elements in $\mathbb{C}$. We have to show that $\alpha(x + y) = \alpha x + \alpha y$. We have

$$\begin{aligned} \alpha(x + y) &= \alpha((x_1 + x_2\mathrm{i}) + (y_1 + y_2\mathrm{i})) && \text{substituting for } x,\, y \\ &= \alpha((x_1 + y_1) + (x_2 + y_2)\mathrm{i}) && \text{by definition of addtion in } \mathbb{C} \\ &= \alpha(x_1 + y_1) + \alpha(x_2 + y_2)\mathrm{i} && \text{by definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\ &= (\alpha x_1 + \alpha y_1) + (\alpha x_2 + \alpha y_2)\mathrm{i} && \text{since multiplication distributes over addition in } \mathbb{R} \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha x + \alpha y &= \alpha(x_1 + x_2\mathrm{i}) + \alpha(y_1 + y_2\mathrm{i}) && \text{substituting for } x \text{ and } y \\ &= (\alpha x_1 + \alpha x_2\mathrm{i}) + (\alpha y_1 + \alpha y_2\mathrm{i}) && \text{by definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\ &= (\alpha x_1 + \alpha y_1) + (\alpha x_2 + \alpha y_2)\mathrm{i} && \text{by definition of addition in } \mathbb{C} \end{aligned}$$

Thus we have shown that $\alpha(x + y) = \alpha x + \alpha y$, as desired. This proves that axiom A5 holds in $\mathbb{C}$.

A6. Let α, β be arbitrary real numbers, and $x = x_1 + x_2\mathrm{i}$ be an arbitrary complex number. We need to prove that $(\alpha + \beta)x = \alpha x + \beta x$. Consider

$$\begin{aligned} (\alpha + \beta)x &= (\alpha + \beta)(x_1 + x_2\mathrm{i}) && \text{substituting for } x \\ &= (\alpha + \beta)x_1 + (\alpha + \beta)x_2\mathrm{i} && \text{by definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\ &= (\alpha x_1 + \beta x_1) + (\alpha x_2 + \beta x_2)\mathrm{i} && \text{multiplication distributes over addition in } \mathbb{R} \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha x + \beta x &= \alpha(x_1 + x_2\mathrm{i}) + \beta(x_1 + x_2\mathrm{i}) && \text{substituting for } x \\ &= (\alpha x_1 + \alpha x_2\mathrm{i}) + (\beta x_1 + \beta x_2\mathrm{i}) && \text{definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\ &= (\alpha x_1 + \beta x_2) + (\alpha x_2 + \beta x_2)\mathrm{i} && \text{by definition of addition in } \mathbb{C} \end{aligned}$$

Thus we have shown that A6 holds in $\mathbb{C}$.

A7 Let α, β be arbitrary scalars, and $x = x_1 + x_2\mathrm{i}$ be an arbitrary complex number. We have to show that $(\alpha\beta)x = \alpha(\beta x)$. Notice that on the left hand side we have *only one* multiplication between a scalar and a complex number, while on the right hand side there are *two* such operations. Now,

$$\begin{aligned}
(\alpha\beta)x &= (\alpha\beta)(x_1 + x_2\mathrm{i}) && \text{substituting for } x \\
&= (\alpha\beta)x_1 + (\alpha\beta)x_2\mathrm{i} && \text{definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\
&= \alpha\beta x_1 + \alpha\beta x_2\mathrm{i} && \text{since } \alpha, \beta, x_1, x_2 \in \mathbb{R} \text{ and multiplication of real numbers} \\
& && \text{is associative}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha(\beta x) &= \alpha(\beta(x_1 + x_2\mathrm{i})) && \text{substituting for } x \\
&= \alpha(\beta x_1 + \beta x_2\mathrm{i}) && \text{definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\
&= \alpha\beta x_1 + \alpha\beta x_2\mathrm{i} && \text{definition of scalar multiplication in } \mathbb{C}\text{, see (2)}
\end{aligned}$$

This establishes that A7 holds in $\mathbb{C}$.

A8 We have to show that $1 \cdot x = x$ for all $x = x_1 + x_2\mathrm{i}$ in $\mathbb{C}$. Note that the left hand side of the equation denotes the operation of scalar multiplication between the scalar 1 and the complex number x. We have

$$\begin{aligned}
1 \cdot x &= 1 \cdot (x_1 + x_2\mathrm{i}) && \text{substituting for } x \\
&= 1 \cdot x_1 + 1 \cdot x_2\mathrm{i} && \text{definition of scalar multiplication in } \mathbb{C}\text{, see (2)} \\
&= x_1 + x_2\mathrm{i} && \text{since } x_1,\ x_2 \in \mathbb{R}\text{, and 1 is the multiplicative identity in } \mathbb{R} \\
&= x
\end{aligned}$$

Thus the last axiom, A8, holds in $\mathbb{C}$.

We have proved that all eight axioms hold in $\mathbb{C}$. Hence the set $\mathbb{C}$, with addition and scalar multiplication defined by (1) and (2), respectively, is a vector space.

§3.1, p. 116, #10 Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2) \tag{5}$$

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0). \tag{6}$$

Show that S, with the ordinary scalar multiplication and addition operation $\oplus$, is not a vector space. Which of the eight axioms fail to hold?

Solution: Since scalar multiplication has been defined in the usual way, we should concentrate on axioms that involve addition, as defined by (6). We first notice that when we add with this new rule, the second coordinate always comes out 0. So for example,

$$(1, 1) \oplus (a, b) = (1 + a, 0) \neq (1, 1).$$

no matter what the values of a, b are. Thus there cannot be any vector that serves as an additive identity, and A3 fails.

§3.1, p. 116, #11 Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \tag{7}$$

and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2) \tag{8}$$

Is V a vector space with these operations? Justify your answer.

Solution: Since vector addition is defined in the ordinary way, axioms A1 – A4, which involved just addition alone, will be satisfied.

Let's examine the new definition of scalar multiplication. The second coordinate is unaffected, no matter what scalar we multiply the vector by. Aha! Thus, in particular,

$$0 \circ (1, 1) = (0 \cdot 1, 1) = (0, 1) \tag{9}$$

Now we have a theorem that says that in any vector space the equation

$$0 \cdot \mathbf{x} = \mathbf{0} \tag{10}$$

is always true, where $\cdot$ represents the scalar multiplication, and $\mathbf{0}$ is the (unique) additive identity in the vector space. Since the rule for addition is the usual rule, the additive identity here is the same as usual, that is $\mathbf{0} = (\mathbf{0}, \mathbf{0})$.

Equation (9) shows that equation (10) does not hold when scalar multiplication is given by $\circ$. Hence V with addition and scalar multiplication defined by and fails to be a vector space.

Remark: Alternatively, you can check that A6 is not satisfied. Interestingly, all the other axioms are satisfied!

§3.2, p. 125, #5 Determine whether the following are subspaces of P_4.

First, we recall that P_4 consists of polynomials of degree less than 4. We can specify the set P_4 as follows:

$$P_4 = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ : \ a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}.$$

(a) **Solution:** Let S be the set of polynomials in P_4 of even degree.
Let $p_1(x) = x^2 + 2x + 1$ and $p_2(x) = -x^2 + 3x + 1$. Since $p_1(x)$ and $p_2(x)$ are of degree 2, they are both in S. However, $p_1(x) + p_2(x) = 5x + 2$ has odd degree and thus is not in S. We conclude that S is not closed under $+$ and so is not a subspace of P_4.

(c) **Solution:** Let S be the set of polynomials $p(x)$ in P_4 such that $p(0) = 0$.
That is, the subset S consists of those polynomials in P_4 that evaluate to zero when $x = 0$. You may notice that this means

$$S = \left\{ a_1 x + a_2 x^2 + a_3 x^3 \ : \ a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}.$$

We have to determine if S is a subspace of P_4.

(a) Is S non-empty? Consider, for example, the polynomial $x + x^2$. This is a polynomial of degree less than 4, and takes on the value 0 when $x = 0$, so $x + x^2$ is in S, and S is non-empty.

(b) Next we have to check if S is closed under addition. Let p_1 and p_2 be polynomials in S. Does $p_1 + p_2$ belong to S? We need to check the value of $p_1 + p_2$ at the input 0. So we consider

$$\begin{aligned} (p_1 + p_2)(0) &= p_1(0) + p_2(0) && \text{by definition of polynomial addition} \\ &= 0 + 0 && \text{since } p_1, p_2 \in S, \text{ so } p_1(0) = 0, p_2(0) = 0 \\ &= 0 \end{aligned}$$

Since $p_1 + p_2$ is zero when the input is 0, $p_1 + p_2 \in S$, and so S is closed under $+$.

(c) Next we check if S is closed under scalar multiplication. So let α be a scalar, and let $p \in S$. To check if $\alpha p \in S$, we have to evaluate αp at the input 0.

$$\begin{aligned} (\alpha p)(0) &= \alpha p(0) && \text{by definition of scalar multiplication in } P_4 \\ &= \alpha 0 && \text{since } p \in S, \text{ so } p(0) = 0 \\ &= 0 \end{aligned}$$

Since αp evaluates to zero when the input is 0, $\alpha p \in S$, and S is closed under $\cdot$.

Since S non-empty and closed under both operations, S is a subspace of P_4.

§3.2, p. 126, #19 Let A be an $n \times n$ matrix. Prove that the following statements are equivalent.

(a) $\mathcal{N}(A) = \{0\}$.

(b) A is nonsingular.

(c) For each $b \in \mathbb{R}^n$, the system $Ax = b$ has a unique solution.

Solution: Our strategy is to show that (a) implies (b), that (b) implies (c), and that (c) implies (a). Once this chain of implications is established, we can prove that any one statements implies the other two. For example, (b) $\implies$ (a) will be true because we have (b) $\implies$ (c) $\implies$ (a).

The proof required consists thus of three parts. For each part, we must keep in mind what is assumed and what is to be proven.

1. $(a) \implies (b)$

   Proof. **Assumption: $\mathcal{N}(A) = \{0\}$.**
   Recall that the nullspace of A is the set of all solutions to the homogenous equation $Ax = 0$. $\mathcal{N}(A) = \{0\}$ means the system $Ax = 0$ has only the trivial solution. Hence when A is reduced to echelon form, there are no zero rows. Otherwise, we would have a free variable, and hence a non-trivial solution to $Ax = 0$. Since A is $n \times n$ and its echelon form has no zero rows, A is row equivalent to I. Hence A is invertible (nonsingular). □

2. $(b) \implies (c)$

   Proof. **Assumption: A is nonsingular.**
   Hence there exists an $n \times n$ matrix A^{-1} with the property that $AA^{-1} = A^{-1}A = I_{n\times n}$. We use A^{-1} to solve $Ax = b$ as follows:

$$
\begin{aligned}
Ax = b \implies A^{-1}(Ax) &= A^{-1}(b), && \text{multiplying on the left by } A^{-1} \\
\implies (A^{-1}A)x &= A^{-1}b, && \text{by associativity of matrix multiplication} \\
\implies Ix &= A^{-1}b, && \text{since } A^{-1}A = I \\
\implies x &= A^{-1}b, && \text{by a property of } I
\end{aligned}
$$

   Since A^{-1} is unique, $A^{-1}b$ is unique and the system $Ax = b$ has a unique solution, namely $x = A^{-1}b$. Note no assumption was needed on b, so this argument is valid for every $b \in \mathbb{R}^n$. □

3. $(c) \implies (a)$

   Proof. **Assumption: For each $b \in \mathbb{R}^n$, the system $Ax = b$ has a unique solution.**
   In particular, we note that the zero vector is a vector in $\mathbb{R}^n$, so by assumption the system $Ax = 0$ has a unique solution. We know that $x = 0$ is always one solution of $Ax = 0$. Our assumption tells us that this is the only solution. This means $N(A) = \{0\}$. □

§3.3, p. 138, #15 Prove that any nonempty subset of a linearly independent set of vectors is also linearly independent.

Proof. Let S be a linearly independent set of vectors. Let T be a nonempty subset of S. We must prove that T is also a linearly independent set of vectors. We will prove this result by contradiction.

Suppose on the contrary, that T is a linearly *dependent* set. If we denote the vectors in T by $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$, then linear dependency implies that there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$, *not all zero* such that

$$\alpha_1 \mathbf{v_1} + \alpha_\mathbf{2} \mathbf{v_2} + \ldots + \alpha_\mathbf{k} \mathbf{v_k} = \mathbf{0}. \tag{11}$$

If we let $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ denote the remaining vectors in S (remember T was contained in S), then we can silently sneak these $\mathbf{w_i}$'s into (11) by rewriting this equation as

$$\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \ldots + \alpha_k \mathbf{v_k} + 0 \cdot \mathbf{w_1} + 0 \cdot \mathbf{w_2} + \ldots + 0 \cdot \mathbf{w_k} = \mathbf{0}. \tag{12}$$

Since not all the α_i's are zero, we have a non-trivial linear combination of all the vectors in S equalling the zero vector. This says that S is a linearly dependent set. But this contradicts what we were given! So our assumption that T is a linearly dependent set if false. Hence T is a linearly independent set. Since T was an arbitrary subset of S, we conclude that every subset of a linearly independent set is also linearly independent. □

§3.3, p. 138, #19 Let $v_1, v_2, \cdots, v_n$ be a spanning set for the vector space V, and let v be any other vector in V. Show that $v, v_1, \cdots, v_n$ are linearly dependent.

Solution: Since $v_1, \cdots, v_n$ span V, and $v \in V$, we can express v as a linear combination of $v_1, \cdots, v_n$, i.e.,

$$\begin{aligned} v &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \quad \text{for some scalars} \quad \alpha_1, \alpha_2, \cdots, \alpha_n, \\ \Longrightarrow v - v &= -v + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \quad \text{adding } -v \text{ to both sides} \\ \Longrightarrow \mathbf{0} &= -v + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n. \end{aligned}$$

Now we have a non-trivial linear combination (since v is scaled by -1) of the vectors $v, v_1, v_2, \cdots, v_n$ equal to the zero vector. Hence these vectors are linearly dependent.

§3.3, p. 138, #20 Let $v_1, v_2, \cdots, v_n$ be linearly independent vectors in a vector space V. Show that $v_2, \cdots, v_n$ cannot span V.

Solution: Assume the contrary, that $\text{span}\{v_2, \cdots, v_n\} = V$. Then v_1 is expressible as a linear combination of $v_2, \cdots, v_n$, i.e.,

$$\begin{aligned} v_1 &= \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n \quad \text{for some scalars} \quad \alpha_1, \alpha_2, \cdots, \alpha_n, \\ \Longrightarrow v_1 - v_1 &= -v_1 + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \quad \text{adding } -v_1 \text{ to both sides} \\ \Longrightarrow \mathbf{0} &= -v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n \end{aligned}$$

But now we have a non-trivial linear combination (since v_1 is scaled by -1) of the vectors $v_1, v_2, \cdots, v_n$ equal to the zero vector. Hence $v_1, v_2, \cdots, v_n$ are not linearly independent, a contradiction. Thus $v_2, \cdots, v_n$ cannot span V.

§3.4, p. 151, #12 In Exercise 3 of Section 2 (p. 125), some of the sets formed subspaces of $\mathbb{R}^{2\times 2}$. In each of these cases, find a basis for the subspace and determine its dimension.

Solution:

(a) Let $\mathcal{D}_2$ denote the set of all 2×2 diagonal matrices, i.e.,

$$\mathcal{D}_2 := \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Observe that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus

$$\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a spanning set for $\mathcal{D}_2$. Next, observe that $\mathcal{B}$ is clearly a linearly independent set. Since $\mathcal{B}$ is a linearly independent set that spans $\mathcal{D}_2$, $\mathcal{B}$ is a basis for $\mathcal{D}_2$. Since $\mathcal{B}$ contains 2 matrices, the subspace $\mathcal{D}_2$ has dimension 2.

(e) The set of all 2×2 matrices whose (1,1) entry is zero can be represented by

$$\mathcal{S} := \left\{ \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Observe that $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $\mathcal{S}$, since $\mathcal{B}$ is clearly a linearly independent set that spans $\mathcal{S}$. Since $\mathcal{B}$ contains 3 matrices, $\mathcal{S}$ is a 3-dimensional subspace of the vector space of all 2×2 matrices.

§3.6, p. 160, #16 Let A be a 5×8 matrix with rank equal to 5 and let b be any vector in $\mathbb{R}^5$. Explain why the system $Ax = b$ must have infinitely many solutions.

Solution: Since A has rank 5, there are 5 non-zero pivots. The augmented matrix $A|b$ can be reduced to

$$\begin{array}{cccccccc|c} 1 & \times & \times & \times & \times & \times & \times & \times & \times \\ 0 & 1 & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 1 & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 1 & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 1 & \times & \times & \times & \times \end{array}$$

Since the echelon form of A has no zero rows, this system will always be consistent, regardless of the choice of vector on the right hand side. Since there are 3 free variables, this system has infinitely many solutions.