Apr 2006 Selected Problems, Chapter 4 Math 230(N. Mackey)
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§4.1, p.173, #6a Is the following map L from $\mathbb{R}^2$ to $\mathbb{R}^3$ a linear transformation?

$$L(x) = (x_1, x_2, 1)^T$$

Solution: If L is a linear transformation, L must respect both vector + and scalar multiplication. This means it must be true that $L(x + y) = L(x) + L(y)$ and $L(\alpha x) = \alpha L(x)$ for all $x, y \in \mathbb{R}^2$ and for all α in $\mathbb{R}$. We begin our analysis by investigating whether L respects +.

$$\begin{aligned} L(x+y) &= L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) && \text{substituting for } x,\ y, \text{ since they are in } \mathbb{R}^2, \\ &= L\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) && \text{by definition of addition in } \mathbb{R}^2, \\ &= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 1 \end{bmatrix} && \text{by definition of } L. \end{aligned}$$

On the other hand,

$$\begin{aligned} L(x) + L(y) &= L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) && \text{substituting for } x,\ y, \text{ since they are in } \mathbb{R}^2, \\ &= \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} && \text{by definition of } L, \\ &= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 2 \end{bmatrix} && \text{by definition of addition in } \mathbb{R}^2. \end{aligned}$$

Now we see that $L(x + y) \neq L(x) + L(y)$ since their third coordinates are not equal. Hence L is not a linear transformation.

§4.1, p. 174, #17c Determine the kernel and range of the following linear operator in $\mathbb{R}^3$.

$$L(x) = (x_1, x_1, x_1)^T$$

Solution: The kernel of L, denoted $\ker(L)$, consists of all those vectors in $\mathbb{R}^3$ that are killed by L, that is, all vectors that get mapped to the zero vector by L. Thus our strategy for finding $\ker(L)$ is always to solve $L(x) = 0$ for x. In the present case, we have

$$L(x) = 0 \implies \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly, this tells us $x_1 = 0$, and there are no restrictions on x_2 and x_3. Thus

$$\ker(L) = \left\{ \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\},$$

which we can think of as the yz-plane in $\mathbb{R}^3$. We also see from the last displayed equation that a basis for $\ker(L)$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and so $\dim(\ker(L)) = 2$, making $\ker(L)$ a 2-dimensional subspace of $\mathbb{R}^3$.

The range of L, denoted range(L), consists of all vectors in $\mathbb{R}^3$ that are hit by L, that is, all vectors v in the codomain of L for which there exists a corresponding vector u in the domain of L such that $L(u) = v$. Our strategy for finding range(L) is to describe $L(x)$ in terms of x_1, x_2, x_3. In this case, $L(x) = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This shows that

$$\text{range}(L) = \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : x_1 \in \mathbb{R} \right\},$$

so a basis for range(L) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Hence $\dim(\text{range}(L)) = 1$, making range($L$) a 1-dimensional subspace of $\mathbb{R}^3$.

Finally, we note that $\dim(\text{range}(L)) + \dim(\ker(L)) = 1 + 2 = 3 = \dim(\mathbb{R}^3)$, as expected.

§4.2, p. 189, #14a Let $L : P_3 \to P_2$ be the linear transformation given by

$$L(p(x)) = p'(x) + p(0)$$

Find the matrix representation for L with respect to the bases $\mathcal{S} := [x^2, x, 1]$ and $\mathcal{T} := [2, 1 - x]$ of P_3 and P_2 respectively.

Also find the coordinates of $L(p(x))$ with respect to the ordered basis $\mathcal{T}$, when $p(x) = x^2 + 2x - 3$.

Solution: Let A denote the matrix representation of L relative to the specified bases $\mathcal{S}$ and $\mathcal{T}$. The columns of A correspond to the coordinate vectors representing $L(x^2)$, $L(x)$, and $L(1)$ in the basis $\mathcal{T}$. So we begin by using the definition of L to calculate these images. Since $L(p(x)) = p'(x) + p(0)$, we get

$$\begin{aligned} L(x^2) &= 2x + 0 = 2x, && (1) \\ L(x) &= 1 + 0 = 1 && (2) \\ \text{and} \quad L(1) &= 0 + 1 = 1. && (3) \end{aligned}$$

Then

$$A = \Big[\big[L(x^2)\big]_{\mathcal{T}} \quad \big[L(x)\big]_{\mathcal{T}} \quad \big[L(1)\big]_{\mathcal{T}} \Big] = \Big[\big[2x\big]_{\mathcal{T}} \quad \big[1\big]_{\mathcal{T}} \quad \big[1\big]_{\mathcal{T}} \Big].$$

Next, we must find $\big[2x\big]_{\mathcal{T}}$ and $\big[1\big]_{\mathcal{T}}$. Recall that $\big[2x\big]_{\mathcal{T}}$ stands for the coordinate vector of $2x$ relative to the basis $\mathcal{T}$ of P_2. To find the coordinates of $2x$ we must we must determine how to express $2x$ as a linear combination of the "vectors" (polynomials) in basis $\mathcal{T}$.

Sometimes, as here, this may be easy to see directly by inspection. (But we also offer a general approach for more complex situations.) Observe that

$$2x = 1 \cdot \mathbf{2} + (-2)(\mathbf{1} - \mathbf{x}). \tag{4}$$

Hence, relative to the basis $\mathcal{T}$, the polynomial $2x$ is represented by the coordinate vector $\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]_{\mathcal{T}}$. This will be the first column of A, the matrix representing L in the specified bases.

What if we are faced with a more complex situation or if can't see (4) simply by inspection? Well, we are after scalars c_1, c_2 such that

$$c_1 \cdot \mathbf{2} + c_2 \cdot (\mathbf{1} - \mathbf{x}) = 2x. \tag{5}$$

The coordinate vector of $2x$ relative to the basis $\mathcal{T} := [2, 1 - x]$ will be $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]_{\mathcal{T}}$, and this in turn will become the first column of A. We need to solve (5) for c_1, c_2. Expand the left hand side and collect like terms:

$$\begin{aligned}&(2c_1+c_2)-c_2x=2x,\\ \implies &(2c_1+c_2)+(-c_2-2)x=0,\\ \implies &(2c_1+c_2)\mathbf{1}+(-c_2-2)\mathbf{x}=\mathbf{0}.\end{aligned}$$

But now we have a linear combination of the linearly independent polynomials **1** and **x** equal to the zero polynomial. This means that

$$\begin{aligned}2c_1+c_2&=0\\ -c_2-2&=0\end{aligned}$$

We must solve this linear system for the unknowns c_1, c_2. Make sure to write the system in standard form before proceeding:

$$\begin{aligned}2c_1+c_2&=0\\ -c_2&=2\end{aligned}$$

The system is already in echelon form. Back substitution yields $c_2=-2$ and $c_1=1$. Thus $[2x]_{\mathcal{T}}=\begin{bmatrix}c_1\\c_2\end{bmatrix}_{\mathcal{T}}=\begin{bmatrix}1\\-2\end{bmatrix}_{\mathcal{T}}$.

Next, (2) and (9) tell us that we must express 1 as a linear combination of the polynomials in the basis $\mathcal{T}$. Clearly, $1=\frac{1}{2}\cdot 2+0\cdot(1-x)$, so that $[1]_{\mathcal{T}}=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}_{\mathcal{T}}$.

Thus

$$A=\begin{bmatrix}[2x]_{\mathcal{T}} & [1]_{\mathcal{T}} & [1]_{\mathcal{T}}\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ -2 & 0 & 0\end{bmatrix}.$$

Now to find $[L(p(x))]_{\mathcal{T}}$, we can exploit the matrix A.

$$\begin{aligned}[L(p(x))]_{\mathcal{T}}&=A[p(x)]_{\mathcal{S}}\\ &=A\begin{bmatrix}1\\2\\-3\end{bmatrix}\\ &=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ -2 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\-3\end{bmatrix}\\ &=\begin{bmatrix}1\\-2\end{bmatrix}+2\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}-3\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\\ &=\begin{bmatrix}\frac{1}{2}\\-2\end{bmatrix}_{\mathcal{T}}\end{aligned}$$

We can check this answer.

$$\begin{bmatrix}\frac{1}{2}\\-2\end{bmatrix}_{\mathcal{T}}=\frac{1}{2}\cdot\mathbf{2}+(-2)\cdot\mathbf{(1\text{-}x)}=2x-1.$$

Then calculate $L(p(x))$ when $p(x) = x^2 + 2x - 3$, directly from the definition.

$$\begin{aligned} L(p(x)) &= p'(x) + p(0), && \text{by definition of } L \\ &= 2x + 2 - 3, && \text{since } p'(x) = 2x + 2 \text{ and } p(0) = -3 \\ &= 2x - 1 \end{aligned}$$

which agrees with what was obtained by using the matrix encoding of A.

§4.2, p.198, #18a Let $E = [u_1, u_2, u_3]$ and $F = [b_1, b_2]$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, where

$$u_1 = (1, 0, -1)^T, \quad u_2 = (1, 2, 1)^T, \quad u_3 = (-1, 1, 1)^T,$$

and

$$b_1 = (1, -1)^T, \quad b_2 = (2, -1)^T.$$

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $L(x) = (x_3, x_1)^T$. Find the matrix representation of L with respect to the bases E and F.

Solution: Let A denote the matrix representation of L relative to the specified bases E and F. The columns of A correspond to the coordinate vectors representing $L(u_1)$, $L(u_2)$, and $L(u_3)$ in the basis F. We begin by using the definition of L to calculate these images. By definition,

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix}. \tag{6}$$

Note that coordinates in (6) are with respect to the standard bases of $\mathbb{R}^3$and $\mathbb{R}^2$. Using (6) we get

$$L(u_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{7}$$

$$L(u_2) = L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8}$$

$$\text{and} \quad L(u_3) = L\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{9}$$

We must rewrite each of $L(u_1)$, $L(u_2)$, and $L(u_3)$ as linear combinations of the vectors in the basis $F = [\mathbf{b_1}, \mathbf{b_2}]$. We begin with $L(u_1)$. We want to find scalars c_1, c_2 such that

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} = L(u_1) \tag{10}$$

$$\implies c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{11}$$

$$\implies \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{12}$$

If you use the column-wise view of matrix multiplication, you will be able to see by inspection that $c_1 = -1$ and $c_2 = 0$ is a solution to this system. Moreover, we can argue that this solution must be unique, because the coefficient matrix is invertible, since its columns come from the basis F, so the columns are linearly independent. (Observe that this argument justifying the invertibility of the coefficient matrix can be applied just as easily even if we were dealing with a higher dimensional co-domain!) If you didn't see the solution by inspection, set up the usual gig, reduce to echelon form, then do back substitution:

$$\begin{array}{rr|rl} 1 & 2 & -1 & \\ -1 & -1 & 1 & \quad R_2 \leftarrow R_2 + R_1 \\ \hline 1 & 2 & -1 & \\ 0 & 1 & 0 & \end{array}$$

From R_2 we see that $c_2 = 0$, and then substituting in R_1 yields $c_1 = -1$. Thus the first column of A has been found:

$$[L(u_1)]_F = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_F = \begin{bmatrix} -1 \\ 0 \end{bmatrix}_F. \tag{13}$$

Similarly, we find $[L(u_2)]_F$ and $[L(u_3)]_F$ by setting up and solving appropriate linear systems. Observe that the coefficient matrix will be the same, only the RHS will change.

$$[L(u_2)]_F : \begin{array}{rr|r} 1 & 2 & 1 \\ -1 & -1 & 1 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 2 \end{array} \quad R_2 \leftarrow R_2 + R_1 \qquad [L(u_3)]_F : \begin{array}{rr|r} 1 & 2 & 1 \\ -1 & -1 & -1 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 0 \end{array} \quad R_2 \leftarrow R_2 + R_1$$

Back substitution yields

$$[L(u_2)]_F = \begin{bmatrix} -3 \\ 2 \end{bmatrix}_F, \qquad [L(u_3)]_F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_F.$$

Thus

$$A = \left[[L(u_1)]_F \quad [L(u_2)]_F \quad [L(u_3)]_F \right] = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

is the matrix encoding of L relative to the basis E in the domain, and F in the co-domain.

Chapter 4, Test A, p.196 Answer *true* if the statement is always true, and *false* otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.

#1 Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If $L(\mathbf{x}) = L(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Solution: We begin by using what we are given, that L is a linear transformation and that $L(\mathbf{x}) = L(\mathbf{y})$.

$$\begin{array}{rll} & L(\mathbf{x}) = L(\mathbf{y}) & \text{by assumption,} \\ \Longrightarrow & L(\mathbf{x}) - L(\mathbf{y}) = 0 & \text{subtracting } L(\mathbf{y}) \text{ from both sides,} \\ \Longrightarrow & L(\mathbf{x} - \mathbf{y}) = 0 & \text{since } L \text{ is linear.} \end{array}$$

Thus L kills the vector $\mathbf{x} - \mathbf{y}$. If the only vector that L kills is the zero vector (in other words, if $\ker(L) = \{0\}$), then we can conclude that $\mathbf{x} - \mathbf{y} = 0$ and hence $\mathbf{x} = \mathbf{y}$. However, if $\ker(L)$ is nontrivial, then there will be a nonzero vector $\mathbf{u}$ mapped to zero by L. In this case, we can set $\mathbf{x} - \mathbf{y} = \mathbf{u} \neq 0$, and then it will follow that $\mathbf{x} \neq \mathbf{y}$. This suggests a counterexample.

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the x_1−axis, that is, $L\left(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right]$. We have shown elsewhere that L is a linear transformation. We can see that any two points in the plane that lie on the same vertical line will get mapped by L to the same point on the x-axis. We choose 2 such points.

$$L\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \qquad \text{and} \qquad L\left(\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right],$$

but $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$. Thus we conclude that the statement is false in general.

$\boxed{\#2}$ If L_1 and L_2 are linear operators on a vector space V, then $L_1 + L_2$ is also a linear operator on V, where $L_1 + L_2$ is defined as

$$(L_1 + L_2)(x) := L_1(x) + L_2(x), \quad \text{for all } x \in V. \tag{14}$$

Solution: We must investigate whether $L_1 + L_2$ respects vector addition and scalar multiplication. Read (14) with care —the plus sign on the LHS is between *maps* while the plus sign on the RHS is between *vectors*.

1. We want to know if $L_1 + L_2$ respects vector plus, that is

$$(L_1 + L_2)(x_1 + x_2) = (L_1 + L_2)(x_1) + (L_1 + L_2)(x_2), \quad \text{for all } x_1, x_2 \in V. \tag{15}$$

   Start with the LHS. Let $x_1,\ x_2 \in V$.

$$\begin{aligned}
(L_1 + L_2)(x_1 + x_2) &= L_1(x_1 + x_2) + L_2(x_1 + x_2) && \text{by definition (14) of } L_1 + L_2, \\
&= L_1(x_1) + L_1(x_2) + L_2(x_1) + L_2(x_2) && \text{since } L_1 \text{ and } L_2 \text{ are linear,} \\
&= L_1(x_1) + L_2(x_1) + L_1(x_2) + L_2(x_2) && \text{since vector + is commutative,} \\
&= \big(L_1(x_1) + L_2(x_1)\big) + \big(L_1(x_2) + L_2(x_2)\big) && \text{since vector + is associative,} \\
&= (L_1 + L_2)(x_1) + (L_1 + L_2)(x_2) && \text{by definition (14) of } L_1 + L_2.
\end{aligned}$$

   Thus (15) holds, and $L_1 + L_2$ respects vector $+$.

2. Next we check whether

$$(L_1 + L_2)(\alpha x) = \alpha(L_1 + L_2)(x), \quad \text{for all } x \in V, \text{ and for all } \alpha \in \mathbb{R}. \tag{16}$$

   So start with the LHS. Let α be any scalar, and x any vector in V.

$$\begin{aligned}
(L_1 + L_2)(\alpha x) &= L_1(\alpha x) + L_2(\alpha x) && \text{by definition (14) of } L_1 + L_2, \\
&= \alpha L_1(x) + \alpha L_2(x) && \text{since } L_1 \text{ and } L_2 \text{ are both linear,} \\
&= \alpha\big(L_1(x) + L_2(x)\big) && \text{since } \cdot \text{ distributes over + in any vector space,} \\
&= \alpha(L_1 + L_2)(x) && \text{by definition (14) of } L_1 + L_2.
\end{aligned}$$

   Thus (16) hold, and $L_1 + L_2$ respects scalar multiplication.

Since $L_1 + L_2$ respects both operations, $L_1 + L_2$ is a always linear transformation. In other words, the sum of two linear transformations defined on V is also a linear transformation on V.

#3 Let $L : V \to V$ be a linear operator. If $x \in \ker(L)$, then $L(v + x) = L(v)$, for all $v \in V$.
Solution: This is patently true, as we show. Let $v \in V$. Then we have

$$\begin{aligned} L(v+x) &= L(v) + L(x) && \text{since } L \text{ is linear,} \\ &= L(v) + 0 && \text{since } x \in \ker(L), \\ &= L(v) && \text{since 0 is the additive identity.} \end{aligned}$$

#4 If L_1 rotates each vector x in $\mathbb{R}^2$ by 60 degrees and then reflects the resulting vector about the x-axis, and L_2 is a transformation that does the same two operations, but in the reverse order, then $L_1 = L_2$.
Solution: Since the direction of the rotation has not been specified, we choose a direction, say counterclockwise. Let R denote the linear transformation that rotates each vector in $\mathbb{R}^2$ by 60 degrees counterclockwise, and S denote the linear transformation that reflects each vector in $\mathbb{R}^2$ about the $x-$axis. Then $L_1 := S \circ R$, while $L_2 := R \circ S$. Thus $L_1 = L_2$ if and only if $S \circ R = R \circ S$. Let's find the matrix representations of R and S relative to the standard basis for $\mathbb{R}^2$.

$$[R] = \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix}, \quad [S] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{17}$$

Now $S \circ R = R \circ S$ if and only if the matrices that represent them are equal, that is, if and only if the matrix products $[S][R]$ and $[R][S]$ are equal. Now,

$$[S][R] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix} = \begin{bmatrix} \cos 60 & -\sin 60 \\ -\sin 60 & -\cos 60 \end{bmatrix}.$$

On the other hand,

$$[R][S] = \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos 60 & \sin 60 \\ \sin 60 & -\cos 60 \end{bmatrix}.$$

Since $[S][R] \neq [R][S]$, we conclude that $S \circ R \neq R \circ S$, and hence $L_1 \neq L_2$.

#7 Let $E = [x_1, x_2, \ldots, x_n]$ be an ordered basis for $\mathbb{R}^n$. If $L_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^n$ have the same matrix representation with respect to E, then $L_1 = L_2$.
Solution: The notation $L_1 = L_2$ means that L_1 and L_2 represent the *same function*, that is, L_1 and L_2 yield the same output for every possible input. Thus we need to determine whether $L_1(v) = L_2(v)$ for all $v \in V$. Since L_1 and L_2 are represented by same matrix, call it A, with respect to the basis E, we have

$$[L_1(v)]_E = A[v]_E \quad \text{and} \quad [L_2(v)]_E = A[v]_E, \quad \text{for all } v \in \mathbb{R}^n$$

Thus $[L_1(v)]_E = [L_2(v)]_E$ for all $v \in \mathbb{R}^n$. This means that for every v, $L_1(v)$ and $L_2(v)$ are expressed by the same linear combination of the basis vectors $x_1, x_2, \ldots, x_n$. Thus $L_1(v) = L_2(v)$ for every $v \in \mathbb{R}^n$. Consequently, $L_1 = L_2$ and the statement is true.