Last Revised: December 5, 2011

Problem Let S be the 2-dimensional subspace of $\mathbb{R}^3$ spanned by

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

(a) Find a basis for $S^\perp$.

Solution: Recall that the nullspace and rowspace of a matrix are orthogonal complements of each other. So, if make x_1 and x_2 into the rows of a matrix, then the rowspace of this matrix will be S, and the nullspace will be $S^\perp$. Thus all we have to do is find a basis for $\mathcal{N}(A)$, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

So we begin by finding all solutions to the homogeneous system $Ax = \mathbf{0}$.

$$\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ \hline \end{array}$$

The matrix A is already in echelon form, so all we have to do is back susbstitution. From the second row, we see that x_3 is free. We find x_1 and x_2 in terms of the free variable x_3. From row 2, $x_2 = -2x_3$. Plugging this into row 1 yields $x_1 = -2x_3$. Thus

$$S^\perp = \mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}.$$

So a basis for $S^\perp$ is $\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) Find a geometric meaning for S and $S^\perp$.

Solution: S is spanned by two linearly independent vectors x_1 and x_2, so S is a plane in $\mathbb{R}^3$ passing through the origin. $S^\perp$ has a basis consisting of just one vector, so $S^\perp$ is a one-dimensional subspace of $\mathbb{R}^3$ that is, a line through the origin. *This line $S^\perp$ is perpendicular to the plane S.*