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Problem Let A be a nonsingular $n \times n$ matrix and let λ be an eigenvalue of A.

(a) Show that $\lambda \neq 0$.

Solution: Recall that if λ is an eigenvalue of A, then there is some nonzero vector v such that $Av = \lambda v$.

If $\lambda = 0$, then we have a non-zero vector v such that $Av = 0v = \mathbf{0}$. Thus the homogeneous system $Ax = 0$ has a non-trivial solution. Since A is a square matrix, this means that A must be singular. This contradicts the fact that A is nonsingular.

Here's an alternate argument:
If $\lambda = 0$, then we have a non-zero vector v such that $Av = 0v = \mathbf{0}$. So $\mathcal{N}(A)$, the nullspace of A, contains a non-zero vector, v. Hence $\dim(\mathcal{N}(A))$ is at least one. This implies that $\text{rank}(A)$ is strictly less than n, which implies that A is not invertible. This is clearly a contradiction since we know that A is invertible.

Thus it must be that $\lambda \neq 0$.

(b) Show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Solution: Let v be an eigenvector corresponding to the eigenvalue λ of A.
On one hand,

$$\begin{aligned} A^{-1}(Av) &= A^{-1}(\lambda v), && \text{since } v \text{ is an eigenvector associated with } \lambda, \\ &= \lambda A^{-1}v, && \text{since } \lambda \text{ is a scalar} \end{aligned}$$

On the other hand,

$$\begin{aligned} A^{-1}(Av) &= (A^{-1}A)v, && \text{by associativity of matrix multiplication,} \\ &= Iv, && \text{since } A^{-1}A = I, \\ &= v, && \text{by a property of } I. \end{aligned}$$

Thus

$$\begin{aligned} \lambda A^{-1}v &= v, && \text{since each equals } A^{-1}(Av) \text{ by the work above,} \\ \Longrightarrow \quad A^{-1}v &= \tfrac{1}{\lambda}v, && \text{since } \lambda \neq 0 \text{ by part (a), we may divide both sides by } \lambda. \end{aligned}$$

Thus $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Note that v is an eigenvector of A as well as A^{-1}. This says that if we have eigenvalues and eigenvectors of an invertible matrix A, then we can identify eigenvalues and eigenvectors of A^{-1}, *without calculating* A^{-1}.