Math 230 **Final Exam Preparation** Mackey

Last updated: December 5, 2011

The Final exam will be cumulative, with emphasis on the later part of the course. Here's a good preparation strategy:

1. Quizzes and Exams.
   (a) Carefully examine the problems and the solutions.
   (b) Modify the problems, solve the modified problems.
   (c) Memorize definitions that were tested. Clearly understand all of them.
2. Then study the solutions to the problems assigned form the **Chapter Tests, and True/False problems.** Solutions to several of these problems, particularly from the later chapters have been posted on the class web page.
3. Teview all problems discussed in class.
4. Then review solutions to all assigned homework problems.
5. The Student Study Guide is a good resource for identifying the key definition and concepts from each Chapter.

Here is a list of topics for review.

1. Know how to determine if a linear system $Ax = b$ has a unique solution, infinitely many solutions or no solution. Know how to correctly specify the set of all solutions.
2. What property must b have in order for $Ax = b$ to have a solution (that is for the system to be consistent)?
3. If a homogeneous system $Ax = 0$ has a nontrivial solution, what does that say about the columns of A?
4. How do row operations of type I, II and II affect the value of the determinant of a matrix? What is the connection between $\det(A)$ and $\det(\alpha A)$ where $\alpha \in \mathbb{R}$?
5. Can $\det(A + B)$ be expressed in terms of $\det(A)$ and $\det(B)$? Can $\det(AB)$ be? What about $\det(A^{-1})$? Or $\det(A^{-3}B^T)$?
6. Know the connection between the rank of a matrix and the dimension of its nullspace.(The Rank-Nullity Theorem.)
7. Know how to calculate a basis for and the dimension of the rowspace, columnspace and nullspace of an $n \times m$ matrix.
8. If a matrix is invertible, is its inverse unique? Why? Is a product of invertible matrices also invertible? Why? (The answer is yes to both questions, and you should be able to prove proofs! Check your class notes!)

9. Know the various conditions equivalent to invertibility of an $n \times n$ matrix (Thus if any of these conditions is violated, A cannot be invertible):
   - $\text{rank}(A) = n$
   - A has linearly independent rows, so the rows of A form a basis for $\mathbb{R}^n$
   - A has linearly independent columns, so the columns of A form a basis for $\mathbb{R}^n$
   - $Ax = 0$ has only the trivial solution
   - The nullspace of A contains only the zero vector
   - The scalar zero cannot be an eigenvalue of A.
   - $\det(A) \neq 0$.
10. Know what the standard vector spaces are along with their standard bases, and their dimensions.
11. How can one determine if a given vector is in the span of a given set of vectors?
12. How does one determine if a given set of vectors in is linearly independent?
13. Can a set of 10 vectors span $\mathbb{R}^{10}$? Under what conditions?
    Can fewer than 10 vectors span $\mathbb{R}^{10}$? Why not?
    Can a set containing more than n vectors in $\mathbb{R}^n$ be linearly independent? Explain why not.
14. Given a set of vectors in $\mathbb{R}^n$, know how to determine whether or not they are linearly independent. If they are linearly dependent, know how to reduce them to a linearly dependent set.
15. Know how to determine if a given subset of vectors forms a subspace. Know some standard examples of subpaces of $\mathbb{R}^2$, $\mathbb{R}^3$, and of $n \times n$ real matrices. Know how to find a basis for a subspace.
16. Your friend claims that any subspace of any vector space must always contain the zero vector. Is she correct? Why?
17. How does one find the coordinates of a vector with respect to a given basis?
18. Know how to determine if a given map is a linear transformation.
19. Your friend isn't sure if a linear transformation always takes the zero vector in the domain to the zero vector in the co-domain. Convince him that this is always true.
20. How does one find the matrix representation of a linear transformation with respect to a given basis?
21. If the action of a linear transformation on a basis is specified, how can one deduce its action on any other given vector?
22. Know how to encode rotations, reflections, dilations (stretching every vector by some given factor) and combinations of these via matrices using the standard basis of $\mathbb{R}^2$.

23. Know the Cauchy Schwarz inequality in $\mathbb{R}^n$ and its significance.
24. Know how to calculate the length of a vector in $\mathbb{R}^n$, and how to find the angle between two vectors in $\mathbb{R}^n$. When are two vectors in $\mathbb{R}^n$ said to be orthogonal?
25. How do you calculate the orthogonal projection of a given vector in $\mathbb{R}^n$ along another given vector in $\mathbb{R}^n$?
26. Let A be any $m \times n$ matrix. Your friend claims that the rowspace and the nullspace of A are orthogonal complements. Explain why this is so.
27. If you are given a basis for a subspace S of $\mathbb{R}^n$, how can you find a basis for its orthogonal complement? If $\dim(S) = k$, what will $\dim(S^{\perp})$ be?
28. Know how to calculate the eigenvalues and eigenvectors of 2×2 and 3×3 matrices. Learn to carefully examine matrices to see if some eigenpairs can be guessed by inspection.
29. If (λ, v) is an eigenpair of A, will (λ^2, v) be an eigenpair of A^2? If A is invertible, will (λ^{-1}, v) be an eigenpair of A? (Remember that if A is invertible, zero cannot be an eigenvalue of A).
30. What can be said about the eigenvalues and eigenvectors of a real symmetric matrix?
31. What do we mean when we say that a matrix is diagonalizable?
32. If you are given a spectral decomposition of a matrix, how do you read off eigenpairs? Conversely, if you are given n eigenpairs, how do you construct a spectral decomposition? If the n eigenvectors you are given are orthonormal, does this make the construction of the spectral decomposition simpler? How so?
33. Are all $n \times n$ matrices diagonalizable? Give an example of one that is not.
34. If a matrix has n distinct eigenvalues, must that matrix be diagonalizable? Why?
35. What is the connection between eigenpairs and finding $\lim_{n\to\infty} A^n$?