1. Prove that if A is nonsingular then A^T is nonsingular and

$$(A^T)^{-1} = (A^{-1})^T.$$

Discussion: Lets put into words what are we asked to show in this problem. First, we must show that if a matrix is invertible, then so is its transpose. We must also show that "the inverse of the transpose is the same as the transpose of the inverse." In other words, if we think of inverting and transposing as processes we may perform on square matrices, then for invertible matrices, these two processes performed in either order yield the same result.

Proof: Let A be nonsingular. By definition, there exists A^{-1} such that $A^{-1}A = AA^{-1} = I$. First we note that $I^T = I$ (verify this). Now we see that

$$\begin{aligned} I = I^T &= (A^{-1}A)^T, && \text{since } A^{-1}A = I \\ &= A^T(A^{-1})^T, && \text{since } (AB)^T = B^T A^T. \end{aligned}$$

Thus we have

$$A^T(A^{-1})^T = I \tag{1}$$

Similarly, we can show that

$$(A^{-1})^T A^T = I. \tag{2}$$

Thus A^T and $(A^{-1})^T$ multiply to give the identity matrix. Hence A^T is invertible. Furthermore, (1) and (2) show that $(A^{-1})^T$ is the inverse of A^T. In mathematical notation, this statement becomes the equation

$$(A^{-1})^T = (A^T)^{-1}.$$

Q.E.D.

(Note: Q.E.D. is an abbreviation for the Latin phrase "Quod erat demonstrandum" which means "That which was to be demonstrated", signifying the end of the proof.)

2. Show that if A is a symmetric nonsingular matrix, then A^{-1} is also symmetric.

Proof: Since A is symmetric, $A^T = A$. Since A is nonsingular A^{-1} exists. Now,

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1}, && \text{by §1.3, problem 17} \\ &= A^{-1}, && \text{since } A^T = A \end{aligned}$$

Thus A^{-1} is symmetric, since it is equal to its own transpose. Q.E.D.

3. Let A be an $n \times n$ matrix and let x and y be vectors in $\mathbb{R}^n$. Show that if $Ax = Ay$ and $x \neq y$, then the matrix A must be singular.

Proof (by contradiction): We are given that $Ax = Ay$ with $x \neq y$. We have to argue that this forces A to be singular. Suppose, on the contrary, that A is nonsingular. Then there exists A^{-1} such that $A^{-1}A = AA^{-1} = I$. Now,

$$\begin{aligned} Ax = Ay \quad &\Longrightarrow \quad A^{-1}(Ax) = A^{-1}(Ay), && \text{left multiplying by } A^{-1} \\ &\Longrightarrow \quad (A^{-1}A)x = (A^{-1}A)y, && \text{since matrix multiplication is associative} \\ &\Longrightarrow \quad Ix = Iy && \text{since } A^{-1}A = I \\ &\Longrightarrow \quad x = y \end{aligned}$$

which contradicts $x \neq y$. Hence A must be singular. Q.E.D.

4. Let A be an $n \times n$ matrix and let

$$B = A + A^T \ \text{ and } \ C = A - A^T \tag{3}$$

(a) Show that B is symmetric and C is skew-symmetric.

Proof: To show B is symmetric, we must show that it equals its own transpose.

$$\begin{aligned} B^T &= (A + A^T)^T, && \text{by (3)} \\ &= A^T + (A^T)^T, && \text{since transpose of a sum is the sum of the transposes} \\ &= A^T + A, && \text{since taking the transpose twice yields the original matrix} \\ &= B && \text{since matrix addition is commutative} \end{aligned}$$

Thus $B = B^T$ and so B is symmetric.

To show C is skew-symmetric, we must show that $C = -C^T$.

$$\begin{aligned} C^T &= (A - A^T)^T, && \text{by (3)} \\ &= A^T - (A^T)^T, && \text{since the transpose of a difference is the difference of the transposes} \\ &= A^T - A, && \text{since taking the transpose twice yields the original matrix} \\ &= -(A - A^T), && \text{factoring out the scalar } -1 \\ &= -C, && \text{by (3)} \end{aligned}$$

Thus $C^T = -C$ which implies $C = -C^T$ and we see that C is skew-symmetric.

Q.E.D.

(b) Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and a skew-symmetric matrix.

Proof: Let A be an $n \times n$ matrix. Then A^T exists and is also an $n \times n$ matrix. By part (a), $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. Now we notice that

$$(A + A^T) + (A - A^T) = 2A \tag{4}$$

since matrix addition is associative and commutative.

This is close to what we want, but not exactly what we want. We have a symmetric matrix and a skew-symmetric matrix that add to give $2A$, the matrix A times the scalar 2. We fix the problem by multiplying both sides of (4) by 1/2.

$$\begin{aligned} \tfrac{1}{2}[(A + A^T) + (A - A^T)] &= \tfrac{1}{2}(2A) \\ \Longrightarrow \quad \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T) &= A \end{aligned}$$

since scalar multiplication distributes over matrix addition.

Finally, we note that multiplying a symmetric matrix by a scalar yields a symmetric matrix (the reader should verify this). Similarly, a scalar times a skew-symmetric matrix yields a skew-symmetric matrix (verify). Thus $\frac{1}{2}(A + A^T)$ and $\frac{1}{2}(A - A^T)$ are symmetric and skew-symmetric respectively and we have expressed A as the sum of a symmetric matrix and a skew-symmetric matrix.

Q.E.D.

5. In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, in certain special cases the commutative property does hold. Show that:

(a) If A and B are $n \times n$ diagonal matrices, then $AB = BA$.

$$AB = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & \ddots & \\ & & & b_{nn} \end{bmatrix}$$

We may think of this product *row-wise*: the ith row of AB is a linear combination of the rows of B with scalars coming from the ith row of A. Thus in determining the first row of AB, for instance, we see that the first row of B gets scaled by a_{11} while all the other rows get scaled by 0. So the first row of AB has first entry $a_{11}b_{11}$, all the other entries being 0. The attentive reader should verify that

$$AB = \begin{bmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & \\ & & \ddots & \\ & & & a_{nn}b_{nn} \end{bmatrix}$$

Similarly,

$$BA = \begin{bmatrix} b_{11}a_{11} & & & \\ & b_{22}a_{22} & & \\ & & \ddots & \\ & & & b_{nn}a_{nn} \end{bmatrix}$$

Since multiplication of scalars is commutative, $a_{ii}b_{ii} = b_{ii}a_{ii}$ for $i = 1 \dots n$. Thus $AB = BA$.

(b) If A is an $n \times n$ matrix and

$$B = a_0 I + a_1 A + a_2 A^2 + \dots + a_k A^k \tag{5}$$

where $a_0, a_1, \dots, a_k$ are scalars, then $AB = BA$.

Proof:

$$\begin{aligned} AB &= A(a_0 I + a_1 A + a_2 A^2 + \dots + a_k A^k) && \text{by (5)} \\ &= a_0 A + a_1 A^2 + a_2 A^3 + \dots + a_k A^{k+1} && \text{since matrix multiplication distributes over matrix addition} \\ &= (a_0 I + a_1 A + a_2 A^2 + \dots + a_k A^k)A && \text{factoring out A on the } \textbf{right} \\ &= BA && \text{by (5)} \end{aligned}$$

Q.E.D.

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