MATH 6070 **Homework 2** Spring 2011
Due: Noon, Friday 28 January 2011

1. Let $h > 0$ be fixed, and consider the *symmetric difference* operator SD_h defined by

$$SD_h(f, x) = \frac{f(x+h) - f(x-h)}{2h}.$$

   (a) Prove that SD_h is linear in the first argument, that is

$$SD_h(f+g, x) = SD_h(f, x) + SD_h(g, x) \quad \text{for all} \quad f, g \tag{1}$$
$$SD_h(\alpha f, x) = \alpha SD_h(f, x) \quad \text{for all scalars} \quad \alpha \tag{2}$$

   (b) Prove that

$$\mathrm{SD}_h(f, a) = f'(a)$$

   whenever f is a polynomial in x of degree at most 2. (Tip: A nice proof would exploit the linearity result of the previous part.)

   (c) Suppose $f(x)$ is a polynomial of degree 3 in x. Calculate $SD_h(f, x)$. (Tip: A nice solution would use the results of the previous parts.)

   Remark: Note that (b), (c) have to be established for *all* quadratics or cubics.

2. In class, we derived a 3 point, $O(h^2)$ approximation for $f'(x)$ at a left endpoint of a set of data. Derive the analogue for the right end-point of the set of data.

3. Use Taylor series to derive an $O(h^2)$ approximation for $f'(x)$ based on the values of f at the unequally spaced points $x-h$, x, $x+2h$. Such formulas are needed when some data is missing.

4. Use Taylor series to derive an $O(h^4)$ five-point approximation for $f'(x)$ based on the values of f at the points $x-2h$, $x-h$, x, $x+h$, $x+2h$.

5. **Existence is Important** Consider the following result about the existence of a solution to a first order IVP.

   Theorem 1 *If f is continuous in a rectangle R centered at (t_0, x_0), say*

$$R = \{(t, x) : |t - t_0| \le \alpha, \quad |x - x_0| \le \beta\}$$

   then the initial-value problem

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0$$

   has a solution $x(t)$ for $|t - t_0| \le \min(\alpha, \beta/M)$, where M is the maximum of $|f(t, x)|$ in the rectangle R.

Now consider the IVP $\dfrac{dx}{dt} = 1 + x^2, \quad x(0) = 0.$

(a) Solve the IVP exactly. Graph the solution. A neat properly labeled graph by hand is okay, or you may use MATLAB.

(b) Clearly predict what Euler's method will do, if it runs until $t = 1.6$. What effect does step size have on your prediction?

(c) Submit relevant numerical evidence for part (b), and a plot of the path of Euler. Make sure you annotate the numerical evidence and the labeled plot appropriately.

(d) What is the largest interval around $t_0 = 0$ for which Theorem 1 guarantees the existence of a solution? Show your work. This argument is delicate. Be careful, and be clear. If you are aware of a gap in your reasoning, then point it out. A good scientist doesn't fudge, or knowingly present a fake argument.