1. (a) Define what is meant by a fixed point of a function.
   (b) Let $f : [a, b] \to [a, b]$ be a continuous function. Must f have a fixed point? Explain.
   (c) Define what is meant by a contraction.
   (d) Give an example of functions $f, g : [0, 1] \to [0, 1]$ such that f is a contraction, and g is not. Justify your answer.
   (e) Prove or disprove: Every contraction is continuous.
   (f) State and prove the contraction mapping theorem.
2. You should be able to give a precise definition of linear and quadratic convergence.
3. What is the asymptotic rate at which a functional iteration converges to a fixed point? What is the asymptotic error constant? Justify your answer.
4. Explain how Newton's method for finding a solution to $f(x) = 0$ can be viewed as a fixed point iteration.
5. Near a simple root of f, Newton's method converges quadratically. Why does this not contradict the fact that the fixed point iteration method converges asymptotically at a linear rate?
6. (a) Know the definition of a vector norm (p. 186-7), and examples of vector norms.
   (b) Show that a norm defined on $\mathbb{R}^n$ must involve all the components of a vector in some way.
   (c) Know the definition of subordinate matrix norms. What is a practical way to calculate the matrix 1-norm and matrix ∞-norm? (see p.193, problem 11)
   (d) p. 193, problems 2, 7a, 16, 26, 28, 40
7. Let $A \in \mathbb{R}^{n\times n}$. Define $\rho(A)$, the spectral radius A. State the precise connection between $\rho(A)$ and $\|A\|$, where $\|\cdot\|$ is any matrix norm induced by a vector norm.
8. Give a precise statement of the Theorem on Neumann Series.
9. What is meant by a **direct method** for solving $Ax = b$? What is the standard direct method, and what is it's cost? Describe some **iterative methods** for solving $Ax = b$. When and why would you use iterative methods?
10. Given $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and an intial vector $x \in \mathbb{R}^n$, know how to compute (by hand) the first few iterates of Gauss-Jacobi and Gauss-Seidel. See for example, p. 245 problem 13a,b.
11. Let A be a non-singular matrix. Denote the strict lower triangular part of A by L, the diagonal part of A by D and the strict upper triangular part of A by U. Note A, L, D, and U are all matrices.

    (a) Show how the Gauss-Jacobi iteration scheme for solving $Ax = b$ can be viewed as a fixed point iteration. What is a necessary and sufficient condition for its convergence?
    (b) Do the same for the Gauss-Seidel method.

(c) What property of the matrix A guarantees convergence of the Gauss-Jacobi and Gauss Seidel iterative methods?

12. Let A be an $n \times n$ matrix, $x \in \mathbb{R}^n, b \in \mathbb{R}^n$.

    (a) Under what conditions on A is there a connection between the function $q(x) = \langle x, Ax \rangle - \langle x, b \rangle$ and the problem of finding a solution to $Ax = b$? Explain the connection, giving complete mathematical justification.

    (b) p. 244, problem 1, 2.

    (c) Explain the steepest descent method for solving $Ax = b$.