MATH 637 **Homework 3** Spring 2012
Due Date: Jan 24, 2012.

Notation: TB=Trefethen & Bau. HJ=Horn & Johnson handout.

Always include justification.

1. HJ, p.261, Exercise at the top of the page, reproduced here for your convenience: Let $D = \text{diag}(d_1, d_2 \dots d_n)$, and define

$$(x, y) \equiv y^* D x \qquad \forall x, y \in \mathbb{C}^n$$

   (a) Which of the axioms for an inner product does $(\cdot\,, \cdot)$ satisfy?

   (b) Under what conditions on D is $(\cdot\,, \cdot)$ an inner product on $\mathbb{C}^n$?

   Include a definition of inner product at the start of your solution, that requires linearity in the first argument (as does HJ).

2. HJ, p.263, Problem 4: Let V be an arbirary real or complex vector space. Show that any vector norm on V derived from an inner product, that is

$$||x|| := \sqrt{\langle x, x \rangle}$$

   must satisfy the *parallelogram identity*

$$\frac{1}{2}\left(||x+y||^2 + ||x-y||^2\right) = ||x||^2 + ||y||^2$$

   Why is this inequality so named?

3. HJ, p.263, Problem 5: Consider the function

$$||x||_\infty := \max_{1 \le i \le n} |x_i|$$

   defined on $\mathbb{C}^n$ Show that

   (a) $||x||_\infty$ is a vector norm on $\mathbb{C}^n$.

   (b) $||x||_\infty$ does not arise from any inner product on $\mathbb{C}^n$.

4. HJ, p.263, Problem 6: Let V be an arbitrary real or complex vector space, and let $||\cdot||$ be a vector norm on V derived from an inner product $\langle \cdot\,, \cdot \rangle$.

   (a) Show that

$$\text{Re}\langle x, y \rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

   This is known as the *polarization identity.*

   (b) Show that

$$\text{Re}\langle x, y \rangle = \frac{1}{2}\left(||x+y||^2 - ||x||^2 - ||y||^2\right)$$

5. HJ, p.263, Problem 7: For $x \in \mathbb{C}^n$, let $|||x||_1 := |x_1| + |x_2| + \cdots + |x_n|$.

   (a) Show that this defines a norm on $\mathbb{C}^n$. This is called the ℓ_1 norm.

   (b) Show that the ℓ_1 norm does not obey the polarization identity. Hence the ℓ_1 norm is not derived from any inner product on $\mathbb{C}^n$.

6. HJ, p.265, 2nd Exercise: A norm $||\cdot||$ is said to be *unitarily invariant* if $||Ux|| = ||x||$ for all $x \in \mathbb{C}^n$, and all unitary matrices $U \in \mathbb{C}^{n\times n}$. Show that the Euclidean norm $||\cdot||_2$ is unitarily invariant.

7. Vector norms are related by various inequalities. For each of the following prove the inequality holds, and give an example of a nonzero vector for which equality is achieved. Here $x \in \mathbb{C}^n$.

   (a) $||x||_\infty \leq ||x||_2$

   (b) $||x||_2 \leq \sqrt{n}\,||x||_\infty$