Math 637 **Homework 4** N. Mackey

Turn in: 2, 4, 6, 7, 9, 11. **Due: 2 Feb 2012.**

Notation: TB=Trefethen & Bau.

1. Prove $\|A\|_F = \sqrt{\text{trace}(A^T A)}$, for all $A \in \mathbb{R}^{m\times n}$.
2. Define
$$\|A\|_{\max} = \max_{1\le i\le m,\, 1\le j\le n} |a_{ij}| \quad \text{for all} \quad A \in \mathbb{R}^{m\times n}.$$
Does $\|\cdot\|_{\max}$ yield a matrix norm? Prove or disprove your claim.
3. For $A \in \mathbb{C}^{m\times n}$, prove that
$$\|A\|_\infty = \max_i \sum_j |a_{ij}| \, .$$
(Hence the matrix ∞-norm is also called the *max-row-sum norm*).
4. Let I denote the $n \times n$ identity matrix. Prove that
   (a) $\|I\| = 1$ when $\|\cdot\|$ is any matrix norm induced by a vector norm.
   (b) $\|I\|_F = \sqrt{n}$.
5. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots \mathbf{a}_n] \in \mathbb{C}^{m\times n}$ be partitioned by columns. Prove that
$$\|A\|_F^2 = \|\mathbf{a}_1\|_2^2 \ + \ \|\mathbf{a}_2\|_2^2 \ \ldots \ \|\mathbf{a}_n\|_2^2 \, .$$
6. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Use problem 5 to prove that
   (a) $\|AB\|_F \le \|A\|_2 \, \|B\|_F$,
   (b) $\|AB\|_F \le \|A\|_F \|B\|_2$.
7. Let $A \in \mathbb{C}^{n\times n}$.
   (a) $\|A\|_2 \le \|A\|_F$. (Hint: Look at $||Ax||_2$ entrywise, and find a way to use Cauchy Schwartz).
   (b) $\|A\|_F \le \sqrt{n}\, \|A\|_2$. (Hint: Use Problem 6.)

   Thus we have
$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F \, .$$
8. Let $A = \text{diag}[d_1, d_2, \ldots, d_n]$. Prove that $\|A\|_2 = \max_i |d_i|$.

9. TB, p. 24, problem 3.2: Let $||\cdot||$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq ||A||$, where $\rho(A)$ is the *spectral radius* of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue of A.

10. TB, p. 24, problem 3.4: Let A be an $m \times n$ matrix and let B be a submatrix of A, that is, a $r \times s$ matrix, with $r \leq m$, $s \leq n$, obtained by selecting certain rows and columns of A.

    (a) Explain how B can be obtained from A by multiplying A by certain row and column "deletion matrices" as was done in a previous homework exercise (TB, Exercise 1.1).

    (b) Using this product, show that $||B||_p \leq ||A||_p$ for any p with $1 \leq p \leq \infty$.

11. TB, p. 24, problem 3.5: Example 3.6 on p. 22 of TB shows that if $E = uv^*$ is an outer product, then $||E||_2 = ||u||_2||v||_2$. Is the same true of the Frobenius norm, i.e., $||E||_F = ||u||_F||v||_F$? Prove it or give a counterexample.