Series RLC Circuit Response as R is Varied

[Based on Example 8.7 of *Fundamentals of Circuit Analysis* by Alexander and Sadiku, 6th edition]

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Corrections to damon.miller@wmich.edu

Notice how Mathematica avoids round-off error by not using finite-precision representations of numbers

```
In[1]:= Sqrt[2]
```

Out[1]= $\sqrt{2}$

Numerical representation to 300 digits

```
In[2]:= N[Sqrt[2], 300]

Out[2]= 1.414213562373095048801688724209698078569671875376948073176679737990732478462107038850387534327641572735013846230912297024924836055850737212644121497099935831413222665927505592755799950501152782060571470109559971605970274534596862014728517418640889198609552329230484308714321450839762603627995251407990
```

CASE 1 (overdamped)

```
In[3]:= R = 5; L = 1; c = 1 / 4;

In[4]:= charpoly = s^2 + R / L s + 1 / (L c)
```

Out[4]= $4 + 5\,s + s^2$

```
In[5]:= w0 = Sqrt[1 / (L c)]
```

Out[5]= 2

```
In[6]:= α = R / ( 2 L)
```

Out[6]= $\frac{5}{2}$

```
In[7]:= Solve[charpoly == 0, s]

Out[7]= {{s → -4}, {s → -1}}

In[8]:= {I0 = 24 / (R + 1), vC0 = 24 / (R + 1), dvC0 = I0 / c}

Out[8]= {4, 4, 16}

In[9]:= Clear[vC];

In[10]:= soln = DSolve[{vC''[t] + R / L vC'[t] + vC[t] / (L c) == 24 / (L c),
             vC[0] == vC0, vC'[0] == dvC0}, vC, t];
```

In[11]:=
```
vC[t_] = vC[t] /. soln[[1]]
```

Out[11]= $\frac{4}{3} e^{-4t} \left(1 - 16 e^{3t} + 18 e^{4t}\right)$

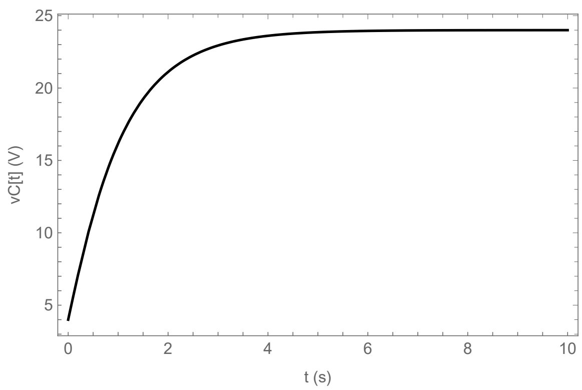
matches (8.7.5) of text

In[12]:=
```
Expand[vC[t]]
```

Out[12]= $24 + \frac{4 e^{-4t}}{3} - \frac{64 e^{-t}}{3}$

In[13]:=
```
Plot[vC[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
 FrameLabel → {"t (s)", "vC[t] (V)"}]
```

Out[13]=



In[14]:=
```
i[t_] = c D[vC[t], t];
```

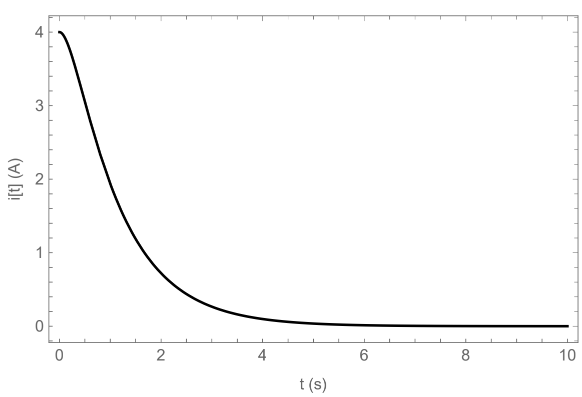
matches (8.7.6) of text

In[15]:=
```
Expand[i[t]]
```

Out[15]= $-\frac{4}{3} e^{-4t} + \frac{16 e^{-t}}{3}$

In[16]:=
```
Plot[i[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
 FrameLabel → {"t (s)", "i[t] (A)"}]
```

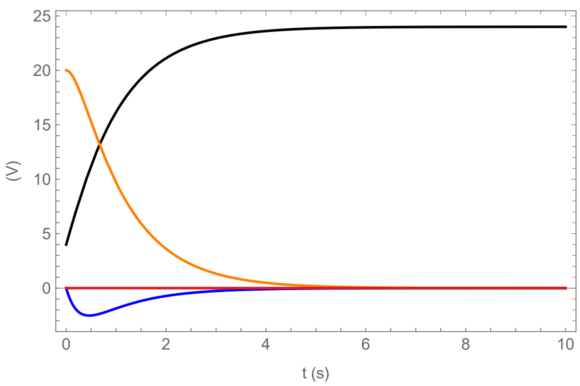
Out[16]=



In[17]:=
```
vL[t_] = L D[i[t], t];
```

In[18]:=
```
Expand[vL[t]]
```
Out[18]= $\frac{16\,e^{-4t}}{3} - \frac{16\,e^{-t}}{3}$

In[19]:=
```
vR[t] = R i[t];
```
In[20]:=
```
Expand[vR[t]]
```
Out[20]= $-\frac{20}{3}e^{-4t} + \frac{80\,e^{-t}}{3}$

In[21]:=
```
Plot[{vC[t], vL[t], vR[t], vC[t] + vL[t] + vR[t] - 24}, {t, 0, 10},
 PlotRange → All, Frame → True, PlotStyle → {Black, Blue, Orange, Red},
 FrameLabel → {"t  (s)", "(V)"}]
```
Out[21]=


CASE 2 (critically damped)

In[22]:=
```
R = 4;
```
In[23]:=
```
charpoly = s^2 + R / L s + 1 / (L c)
```
Out[23]= $4 + 4s + s^2$

In[24]:=
```
w0 = Sqrt[1 / (L c)]
```
Out[24]= 2

In[25]:=
```
α = R / (2 L)
```
Out[25]= 2

In[26]:=
```
Solve[charpoly == 0, s]
```
Out[26]= {{s → -2}, {s → -2}}

In[27]:=
```
Clear[vC];
```
In[28]:=
```
{I0 = 24 / (R + 1), vC0 = 24 / (R + 1), dvC0 = I0 / c}
```
Out[28]= $\left\{\frac{24}{5}, \frac{24}{5}, \frac{96}{5}\right\}$

In[29]:=
```
soln = DSolve[{vC''[t] + R / L vC'[t] + vC[t] / (L c) == 24 / (L c),
    vC[0] == vC0, vC'[0] == dvC0}, vC, t];
```

In[30]:= `vC[t_] = vC[t] /. soln[[1]]`

Out[30]= $\frac{24}{5} e^{-2t} \left(-4 + 5 e^{2t} - 4t\right)$

matches (8.7.11) of text

In[31]:= `Expand[vC[t]]`

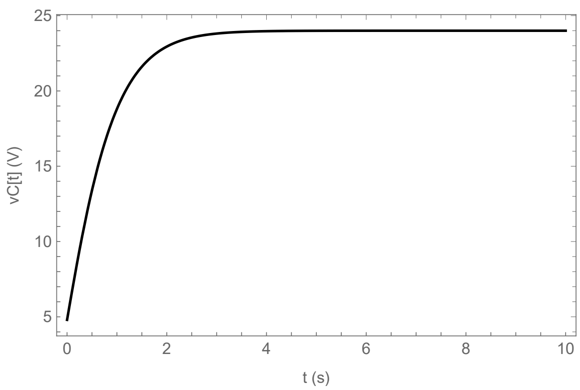
Out[31]= $24 - \frac{96 e^{-2t}}{5} - \frac{96}{5} e^{-2t} t$

In[32]:= `N[96 / 5]`

Out[32]= 19.2

In[33]:= `Plot[vC[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black}, FrameLabel → {"t (s)", "vC[t] (V)"}]`

Out[33]=



In[34]:= `i[t_] = c D[vC[t], t];`

matches (8.7.12) of text

In[35]:= `Expand[i[t]]`

Out[35]= $\frac{24 e^{-2t}}{5} + \frac{48}{5} e^{-2t} t$

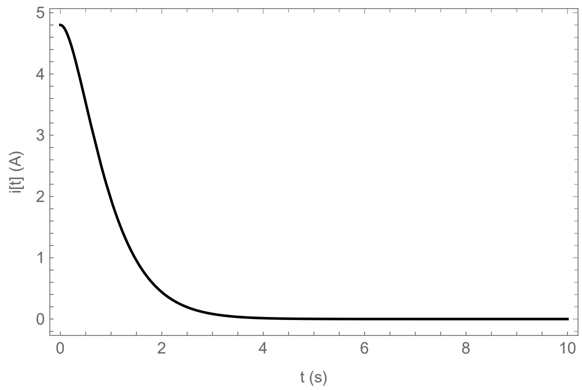
In[36]:= `N[48 / 5]`

Out[36]= 9.6

In[37]:= `N[24 / 5]`

Out[37]= 4.8

In[38]:= `Plot[i[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black}, FrameLabel → {"t (s)", "i[t] (A)"}]`

Out[38]=


In[39]:= `vL[t_] = L D[i[t], t];`

In[40]:= `Expand[vL[t]]`

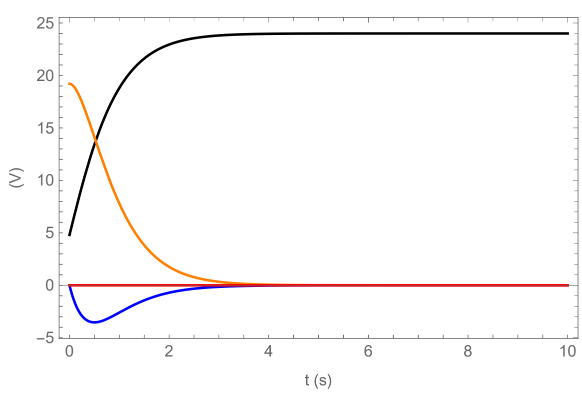
Out[40]= $-\frac{96}{5} e^{-2t} t$

In[41]:= `vR[t] = R i[t];`

In[42]:= `Expand[vR[t]]`

Out[42]= $\frac{96 e^{-2t}}{5} + \frac{192}{5} e^{-2t} t$

In[43]:= `Plot[{vC[t], vL[t], vR[t], vC[t] + vL[t] + vR[t] - 24}, {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black, Blue, Orange, Red}, FrameLabel → {"t (s)", "(V)"}]`

Out[43]=


CASE 3 (underdamped)

In[44]:= `R = 1;`

In[45]:= `charpoly = s^2 + R / L s + 1 / (L c)`

Out[45]= $4 + s + s^2$

In[46]:= `w0 = Sqrt[1 / (L c)]`

Out[46]= 2

In[47]:= α = R / (2 L)

Out[47]= $\frac{1}{2}$

In[48]:= Solve[charpoly == 0, s]

Out[48]= $\left\{\left\{s \to \frac{1}{2}\left(-1 - i\sqrt{15}\right)\right\}, \left\{s \to \frac{1}{2}\left(-1 + i\sqrt{15}\right)\right\}\right\}$

In[49]:= N[%]

Out[49]= {{s → -0.5 - 1.93649 ⅈ}, {s → -0.5 + 1.93649 ⅈ}}

In[50]:= Clear[vC];

In[51]:= {I0 = 24 / (R + 1), vC0 = 24 / (R + 1), dvC0 = I0 / c}

Out[51]= {12, 12, 48}

```
In[52]:= soln = DSolve[{vC''[t] + R / L vC'[t] + vC[t] / (L c) == 24 / (L c),
             vC[0] == vC0, vC'[0] == dvC0}, vC, t];
```

In[53]:= vC[t_] = vC[t] /. soln[[1]]

Out[53]= $\frac{4}{5} e^{-t/2} \left(30 e^{t/2} - 15 \cos\left[\frac{\sqrt{15}\, t}{2}\right] + 7\sqrt{15} \sin\left[\frac{\sqrt{15}\, t}{2}\right]\right)$

matches (8.7.16) of text

In[54]:= Expand[vC[t]]

Out[54]= $24 - 12 e^{-t/2} \cos\left[\frac{\sqrt{15}\, t}{2}\right] + 28\sqrt{\frac{3}{5}}\, e^{-t/2} \sin\left[\frac{\sqrt{15}\, t}{2}\right]$

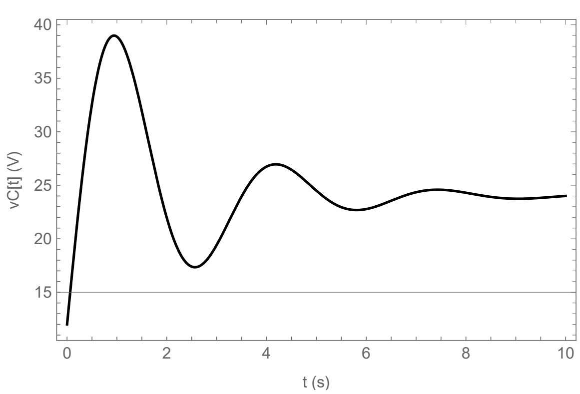
In[55]:= N[Sqrt[15] / 2]

Out[55]= 1.93649

In[56]:= N[28 Sqrt[3 / 5]]

Out[56]= 21.6887

```
In[57]:= Plot[vC[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
          FrameLabel → {"t (s)", "vC[t] (V)"}]
```

Out[57]= 

In[58]:= i[t_] = c D[vC[t], t];

matches (8.7.18) of the text

In[59]:=
```
Expand[i[t]]
```

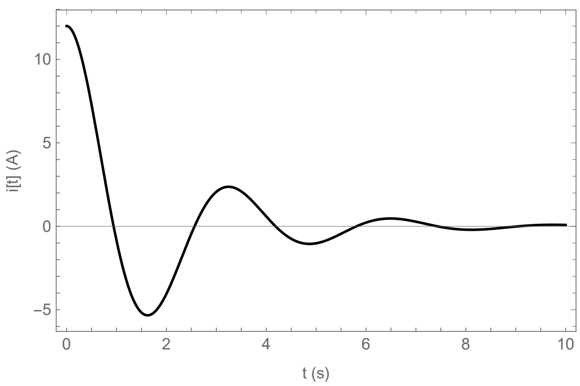
Out[59]= $12\, e^{-t/2} \,\text{Cos}\left[\frac{\sqrt{15}\, t}{2}\right] + 4\sqrt{\frac{3}{5}}\, e^{-t/2}\, \text{Sin}\left[\frac{\sqrt{15}\, t}{2}\right]$

In[60]:=
```
N[4 Sqrt[3 / 5]]
```

Out[60]= 3.09839

In[61]:=
```
Plot[i[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
  FrameLabel → {"t  (s)", "i[t]  (A)"}]
```

Out[61]=



In[62]:=
```
vL[t_] = L D[i[t], t];
```

In[63]:=
```
Expand[vL[t]]
```

Out[63]= $-32\sqrt{\frac{3}{5}}\, e^{-t/2}\, \text{Sin}\left[\frac{\sqrt{15}\, t}{2}\right]$

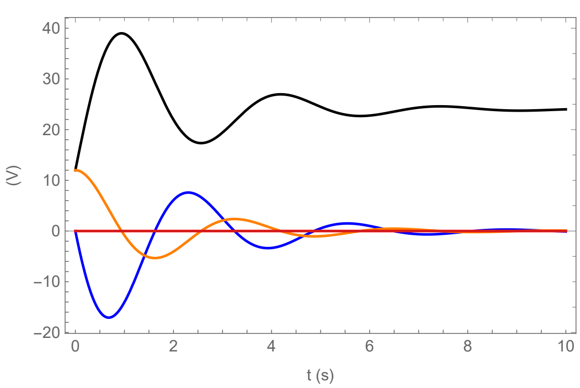
In[64]:=
```
vR[t] = R i[t];
```

In[65]:=
```
Expand[vR[t]]
```

Out[65]= $12\, e^{-t/2} \,\text{Cos}\left[\frac{\sqrt{15}\, t}{2}\right] + 4\sqrt{\frac{3}{5}}\, e^{-t/2}\, \text{Sin}\left[\frac{\sqrt{15}\, t}{2}\right]$

In[66]:=
```
Plot[{vC[t], vL[t], vR[t], vC[t] + vL[t] + vR[t] - 24}, {t, 0, 10},
  PlotRange → All, Frame → True, PlotStyle → {Black, Blue, Orange, Red},
  FrameLabel → {"t  (s)", "(V)"}]
```

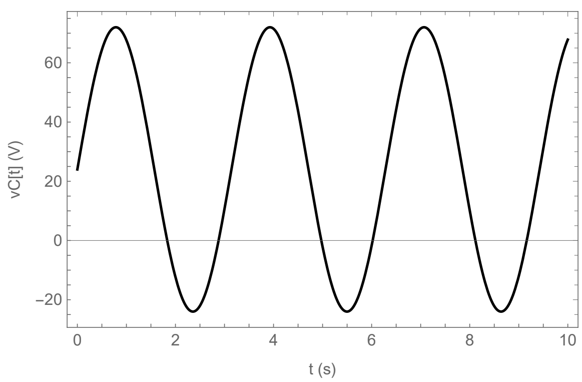
Out[66]=



CASE 4 (oscillatory)

```
In[67]:= R = 0;

In[68]:= charpoly = s^2 + R / L s + 1 / (L c)
Out[68]= 4 + s^2

In[69]:= w0 = Sqrt[1 / (L c)]
Out[69]= 2

In[70]:= α = R / ( 2 L)
Out[70]= 0

In[71]:= Solve[charpoly == 0, s]
Out[71]= {{s → -2 ⅈ}, {s → 2 ⅈ}}

In[72]:= Clear[vC];

In[73]:= {I0 = 24 / (R + 1), vC0 = 24 / (R + 1), dvC0 = I0 / c}
Out[73]= {24, 24, 96}

In[74]:= soln = DSolve[{vC''[t] + R / L vC'[t] + vC[t] / (L c) == 24 / (L c),
            vC[0] == vC0, vC'[0] == dvC0}, vC, t];

In[75]:= vC[t_] = vC[t] /. soln[[1]]
Out[75]= 24 (1 + 2 Sin[2 t])

In[76]:= Plot[vC[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
          FrameLabel → {"t (s)", "vC[t] (V)"}]
```

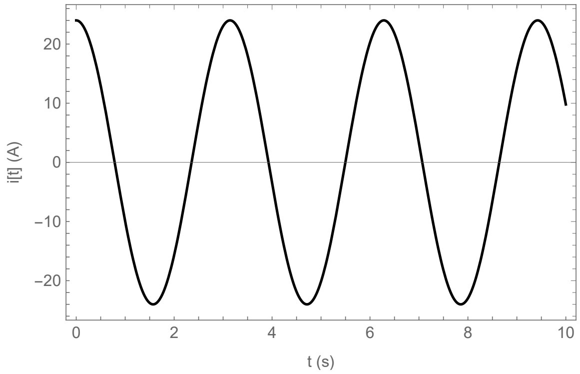
Out[76]=




```
In[77]:= i[t_] = c D[vC[t], t];

In[78]:= Expand[i[t]]
Out[78]= 24 Cos[2 t]
```

In[79]:=
```
Plot[i[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
 FrameLabel → {"t (s)", "i[t] (A)"}]
```

Out[79]=



In[80]:=
```
vL[t_] = L D[i[t], t];
```

In[81]:=
```
Expand[vL[t]]
```

Out[81]= −48 Sin[2 t]

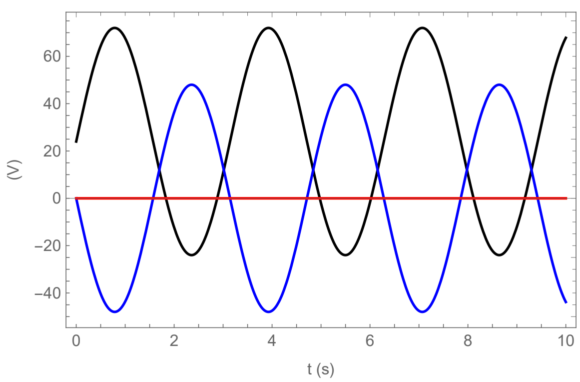
In[82]:=
```
vR[t] = R i[t];
```

In[83]:=
```
Expand[vR[t]]
```

Out[83]= 0

In[84]:=
```
Plot[{vC[t], vL[t], vR[t], vC[t] + vL[t] + vR[t] - 24}, {t, 0, 10},
 PlotRange → All, Frame → True, PlotStyle → {Black, Blue, Orange, Red},
 FrameLabel → {"t (s)", "(V)"}]
```

Out[84]=



CASE 5 (unstable)

In[85]:=
```
R = -1 + 1 / 100;
```

In[86]:=
```
charpoly = s^2 + R / L s + 1 / (L c)
```

Out[86]= $4 - \frac{99\,s}{100} + s^2$

In[87]:=
```
w0 = Sqrt[1 / (L c)]
```

Out[87]= 2

In[88]:= α = R / (2 L)

Out[88]= $-\frac{99}{200}$

In[89]:= Solve[charpoly == 0, s]

Out[89]= $\left\{\left\{s \to \frac{1}{200}\left(99 - i\sqrt{150199}\right)\right\}, \left\{s \to \frac{1}{200}\left(99 + i\sqrt{150199}\right)\right\}\right\}$

In[90]:= N[%]

Out[90]= {{s → 0.495 - 1.93778 i}, {s → 0.495 + 1.93778 i}}

In[91]:= Clear[vC]

In[92]:= {I0 = 24 / (R + 1), vC0 = 24 / (R + 1), dvC0 = I0 / c}

Out[92]= {2400, 2400, 9600}

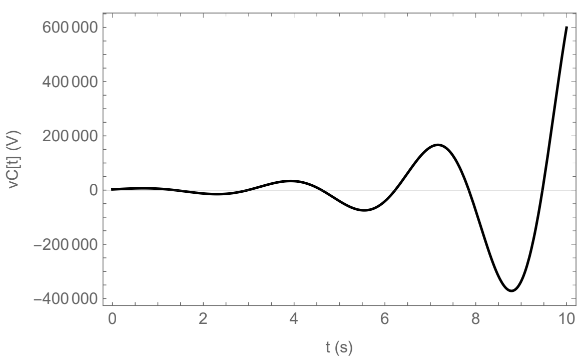
In[93]:= soln = DSolve[{vC''[t] + R / L vC'[t] + vC[t] / (L c) == 24 / (L c),
vC[0] == vC0, vC'[0] == dvC0}, vC, t];

In[94]:= vC[t_] = vC[t] /. soln[[1]]

Out[94]= $\frac{1}{150199} 24 \left(150199 + 14869701 e^{99 t/200} \operatorname{Cos}\left[\frac{\sqrt{150199}\, t}{200}\right] + 70199 \sqrt{150199}\, e^{99 t/200} \operatorname{Sin}\left[\frac{\sqrt{150199}\, t}{200}\right]\right)$

In[95]:= Plot[vC[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black},
FrameLabel → {"t (s)", "vC[t] (V)"}]

Out[95]=



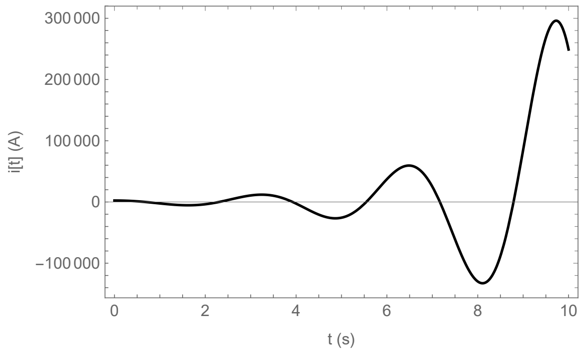
In[96]:= i[t_] = c D[vC[t], t];

In[97]:= Expand[i[t]]

Out[97]= $2400\, e^{99 t/200} \operatorname{Cos}\left[\frac{\sqrt{150199}\, t}{200}\right] - \frac{237600\, e^{99 t/200} \operatorname{Sin}\left[\frac{\sqrt{150199}\, t}{200}\right]}{\sqrt{150199}}$

In[98]:= `Plot[i[t], {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black}, FrameLabel → {"t (s)", "i[t] (A)"}]`

Out[98]=



In[99]:= `vL[t_] = L D[i[t], t];`

In[100]:= `Expand[vL[t]]`

Out[100]= $$-\frac{1920000\, e^{99\, t/200}\, \mathrm{Sin}\left[\frac{\sqrt{150199}\, t}{200}\right]}{\sqrt{150199}}$$

In[101]:= `vR[t] = R i[t];`

In[102]:= `Expand[vR[t]]`

Out[102]= $$-2376\, e^{99\, t/200}\, \mathrm{Cos}\left[\frac{\sqrt{150199}\, t}{200}\right] + \frac{235224\, e^{99\, t/200}\, \mathrm{Sin}\left[\frac{\sqrt{150199}\, t}{200}\right]}{\sqrt{150199}}$$

In[103]:= `Plot[{vC[t], vL[t], vR[t], vC[t] + vL[t] + vR[t] - 24}, {t, 0, 10}, PlotRange → All, Frame → True, PlotStyle → {Black, Blue, Orange, Red}, FrameLabel → {"t (s)", "(V)"}]`

Out[103]=