1 Parametrizing characters

1.1 Parametrizing characters of real tori

Let $T$ be a torus over $\mathbb{C}$, with Cartan involution $\theta$, and $T(\mathbb{R})$ the corresponding real torus, i.e., the fixed points of the Galois action obtained by composing $\theta$ with inversion and complex conjugation. We parametrize continuous characters $\pi$ of $T(\mathbb{R})$ by pairs $(\lambda, \kappa)$, where $\lambda = d\pi \in \mathfrak{t}^*$, and $\kappa \in X^*(T)$, and satisfying

$$\lambda - ^\vee \theta(\lambda) = \kappa - ^\vee \theta(\kappa). \tag{1}$$

Two pairs $(\lambda, \kappa)$ and $(\lambda', \kappa')$ parametrize the same character if and only if

$$\lambda = \lambda' \text{ and } \kappa = \kappa' + (\eta + ^\vee \theta(\eta)) \text{ for some } \eta \in X^*(T). \tag{2}$$

The parameter $\kappa$ may be obtained from $\pi$ by extending the restriction $\pi|_{T(\mathbb{R})}$ of $\pi$ to the compact part of the torus to an algebraic character of the complex torus $T$. This extension is in general not unique. (This ambiguity is reflected in condition (2).)

Example 1 Let $T = \mathbb{C}^\times \times \mathbb{C}^\times$. Then $\mathfrak{t}^* \simeq \mathbb{C} \oplus \mathbb{C}$, and $X^*(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Consider $\theta$ given by $\theta(t_1, t_2) = (t_1, t_2^{-1})$, so that $T(\mathbb{R}) = S^1 \times \mathbb{R}^\times$. If $\lambda = (z_1, z_2)$ and $\kappa = (k_1, k_2)$, then condition (1) says that $(2z_1, 0) = (2k_1, 0)$, or $z_1 = k_1$. If $\eta = (n_1, n_2) \in X^*(T)$, then $\eta + ^\vee \theta(\eta) = (0, 2n_2)$, so condition (2) says that only the parity of $k_2$ matters for $\pi$. Given $(\lambda, \kappa)$ as specified, the character $\pi$ is then given by

$$\pi(e^{i\varphi}, r) = e^{i(k_1 + n_1)\varphi} |r|^{z_2} \text{sgn}(r)^{k_2} \text{ for } \varphi \in \mathbb{R} \text{ and } r \in \mathbb{R}^\times. \tag{3}$$

Example 2 Again let $T = \mathbb{C}^\times \times \mathbb{C}^\times$, but this time let $\theta$ be given by $\theta(t_1, t_2) = (t_2, t_1)$, so that $T(\mathbb{R}) = \{ (t, t^{-1}) : t \in \mathbb{C}^\times \} \simeq \mathbb{C}^\times$. Now $^\vee \theta(\lambda) = (-z_2, -z_1)$, so condition (1) says $z_1 + z_2 = k_1 + k_2$, and (2) implies that the character depends on the sum $k_1 + k_2$, rather than the individual values $k_1, k_2$. The character $\pi$ is now given by

$$\pi(re^{i\varphi}, r^{-1}e^{i\varphi}) = r^{z_1 - z_2} e^{i(k_1 + k_2)\varphi} = r^{z_1 - z_2} e^{i(z_1 + z_2)\varphi}. \tag{4}$$

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We are also going to need to specify characters of covers of our real torus, in particular the sign character, and \( C \).

By definition, \( z \) is a complex number and \( \exp z \) is the fundamental group of \( \mathbb{R} \) and the group of characters of \( \pi_1(T)(\mathbb{R}) \) corresponds under this isomorphism to the \( \vee \theta \)-fixed elements of \( \vee T \).

Example 3 We still let \( T = \mathbb{C}^\times \times \mathbb{C}^\times \), and change the Cartan involution once again, to \( \theta(t_1, t_2) = (t_2^{-1}, t_1^{-1}) \). Now \( \mathcal{T}(\mathbb{R}) = \{ (t, t) : t \in \mathbb{C}^\times \} \) is \( \mathbb{C}^\times \), and \( \vee \theta(\lambda) = (z_2, z_1) \). This time, condition (1) says \( z_1 - z_2 = k_1 - k_2 \), and the character will depend on the difference \( k_1 - k_2 \), rather than the individual values. The character \( \pi \) is given by

\[
\pi(re^{i\varphi}, re^{-i\varphi}) = e^{z_1 + z_2 e^{i(k_1-k_2)\varphi}} = e^{z_1 + z_2 e^{i(z_1-z_2)\varphi}}.
\]

1.2 Covers of real tori

We are also going to need to specify characters of covers of our real torus, in particular the \( \rho \) double cover. These covers are obtained as quotients of a canonical covering \( T(\mathbb{R}) \) (see §5 of [3] for details).

By definition,

\[
\mathcal{T}(\mathbb{R}) = \{ X \in t : \exp(X) \in T(\mathbb{R}) \}/(1 + \theta)\pi_1(T).
\]

Here \( \pi_1(T) \) is the fundamental group of \( T \), \( \pi_1(T) = 2\pi iX_1(T) \). The kernel of the covering map \( \mathcal{T}(\mathbb{R}) \rightarrow T(\mathbb{R}) \) is \( \pi_1(T)(\mathbb{R}) = \pi_1(T)/(1 + \theta)\pi_1(T) \). The group of characters of the abelian group \( \pi_1(T) \) is naturally isomorphic with \( \vee T \), and the group of characters of \( \pi_1(T)(\mathbb{R}) \) corresponds under this isomorphism to the \( \vee \theta \)-fixed elements of \( \vee T \).

Example 4 If \( T = \mathbb{C}^\times \) is a one-dimensional torus, then \( \pi_1(T) = 2\pi i\mathbb{Z} \cong \mathbb{Z} \), with character group \( \mathbb{C}^\times = \vee T \). Here \( z \in \vee T \) corresponds to the character \( \chi_z \) given by \( \chi_z(k) = z^k \).

If \( \theta = 1 \) so that \( T(\mathbb{R}) = S^1 \), then \( \mathcal{T}(\mathbb{R}) = i\mathbb{R}/4\pi i\mathbb{Z} \) (a two-fold cover), and \( \pi_1(T)(\mathbb{R}) = 2\pi i\mathbb{Z}/4\pi i\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \). In this case, \( (\vee T)^{\vee \theta} = \{ \pm 1 \} \), and the correspondence is easy: \( \chi_{-1} \) is the sign character, and \( \chi_1 \) the trivial one.

Now let \( \theta \) be inversion, and \( T(\mathbb{R}) = \mathbb{R}^\times \). Then \( \mathcal{T}(\mathbb{R}) = \mathbb{R} + \pi i\mathbb{Z} \), a \( \mathbb{Z} \)-fold cover, and \( \pi_1(T)(\mathbb{R}) = 2\pi i\mathbb{Z} \cong \mathbb{Z} \). The dual of this group is isomorphic to \( \mathbb{C}^\times \) which coincides with \( (\vee T)^{\vee \theta} \), since \( \vee \theta \) is trivial.

We partition the characters of \( \mathcal{T}(\mathbb{R}) \) according to their restriction to \( \pi_1(T)(\mathbb{R}) \); for \( z \in (\vee T)^{\vee \theta} \), we denote by \( \Pi^T(\mathbb{R}) \) the set of characters of the canonical covering (or projective characters of our real torus) whose restriction to \( \pi_1(T)(\mathbb{R}) \) is \( \chi_z \). We call the quotient of \( \mathcal{T}(\mathbb{R}) \) by the kernel of \( \chi_z \), the cover of \( T(\mathbb{R}) \) determined by \( z \). If \( z \) has order 2 then this cover is a two-fold cover.

The set \( \Pi^T(\mathbb{R}) \) may be parametrized in a fashion that is quite analogous to the parametrization of the characters of the real torus in Section 1.1: let \( \xi \in \mathfrak{t}^\times \) be such that \( z = \exp(2\pi i\xi) \). Then an element \( \pi \in \Pi^T(\mathbb{R}) \) may be specified by a pair \( (\lambda, k) \) with \( \lambda = d\pi \in \mathfrak{t}^\times \) and \( k \in \xi + X(1)(T) \), satisfying the same compatibility condition (1) as for characters of \( T(\mathbb{R}) \). Similarly, condition (2) is necessary and sufficient for two such pairs to determine the same character.

Example 5 To continue our example of one-dimensional tori, let \( T(\mathbb{R}) = S^1 \). The cover determined by \( z = 1 \) is the trivial (one-fold) cover, and we get the true characters of the torus. The cover determined by \( z = -1 \) is the canonical covering. We can choose \( \xi = \frac{1}{2} \). Then characters (i.e., genuine characters of the double cover) are given by pairs \( (k, k) \) with \( k \in \frac{1}{2} + \mathbb{Z} \). The corresponding character \( \pi \) is of course then defined by \( \pi(kx + 4\pi ik) = e^{ikx} \).

To get a double cover of \( \mathbb{R}^\times \), we'll pick \( z = -1 \) again. Since \( \chi_{-1}(2\pi ik) = (-1)^k \), the kernel of \( \chi_{-1} \) is \( 4\pi i\mathbb{Z} \subset 2\pi i\mathbb{Z} = \pi_1(T)(\mathbb{R}) \), so the cover determined by \( -1 \) is \( \mathbb{R} + \pi i\mathbb{Z}/4\pi i\mathbb{Z} \cong \mathbb{R}^\times_1 \times \mathbb{Z}/4\mathbb{Z} \).

Elements of \( \Pi^{-1}(\mathbb{R}) \) are given by pairs \( (\lambda, k) \) with \( \lambda \) a complex number and \( k \in \frac{1}{2} + \mathbb{Z} \). The corresponding character \( \pi \) is then given by \( \pi(r, x) = r^{\lambda} e^{\pi i x k} \) for \( r \in \mathbb{R}^\times_1, x = 0, 1, 2, 3 \).
1.3 Parametrizing admissible homomorphisms of the Weil Group

Let $T$ be a complex torus and $\theta$ a Cartan involution specifying a real torus $T(\mathbb{R})$. Let $\check{T}$ be the $L$-group of $T(\mathbb{R})$; i.e., $\check{T}$ is the group generated by $\check{T}$ and an element $\check{\delta} \in \check{T} - \check{T}$ such that

$$\left(\check{\delta}\right)^2 = 1, \quad \text{and} \quad \check{T}t\check{T}^-1 = \check{T}$$

for $t \in \check{T}$. (7)

Recall that the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ is the group generated by $\mathbb{C}^\times$ and an element $j$ satisfying $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$. We want to parametrize admissible homomorphisms from the Weil group into $\check{T}$, up to conjugacy by $\check{T}$ (on the image). These are continuous homomorphisms $\phi$ such that $\phi(\mathbb{C}^\times) \subset \check{T}$ and $\phi(j) \in \check{T} - \check{T}$. Such a map $\phi$ may be specified by giving a homomorphism

$$\phi_0 : \mathbb{C}^\times \rightarrow \check{T}$$

and an element

$$\phi(j) = t_\phi \check{\delta} \quad \text{with} \quad t_\phi \in \check{T}.$$ (9)

Homomorphisms as in (8) are in one-one correspondence with pairs $(\lambda, \mu)$ of elements of $t^*$ such that $\lambda - \mu \in X^*(T)$. The homomorphism corresponding to such a pair is then defined by

$$\phi_0(\exp(z)) = \exp(z\lambda + \tau \mu).$$ (10)

To specify $\phi(j)$ we choose $\tau \in t^*$ such that $\exp(2\pi i \tau) = t_\phi$. For these data to define a homomorphism we need to have $\phi(j)\phi_0(z)\phi(j)^{-1} = \phi_0(\tau)$ and $\phi(j)^2 = \phi_0(-1)$. The first condition easily translates into

$$\mu = \theta(\lambda),$$ (11)

the second gives the condition

$$(\tau + \check{\theta}(\tau)) - \frac{1}{2}(\lambda - \check{\theta}(\lambda)) \in X^*(T).$$ (12)

Consequently, $\phi$ is given by a pair $(\lambda, \tau)$ of elements of $t^*$ satisfying (12). Clearly, $\tau$ is determined only up to an element of the character lattice $X^*(T)$.

The notion of an $E$-group is a generalization of that of the $L$-group. An $E$-group of a torus $T$ as above is a group, also denoted by $\check{T}$, generated by $\check{T}$ and an element $\check{\delta}$ as above, but with $(\check{\delta})^2 = z \in \check{T}$ not required to be the identity. It is not hard to modify the above parametrization to the more general case of an $E$-group determined by $z \in \check{T}$: let $\xi \in t^*$ be such that $\exp(2\pi i \xi) = z$. Then an admissible homomorphism may be given by a pair $(\lambda, \tau)$ of elements of $t^*$ satisfying

$$(\tau + \check{\theta}(\tau)) - \frac{1}{2}(\lambda - \check{\theta}(\lambda)) \in \xi + X^*(T).$$ (13)

If $z \in (\check{T})^\theta$, then there is a canonical one-one correspondence between the set $\Pi^\theta(T(\mathbb{R}))$ of projective characters of the real torus associated to $z$ as in Section 1.2 and the set of $\check{T}$-conjugacy classes of admissible homomorphisms $\phi$ from the Weil group $W_\mathbb{R}$ into the $E$-group of $T(\mathbb{R})$ determined by $z$: if $\pi \in \Pi^\theta(T(\mathbb{R}))$ is given by the pair $(\lambda, \kappa)$ then the corresponding homomorphism $\phi$ is given by $(\lambda, -1/2\kappa)$ (this constitutes a correction of formula (4.8) of [3]). It is easy to check that the compatibility condition (13) is then satisfied, and that the inverse mapping is given by $(\lambda, \tau) \mapsto (\lambda, \kappa)$ with

$$\kappa = \frac{1}{2}(\lambda - \check{\theta}(\lambda)) - (\tau + \check{\theta}(\tau)).$$ (14)
2 Parametrizing Representations

2.1 Inducing Data

To specify a representation \( \pi \) using (real parabolic) inducing data, we write down a standard module which has our representation as an irreducible quotient, subrepresentation, or other kind of distinguished “Langlands” subquotient. Starting with a \( \theta \)-stable real torus \( T(\mathbb{R}) = T(\mathbb{R})_\theta A \) (with \( A = T(\mathbb{R})_s \), the split part of the real torus), we get a Levi subgroup \( MA \) of \( G \) with \( M = \text{Cent}_G(A) \). The data then consist of a discrete series or limit of discrete series representation \( \delta \) of \( M \), and a character \( \exp(\nu) \) of \( A \) with \( \nu \in a^* \) (usually just written \( \nu \)). The unipotent radical \( N \) of our parabolic subgroup \( P = MAN \) can then be chosen so that the induced representation

\[
X(\delta, \nu) = \text{Ind}_P^G(\delta \otimes \nu \otimes 1)
\]

has our representation \( \pi \) as an irreducible quotient or subrepresentation; by choosing our Cartan as compact as possible, we can arrange for our standard module to have a unique irreducible quotient or subrepresentation \( X(\delta, \nu) \simeq \pi \). This is the idea of the final (as opposed to regular) limit characters, \( L \)-data, etc. We’ll mostly use only the final version, illustrating the difference with regular characters only in a few of the examples. We specify a limit of discrete series representation \( \delta \) of \( M \) by a triple \((\lambda_0, \Psi, \tau_0)\), where \( \lambda_0 \) is the Harish-Chandra parameter of \( \delta \), \( \Psi \) system of positive roots making \( \lambda_0 \) dominant, and \( \tau_0 \) a character of the center of \( M \).

In our examples, we follow Fokko’s lead and fix a complex torus \( T \) (whenever possible it will be the diagonal one) in \( G(\mathbb{C}) \), varying the Cartan involution \( \theta \) to pick out the different \( T(\mathbb{R}) \).

Example 6 \( G = \text{SL}(2, \mathbb{R}) \). The representations of \( \text{SL}(2, \mathbb{R}) \) consist of discrete series, limits of discrete series, and principal series representations. For the limits of discrete series, \( T(\mathbb{R}) \) is compact (\( \theta \) in this case is conjugation by the element \( \text{diag}(i, -i) \)), so \( M = G \), and we only specify \( \delta = (\lambda_0, \Psi) \) (\( \tau \) is determined by the other data). We write \( \lambda_0 = (k) \) for an integer \( k \), and \( \Psi = \{2e\} \) if \( k \geq 0 \) and the l.o.d.s. is holomorphic, and \( \Psi = \{-2e\} \) if \( k \leq 0 \) and the l.o.d.s. is antiholomorphic. For future reference (so that we can illustrate the different parametrizations in some specific cases), we fix a few specific representations:

- We let \( \pi_1 \) be the (holomorphic) discrete series with \( \lambda_0 = (3) \); and
- \( \pi_2 \) the antiholomorphic limit of discrete series given by \( \lambda_0 = (0) \) and \( \Psi = \{-2e\} \).

For the principal series representations, \( T(\mathbb{R}) = MA \), so \( M \cong \{\pm 1\} \) and \( A \cong \mathbb{R}_+^* \). We have \( \delta = 1 \) or sign, and \( \nu \) is given by a complex number. We choose

- \( \pi_3 \) to be the trivial representation \( \pi_3 = \overline{x}(1 \otimes 1) \).

Note that if we choose \( \delta = \text{sign} \) and \( \nu = 0 \) we violate the “final” condition; the resulting standard representation will have both limits of discrete series as direct summands.

Example 7 \( G = \text{Sp}(4, \mathbb{R}) \). We denote our four (conjugacy classes of) real tori by

\[ T_k \cong (S^1)^2 \quad (\text{“compact”, } \theta \text{ is conjugation by } \text{diag}(i, i, -i, -i)); \]

\[ T_s \cong (\mathbb{R}_+)^2 \quad (\text{“split”, } \theta \text{ is conjugation by } \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}); \]
The roots of $T_e \cong \mathbb{C}^\times$ : “complex”, $\theta$ is conjugation by \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 0 & 0 \end{pmatrix}: and

$T_m \cong S^1 \times \mathbb{R}^\times$ : “mixed”, $\theta$ is conjugation by \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.

We write elements $\text{diag}(z, w, z^{-1}, w^{-1})$ of $T$ as $(z, w)$, and similarly for elements of $t$ or its dual, and the roots of $G$ with respect to $T$ are $\{ \pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2 \}$. 

**Limit of discrete series.** If $T(\mathbb{R}) = T_k$, then $M = G$, and we get discrete series or limit of discrete series representations of $G$. The Harish-Chandra parameter will be of the form $\lambda_0 = (k, l) \in t^*$ with $k$ and $l$ integers.

- For the first representation $\pi_1$, we choose $\lambda_0 = (2, 1)$ and $\Psi = \{ 2e_1, 2e_2, e_1 - e_2, e_1 + e_2 \}$;
- $\pi_2$ is the discrete series representation with $\lambda_0 = (1, -2)$ and $\Psi = \{ 2e_1, -2e_2, e_1 - e_2, -e_1 - e_2 \}$;
- for $\pi_3$ we choose the limit of discrete series representation with $\lambda_0 = (1, 0)$ and $\Psi = \{ 2e_1, 2e_2, e_1 - e_2, e_1 + e_2 \}$;
- $\pi_4$ is the L.O.D.S. with $\lambda_0 = (1, 0)$ and $\Psi = \{ 2e_1, -2e_2, e_1 - e_2, e_1 + e_2 \}$;
- $\pi_5$ has $\lambda_0 = (1, -1)$ and $\Psi = \{ 2e_1, -2e_2, e_1 - e_2, e_1 + e_2 \}$;
- and $\pi_6$ has $\lambda_0 = (1, -1)$ and $\Psi = \{ 2e_1, -2e_2, e_1 - e_2, -e_1 - e_2 \}$.

**Principal series.** If $T(\mathbb{R}) = T_s$, then $MA = T_s \simeq (\mathbb{R}^\times)^2$, with $M = \{ \pm 1 \} \times \{ \pm 1 \}$ and $A = \mathbb{R}^2$. So a limit of discrete series consists of a character $\delta$ of $\{ \pm 1 \} \times \{ \pm 1 \}$, i.e. $1 \otimes 1, 1 \otimes \text{sign}$, etc. A character of $A$ is given by a pair $\nu = (\nu_1, \nu_2)$ of complex numbers. Then if $a = (x, y) \in \mathbb{R}^2$, $\nu(a) = e^{\nu_1 x + \nu_2 y}$. For our “final” condition we require that if $\nu_1 = 0$ then $\delta$ is trivial on the corresponding $\{ \pm 1 \}$ factor, and if $\nu_1 = \pm \nu_2$ then $\delta$ is either trivial or sign $\otimes \text{sign}$.

- We get the trivial representation $\pi_7 = \text{triv}$ by taking $\delta = \text{triv}$ and $\nu = (1, 2)$.

**Induced from complex Cartan.** If $T(\mathbb{R})$ is the complex Cartan subgroup $T_e$, then $MA \simeq GL(2, \mathbb{R})$, with $M = \text{SL}(2, \mathbb{R})$ and $A = \mathbb{R}_{+}^\times$. Limits of discrete series of $M$ are given by pairs $\lambda_0 = (k)$ and integer and $\Psi = \{ 2e \}$ if $k \geq 0$, or $\Psi = \{ -2e \}$ if $k \leq 0$. (This is a root system in $\text{SL}(2, \mathbb{R})$.) The parameter $\nu$ is specified by a complex number. Our data will be final provided that if $\nu = 0$ then $k$ is odd. (Otherwise we get a sum of limit of discrete series representations; for example, $\pi_5 \oplus \pi_6$ may be obtained by choosing $k = 2$ and $\nu = 0$.)

- We let $\pi_8$ be given by a positive integer $k$ and a generic $\nu$.

**Mixed induced series.** If $T(\mathbb{R}) = T_m$ then $MA \simeq \text{SL}(2, \mathbb{R}) \times \mathbb{R}_0^\times$, so $\delta$ is of the form $\zeta \otimes \chi$ for some limit of discrete series $\zeta$ and a character $\chi$ of $\{ \pm 1 \}$. The character $\nu$ is given by a complex number. For our data to be final we require that if $\nu = 0$ then $\chi = \text{sgn}$. (Otherwise our standard module will be a sum of limits of discrete series of $G$; for instance, if $\zeta$ is the holomorphic discrete series with parameter $(1)$, $\chi = \text{triv}$, and $\nu = 0$, then $X(\delta, \nu) \simeq \pi_3 \oplus \pi_4$.)

- Let $\pi_9$ be given by $\zeta$ the holomorphic discrete series with parameter $(2)$, $\chi = \text{triv}$, and $\nu = 1$.

- For $\pi_{10}$, take $\zeta$ to be the limit of discrete series with $k = 0$, $\Psi = \{ -2e \}$ (with lowest $K$-type $-1$), $\chi = \text{sign}$, and $\nu = 0$ (this representation has infinitesimal character 0).
2.2 Final Limit Characters

These are the parameters \( \gamma = (\Psi, \Gamma, \lambda) \) described in Definition 2.4 of [6]. We will use the notation \( (\Psi_I, \Gamma, \lambda) \) instead. In terms of the inducing data of section 2.1, \( \Psi_I \) is the positive system \( \Psi \), regarded as a system of imaginary roots of \( G \) (rather than \( M \)). The parameter \( \Gamma \) is a character of \( T(\mathbb{R}) \), namely a character such that \( \delta \) has a lowest \((M \cap K)\)-type with highest weight \( \Gamma|_T(\mathbb{R}) \), and \( \Gamma|_A = \exp(\nu) \). We specify \( \Gamma \) by a pair \((\gamma, \eta)\) as in section 1.1, i.e., with \( \gamma \in t^* \), \( \eta \in X^*(T) \), and satisfying the compatibility condition (1). Let \( t_0 \) and \( t_\pm \) be the +1 and \(-1\) eigenspaces of \( \theta \) in \( t \), i.e., the complexified Lie algebras of \( T(\mathbb{R})_c \) and \( A \), respectively. Choose \( \lambda \in t^* \) such that \( \lambda|_{t_0} = \lambda_0 \), and \( \lambda|_{t_\pm} = \nu \). (This is the infinitesimal character of the representation.) Let \( \rho_n \) and \( \rho_c \) be one half the sums of the non-compact and compact roots in \( \Psi_I \), respectively. Then \( \gamma = \lambda + \rho_n - \rho_c \). The parameter \( \eta \) must be chosen accordingly. We work out the examples given in section 2.1.

Example 8 \( G = \text{SL}(2, \mathbb{R}) \).

- For \( \pi_1 \), \( \lambda = (3) \), \( \rho_n = (1) \), and \( \rho_c = 0 \), so \( \gamma = (4) = \eta \).
- For \( \pi_2 \), \( \lambda = (0) \), \( \rho_n = (-1) \), so \( \gamma = (-1) = \eta \).
- For the trivial representation \( \pi_3 \), \( \lambda = (1) = \gamma \), and \( \eta = (0) \).

Example 9 \( G = \text{Sp}(4, \mathbb{R}) \). For limits of discrete series, we have \( \Psi_I = \Psi \), \( \lambda = \lambda_0 \), and \( \gamma = \eta \), so we only need to specify \( \gamma \).

- For \( \pi_1 \), \( \lambda = \lambda_0 = (2, 1) \), \( \rho_n = \left( \frac{3}{2}, \frac{3}{2} \right) \), \( \rho_c = \left( \frac{1}{2}, -\frac{1}{2} \right) \), so \( \gamma = (3, 3) \).
- For \( \pi_2 \), \( \lambda = (1, -2) \), \( \rho_n = \left( \frac{3}{2}, -\frac{3}{2} \right) \), \( \rho_c = \left( \frac{1}{2}, -\frac{1}{2} \right) \), so \( \gamma = (1, -3) \).
- For \( \pi_3 \), \( \lambda = (1, 0) \), \( \rho_n = \left( \frac{3}{2}, \frac{3}{2} \right) \), \( \rho_c = \left( \frac{1}{2}, \frac{1}{2} \right) \), so \( \gamma = (2, 2) \).
- For \( \pi_4 \), \( \lambda = (1, 0) \), \( \rho_n = \left( \frac{3}{2}, -\frac{3}{2} \right) \), \( \rho_c = \left( \frac{1}{2}, -\frac{1}{2} \right) \), so \( \gamma = (2, 0) \).
- For \( \pi_5 \), \( \lambda = (1, -1) \), \( \rho_n = \left( \frac{3}{2}, -\frac{1}{2} \right) \), \( \rho_c = \left( \frac{1}{2}, -\frac{1}{2} \right) \), so \( \gamma = (2, -1) \).
- For \( \pi_6 \), \( \lambda = (1, -1) \), \( \rho_n = \left( \frac{3}{2}, -\frac{3}{2} \right) \), \( \rho_c = \left( \frac{1}{2}, -\frac{1}{2} \right) \), so \( \gamma = (1, -2) \).

For principal series representations, we have \( \Psi_I = \emptyset \), \( \lambda = \gamma = \nu \), and \( \eta \) is only determined up to the parity of the integers.

- The trivial representation \( \pi_7 \) is given by \( \lambda = (1, 2) = \gamma \), and \( \eta = (0, 0) \).

For representations induced from the complex Cartan, we must write the parameters and roots according to the embedding of the torus of \( M \) in our torus \( T \). Notice that the Cartan involution of \( T \) is that of Example 3, so that \( T(\mathbb{R}) = \{ (t, \bar{t}) : t \in \mathbb{C}^* \} \). If \( \Psi = \{2e \} \) then \( \Psi_I = \{ e_1 - e_2 \} \) (now a noncompact root), and analogously for the other choice of positive roots. The Harish-Chandra parameter \( \lambda_0 = (k) \) becomes \( (\frac{k}{2}, -\frac{k}{2}) \in t^* \), and \( \nu \) becomes \( (\frac{k}{2}, \frac{k}{2}) \). So we get \( \gamma = (\frac{\nu + k + \text{sgn}(k)}{2}, \frac{\nu - k - \text{sgn}(k)}{2}) \) (in the case \( k = 0 \), the sign has to match the choice of positive root). If we want to ensure integer entries, we can write \( \eta = (\text{sgn}(k), 0) \) (recall from Example 3 that only the difference of the entries is important here), if not then \( \eta = (\frac{k + \text{sgn}(k)}{2}, -\frac{k - \text{sgn}(k)}{2}) \) is a more symmetric choice. Finally, \( \lambda = (\frac{\nu + k}{2}, \frac{\nu - k}{2}) \). So we have

- for \( \pi_8 \), \( \Psi_I = \{ e_1 - e_2 \} \), \( \lambda = (\frac{\nu + k}{2}, \frac{\nu - k}{2}) \), \( \gamma = (\frac{\nu + k + 1}{2}, \frac{\nu - k - 1}{2}) \), and \( \eta = (\frac{k + 1}{2}, -\frac{k - 1}{2}) \).
Finally, we look at representations attached to the mixed Cartan $T_m$. The torus of the $SL(2, \mathbb{R})$ factor of $M$ coincides with the first factor of our torus $T$, so $\Psi_I = \{2e_1\}$ if $\Psi = \{2e\}$. Similarly, the Harish-Chandra parameter of $\zeta$ provides the first coordinate of $\lambda$, and $\nu$ the second. The first coordinate of $\eta$ is determined by $\lambda_0$, the parity of the second by $\chi$. So we have:

- for $\pi_9$, $\Psi_I = \{2e_1\}$, $\lambda = (2, 1)$, $\gamma = (3, 1)$, and $\eta = (3, 0)$;
- for $\pi_{10}$, $\Psi_I = \{-2e_1\}$, $\lambda = (0, 0)$, $\gamma = (-1, 0)$, and $\eta = (-1, 1)$.

2.3 The parameters $(\Psi, P, \Lambda)$ of $[3]$. 

In these parameters, $\Lambda$ is a character of (the $\rho$-double cover of) our real torus $T(\mathbb{R})$, $P$ a system of positive imaginary roots, and $\Psi$ a system of positive real roots. In order to make our notation more consistent, we write $\Psi_I$ for $P$ and $\Psi_R$ for $\Psi$. As the notation suggests, $\Psi_I$ is the same system of roots as in the limit character parametrization. If a representation is given by final limit character $(\Psi_I, \Gamma, \lambda)$, choose a system $\Psi$ of positive roots containing $\Psi_I$ and such that the following conditions are satisfied: If $\alpha \in \Psi$ is a real or complex root then

$$-\theta\alpha \in \Psi,$$

and

$$\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}, \quad \text{if} \quad \text{Re} \langle \alpha^\vee, \theta\lambda - \lambda \rangle \geq 0.$$  

For example, if $\lambda$ is integral and regular, then $\Psi$ must be such that $\lambda$ is antidominant with respect to all real roots. Let $\rho_R = \rho(\Psi_R)$ and $\rho_{cx}$ be the corresponding half sums of the real and complex roots, respectively. Because of condition (16), $2\rho_{cx}$, like $2\rho_R$, is real valued. Then our representation may be given by $(\Psi_R, \Psi_I, \Lambda)$ with $\Lambda$ the character defined by the pair $(\lambda, \kappa)$, where

$$\kappa = \eta - \rho_R - \rho_{cx} - \rho_n + \rho_c = \eta - \rho(\Psi) + 2\rho_c.$$ 

This character is clearly independent of the choice of complex roots, subject to (16), since whenever $\rho'_{cx}$ corresponds to a second choice, $2\rho_{cx} - 2\rho'_{cx}$ is a sum of expressions of the form $2\alpha - 2\theta\alpha$ for some complex root $\alpha$. Since $-\theta\alpha = ^\vee \theta\alpha$, $\rho_{cx} - \rho'_{cx} = \xi + ^\vee \xi$ for some $\xi \in X^*(T)$. (See condition (2).)

Example 10 $G = SL(2, \mathbb{R})$. For limits of discrete series, we have $\Psi_R = \emptyset$, and $\kappa = \lambda$. More interesting are the principal series:

- For the trivial representation $\pi_3$, $\Psi_I = \emptyset$, $\Psi = \Psi_R = \{-2e\}$, so $\rho(\Psi) = \rho_R = (-1)$, $\lambda = (1)$, and $\kappa = (1)$.

Example 11 $G = Sp(4, \mathbb{R})$. As in the last example, the parameters for limits of discrete series are very easy to determine; we have $\Psi_R = \emptyset$, $\Psi_I$ and $\lambda$ as for final limit parameters, and $\kappa = \lambda$. So we start by looking at the principal series example. Of course, $\Psi_I = \emptyset$ and $\Psi = \Psi_R$ in this case.

- For the trivial representation $\pi_7$ we must choose $\Psi_R = \{-2e_1, -2e_2, -(e_1 \pm e_2)\}$, so $\rho(\Psi) = \rho_R = (-2, -1)$, $\lambda = (1, 2)$, and $\kappa = (2, 1)$.

With respect to the complex Cartan, the real roots are $\pm(e_1 + e_2)$, the complex roots $\pm 2e_1$, and $-\theta(2e_1) = 2e_2$. 

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Since \( \nu \) is assumed to be generic for \( \pi_8 \), condition (17) does not apply here, and we can choose any positive system containing \( \Psi_I = \{ e_1 - e_2 \} \) and satisfying (16), i.e., \( 2e_1 \in \Psi \iff 2e_2 \in \Psi \). We choose \( \Psi = \{ e_1 - e_2, -e_1 - e_2, -2e_1, -2e_2 \} \), which is the system we would need to choose for the case that \( \nu \) is a positive integer. Then \( \Psi_R = \{ -e_1 - e_2 \} \), so \( \rho_R = (-\frac{1}{2}, -\frac{1}{2}) \), \( \rho_{\text{ax}} = (-1, -1) \), \( \rho(\Psi) = (-1, -2) \), and \( \rho_{\zeta} = 0 \), so that \( \kappa = \left( \frac{k+3}{2}, \frac{-k+3}{2} \right) \sim (k, 0) \). Notice that because of (16), \( \rho_R + \rho_{\text{ax}} \) is of the form \( (a, a) \) for some half integer \( a \) and hence \( \kappa \) is independent of the choice of roots.

For the mixed Cartan, the real roots are \( \pm 2e_2 \), the complex roots \( \pm(e_1 \pm e_2) \), and \(-\theta(\pm e_1 \pm e_2) = \mp e_1 \pm e_2 \).

- For \( \pi_9 \), \( \theta \lambda - \lambda = (0, -2) \), so we must take \( \Psi = \{ 2e_1, -2e_2, e_1 - e_2, -e_1 - e_2 \} \), so \( \rho(\Psi) = (1, -2) \), \( \Psi_R = \{ -2e_2 \} \) and \( \kappa = (2, 2) \).

- For \( \pi_{10} \), since \( \lambda = 0 \) we may choose for \( \Psi \) any root system containing \( \Psi_I = \{ -2e_1 \} \) and satisfying (16) (there are four of them); for example, we can choose \( \Psi = \{ -2e_1, -2e_2, e_1 - e_2, -e_1 - e_2 \} \); then \( \Psi_R = \{ -2e_2 \} \) and \( \kappa = (0, 3) \).

### 2.4 Theta stable data \((q, H, \delta, \nu)\)

### 2.5 L-Parameters

The \( L \)-parameter associated to a representation is a conjugacy class of homomorphisms from the Weil group \( W_R \) into a Cartan subgroup \( {}^d T^F \) of the \( L \)-group \( \Gamma^F \) of \( G \). This Cartan subgroup is isomorphic either to the \( L \)-group or an \( E \)-group \( \Gamma^F \) of our real torus \( T(\mathbb{R}) \), depending on whether \( \Lambda \) is a character or projective character of \( T(\mathbb{R}) \).

### References


