The Dictionary of Representations *

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November 2005

1 Parametrizing characters

1.1 Parametrizing characters of real tori

Let T be a torus over $\mathbb{C}$, with Cartan involution θ, and $T(\mathbb{R})$ the corresponding real torus, i. e., the fixed points of the Galois action otained by composing θ with inversion and complex conjugation. We parametrize continuous characters π of $T(\mathbb{R})$ by pairs (λ, κ), where $\lambda = d\pi \in \mathfrak{t}^*$, and $\kappa \in X^*(T)$, and satisfying

$$\lambda -^\vee \theta(\lambda) = \kappa -^\vee \theta(k). \tag{1}$$

Two pairs (λ, κ) and (λ', κ') parametrize the same character if and only if

$$\lambda = \lambda' \text{and } \kappa = \kappa' + (\eta +^\vee \theta(\eta)) \text{ for some } \eta \in X^*(T). \tag{2}$$

The parameter κ may be obtained from π by extending the restriction $\pi|_{T(\mathbb{R})_c}$ of π to the compact part of the torus to an algebraic character of the complex torus T. This extension is in general not unique. (This ambiguity is reflected in condition (2).)

Example 1 *Let $T = \mathbb{C}^\times \times \mathbb{C}^\times$.Then $\mathfrak{t}^* \simeq \mathbb{C} \oplus \mathbb{C}$, and $X^*(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Consider θ given by $\theta(t_1, t_2) = (t_1, t_2^{-1})$, so that $T(\mathbb{R}) = S^1 \times \mathbb{R}^\times$. If $\lambda = (z_1, z_2)$ and $\kappa = (k_1, k_2)$, then condition (1) says that $(2z_1, 0) = (2k_1, 0)$, or $z_1 = k_1$. If $\eta = (n_1, n_2) \in X^*(T)$, then $\eta +^\vee \theta(\eta) = (0, 2n_2)$, so condition (2) says that only the parity of k_2 matters for π. Given (λ, κ) as specified, the character π is then given by*

$$\pi(e^{i\varphi}, r) = e^{ik_1\varphi}|r|^{z_2} sgn(r)^{k_2} \text{ for } \varphi \in \mathbb{R} \text{ and } r \in \mathbb{R}^\times. \tag{3}$$

Example 2 *Again let $T = \mathbb{C}^\times \times \mathbb{C}^\times$, but this time let θ be given by $\theta(t_1, t_2) = (t_2, t_1)$, so that $T(\mathbb{R}) = \{(t, \bar{t}^{-1}) : t \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times$. Now ${}^\vee\theta(\lambda) = (-z_2, -z_1)$, so condition (1) says $z_1 + z_2 = k_1 + k_2$, and (2) implies that the character depends on the sum $k_1 + k_2$, rather than the individual values k_1, k_2. The character π is now given by*

$$\pi(re^{i\varphi}, r^{-1}e^{i\varphi}) = r^{z_1 - z_2} e^{i(k_1+k_2)\varphi} = r^{z_1-z_2} e^{i(z_1+z_2)\varphi}. \tag{4}$$

*This material is based upon work supported by the National Science Foundation under Grant No. 0554278.

Example 3 *We still let $T = \mathbb{C}^\times \times \mathbb{C}^\times$, and change the Cartan involution once again, to $\theta(t_1, t_2) = (t_2^{-1}, t_1^{-1})$. Now $T(\mathbb{R}) = \{(t, \bar{t}) : t \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times$, and ${}^\vee\theta(\lambda) = (z_2, z_1)$. This time, condition (1) says $z_1 - z_2 = k_1 - k_2$, and the character will depend on the difference $k_1 - k_2$, rather than the individual values. The character π is given by*

$$\pi(re^{i\varphi}, re^{-i\varphi}) = r^{z_1+z_2} e^{i(k_1-k_2)\varphi} = r^{z_1+z_2} e^{i(z_1-z_2)\varphi}. \tag{5}$$

1.2 Covers of real tori

We are also going to need to specify characters of covers of our real torus, in particular the ρ double cover. These covers are obtained as quotients of a canonical covering $\widetilde{T}(\mathbb{R})$ (see §5 of [3] for details). By definition,

$$\widetilde{T}(\mathbb{R}) = \{X \in \mathfrak{t} : exp(X) \in T(\mathbb{R})\}/(1+\theta)\pi_1(T). \tag{6}$$

Here $\pi_1(T)$ is the fundamental group of T, $\pi_1(T) = 2\pi i X_*(T)$. The kernel of the covering map $\widetilde{T}(\mathbb{R}) \longrightarrow T(\mathbb{R})$ is $\pi_1(T)(\mathbb{R}) = \pi_1(T)/(1+\theta)\pi_1(T)$. The group of characters of the abelian group $\pi_1(T)$ is naturally isomorphic with ${}^\vee T$, and the group of characters of $\pi_1(T)(\mathbb{R})$ corresponds under this isomorphism to the ${}^\vee\theta$-fixed elements of ${}^\vee T$.

Example 4 *If $T = \mathbb{C}^\times$ is a one-dimensional torus, then $\pi_1(T) = 2\pi i\mathbb{Z} \cong \mathbb{Z}$, with character group $\mathbb{C}^\times = {}^\vee T$. Here $z \in {}^\vee T$ corresponds to the character χ_z given by $\chi_z(k) = z^k$.*

If $\theta = 1$ so that $T(\mathbb{R}) = S^1$, then $\widetilde{T}(\mathbb{R}) = i\mathbb{R}/4\pi i\mathbb{Z}$ (a two-fold cover), and $\pi_1(T)(\mathbb{R}) = 2\pi i\mathbb{Z}/4\pi i\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. In this case, $({}^\vee T)^{{}^\vee\theta} = \{\pm 1\}$, and the correspondence is easy: χ_{-1} is the sign character, and χ_1 the trivial one.

Now let θ be inversion, and $T(\mathbb{R}) = \mathbb{R}^\times$. Then $\widetilde{T}(\mathbb{R}) = \mathbb{R} + \pi i\mathbb{Z}$, a $\mathbb{Z}$-fold cover, and $\pi_1(T)(\mathbb{R}) = 2\pi i\mathbb{Z} \cong \mathbb{Z}$. The dual of this group is isomorphic to $\mathbb{C}^\times$ which coincides with $({}^\vee T)^{{}^\vee\theta}$ since ${}^\vee\theta$ is trivial.

We partition the characters of $\widetilde{T}(\mathbb{R})$ according to their restriction to $\pi_1(T)(\mathbb{R})$; for $z \in ({}^\vee T)^{{}^\vee\theta}$, we denote by $\Pi^z(T(\mathbb{R}))$ the set of characters of the canonical covering (or projective characters of our real torus) whose restriction to $\pi_1(T)(\mathbb{R})$ is χ_z. We call the quotient of $\widetilde{T}(\mathbb{R})$ by the kernel of χ_z *the cover of $T(\mathbb{R})$ determined by z*. If z has order 2 then this cover is a two-fold cover.

The set $\Pi^z(T(\mathbb{R}))$ may be parametrized in a fashion that is quite analogous to the parametrization of the characters of the real torus in Section 1.1: let $\xi \in \mathfrak{t}^*$ be such that $z = exp(2\pi i\xi)$. Then an element $\pi \in \Pi^z(T(\mathbb{R}))$ may be specified by a pair (λ, κ) with $\lambda = d\pi \in \mathfrak{t}^*$ and $\kappa \in \xi + X*(T)$, and satisfying the same compatibility condition (1) as for characters of $T(\mathbb{R})$. Similarly, condition (2) is necessary and sufficient for two such pairs to determine the same character.

Example 5 *To continue our example of one-dimensional tori, let $T(\mathbb{R}) = S^1$. The cover determined by $z = 1$ is the trivial (one-fold) cover, and we get the true characters of the torus. The cover determined by $z = -1$ is the canonical covering. We can choose $\xi = \frac{1}{2}$. Then characters (i. e., genuine characters of the double cover) are given by pairs (k, k) with $k \in \frac{1}{2} + \mathbb{Z}$. The corresponding character π is of course then defined by $\pi(ix + 4\pi i\mathbb{Z}) = e^{ikx}$.*

To get a double cover of $\mathbb{R}^\times$, we'll pick $z = -1$ again. Since $\chi_{-1}(2\pi ik) = (-1)^k$, the kernel of χ_{-1} is $4\pi i\mathbb{Z} \subset 2\pi i\mathbb{Z} = \pi_1(T)(\mathbb{R})$, so the cover determined by -1 is $\mathbb{R} + \pi i\mathbb{Z}/4\pi i\mathbb{Z} \cong \mathbb{R}_+^\times \times \mathbb{Z}/4\mathbb{Z}$. Elements of $\Pi^{-1}(T(\mathbb{R}))$ are given by pairs (λ, k) with λ a complex number and $k \in \frac{1}{2} + \mathbb{Z}$. The corresponding character π is then given by $\pi(r, x) = r^\lambda e^{\pi ixk}$ for $r \in \mathbb{R}_+^\times$, $x = 0, 1, 2, 3$.

1.3 Parametrizing admissible homomorphisms of the Weil Group

Let T be a complex torus and θ a Cartan involution specifying a real torus $T(\mathbb{R})$. Let ${}^\vee T^\Gamma$ be the L-group of $T(\mathbb{R})$; i. e., ${}^\vee T^\Gamma$ is the group generated by ${}^\vee T$ and an element ${}^\vee\delta \in {}^\vee T^\Gamma - {}^\vee T$ such that

$$({}^\vee\delta)^2 = 1, \quad \text{and} \quad {}^\vee\delta t {}^\vee\delta^{-1} = {}^\vee\theta(t) \quad \text{for } t \in {}^\vee T. \tag{7}$$

Recall that the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ is the group generated by $\mathbb{C}^\times$ and an element j satisfying $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$. We want to parametrize admissible homomorphisms from the Weil group into ${}^\vee T^\Gamma$, up to conjugacy by ${}^\vee T$ (on the image). These are continuous homomorphisms ϕ such that $\phi(\mathbb{C}^\times) \subset {}^\vee T$ and $\phi(j) \in {}^\vee T^\Gamma - {}^\vee T$. Such a map ϕ may be specified by giving a homomorphism

$$\phi_0 : \mathbb{C}^\times \longrightarrow {}^\vee T \tag{8}$$

and an element

$$\phi(j) = t_\phi {}^\vee\delta \quad \text{with } t_\phi \in {}^\vee T. \tag{9}$$

Homomorphisms as in (8) are in one-one correspondence with pairs (λ, μ) of elements of $\mathfrak{t}^*$ such that $\lambda - \mu \in X^*(T)$. The homomorphism corresponding to such a pair is then defined by

$$\phi_0(exp(z)) = exp(z\lambda + \overline{z}\mu). \tag{10}$$

To specify $\phi(j)$ we choose $\tau \in \mathfrak{t}^*$ such that $exp(2\pi i\tau) = t_\phi$. For these data to define a homomorphism we need to have $\phi(j)\phi_0(z)\phi(j)^{-1} = \phi_0(\overline{z})$ and $\phi(j)^2 = \phi_0(-1)$. The first condition easily translates into

$$\mu = {}^\vee\theta(\lambda), \tag{11}$$

the second gives the condition

$$(\tau + {}^\vee\theta(\tau)) - \frac{1}{2}(\lambda - {}^\vee\theta(\lambda)) \in X^*(T). \tag{12}$$

Consequently, ϕ is given by a pair (λ, τ) of elements of $\mathfrak{t}^*$ satisfying (12). Clearly, τ is determined only up to an element of the character lattice $X^*(T)$.

The notion of an E-group is a generalization of that of the L-group. An E-group of a torus T as above is a group, also denoted by ${}^\vee T^\Gamma$, generated by ${}^\vee T$ and an element ${}^\vee\delta$ as above, but with $({}^\vee\delta)^2 = z \in {}^\vee T$ not required to be the identity. It is not hard to modify the above parametrization to the more general case of an E-group determined by $z \in {}^\vee T$: let $\xi \in \mathfrak{t}^*$ be such that $exp(2\pi i\xi) = z$. Then an admissible homomorphism may be given by a pair (λ, τ) of elements of $\mathfrak{t}^*$ satisfying

$$(\tau + {}^\vee\theta(\tau)) - \frac{1}{2}(\lambda - {}^\vee\theta(\lambda)) \in \xi + X^*(T). \tag{13}$$

If $z \in ({}^\vee T)^{{}^\vee\theta}$, then there is a canonical one-one correspondence between the set $\Pi^z(T(\mathbb{R})$ of projective characters of the real torus associated to z as in Section 1.2 and the set of ${}^\vee T$-conjugacy classes of admissible homomorphisms ϕ from the Weil group $W_\mathbb{R}$ into the E-group of $T(\mathbb{R})$ determined by z: if $\pi \in \Pi^z(T(\mathbb{R})$ is given by the pair (λ, κ) then the corresponding homomorphism ϕ is given by $(\lambda, -\frac{1}{2}\kappa)$ (this constitutes a correction of formula (4.8) of [3]). It is easy to check that the compatibility condition (13) is then satisfied, and that the inverse mapping is given by $(\lambda, \tau) \mapsto (\lambda, \kappa)$ with

$$\kappa = \frac{1}{2}(\lambda - {}^\vee\theta(\lambda)) - (\tau + {}^\vee\theta(\tau)). \tag{14}$$

2 Parametrizing Representations

2.1 Inducing Data

To specify a representation π using (real parabolic) inducing data, we write down a standard module which has our representation as an irreducible quotient, subrepresentation, or other kind of distinguished "Langlands" subquotient. Starting with a θ-stable real torus $T(\mathbb{R}) = T(\mathbb{R})_c A$ (with $A = T(\mathbb{R})_s$ the split part of the real torus), we get a Levi subgroup MA of G with $M = Cent_G(A)$. The data then consist of a discrete series or limit of discrete series representation δ of M, and a character $\exp(\nu)$ of A with $\nu \in \mathfrak{a}^*$ (usually just written ν). The unipotent radical N of our parabolic subgroup $P = MAN$ can then be chosen so that the induced representation

$$X(\delta, \nu) = Ind_P^G(\delta \otimes \nu \otimes 1) \tag{15}$$

has our representation π as an irreducible quotient or subrepresentation; by choosing our Cartan as compact as possible, we can arrange for our standard module to have a *unique* irreducible quotient or subrepresentation $\overline{X}(\delta, \nu) \simeq \pi$. This is the idea of the *final* (as opposed to *regular*) limit characters, L-data, etc. We'll mostly use only the final version, illustrating the difference with regular characters only in a few of the examples. We specify a limit of discrete series representation δ of M by a triple $(\lambda_0, \Psi, \tau_0)$, where λ_0 is the Harish-Chandra parameter of δ, Ψ system of positive roots making λ_0 dominant, and τ_0 a character of the center of M.

In our examples, we follow Fokko's lead and fix a complex torus T (whenever possible it will be the diagonal one) in $G(\mathbb{C})$, varying the Cartan involution θ to pick out the different $T(\mathbb{R})$.

Example 6 $\mathbf{G} = \mathbf{SL}(\mathbf{2},\mathbb{R})$. *The representations of $SL(2, R)$ consist of discrete series, limits of discrete series, and principal series representations. For the limits of discrete series, $T(\mathbb{R})$ is compact (θ in this case is conjugation by the element $diag(i, -i)$), so $M = G$, and we only specify $\delta = \delta(\lambda_0, \Psi)$ (τ is determined by the other data). We write $\lambda_0 = (k)$ for an integer k, and $\Psi = \{2e\}$ if $k \geq 0$ and the l.o.d.s. is holomorphic, and $\Psi = \{-2e\}$ if $k \leq 0$ and the l.o.d.s. is antiholomorphic. For future reference (so that we can illustrate the different parametrizations in some specific cases), we fix a few specific representations:*

- *We let π_1 be the (holomorphic) discrete series with $\lambda_0 = (3)$; and*
- *π_2 the antiholomorphic limit of discrete series given by $\lambda_0 = (0)$ and $\Psi = \{-2e\}$.*

For the principal series representations, $T(\mathbb{R}) = MA$, so $M \cong \{\pm 1\}$ and $A \cong \mathbb{R}_+^*$. We have $\delta = 1$ or sign, and ν is given by a complex number. We choose

- *π_3 to be the trivial representation $\pi_3 = \overline{X}(1 \otimes 1)$.*

Note that if we choose $\delta = sign$ and $\nu = 0$ we violate the "final" condition; the resulting standard representation will have both limits of discrete series as direct summands.

Example 7 $\mathbf{G} = \mathbf{Sp}(\mathbf{4},\mathbb{R})$. *We denote our four (conjugacy classes of) real tori by*

$T_k \cong (S^1)^2$ ("compact", θ is conjugation by $diag(i, i, -i, -i)$);

$T_s \cong (\mathbb{R}^\times)^2$:"split", $\theta$is conjugation by $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$;

$T_c \cong \mathbb{C}^\times$: *"complex", θ is conjugation by* $\begin{pmatrix} 0&0&0&i \\ 0&0&i&0 \\ 0&i&0&0 \\ i&0&0&0 \end{pmatrix}$*; and*

$T_m \cong S^1 \times \mathbb{R}^\times$: *"mixed", θ is conjugation by* $\begin{pmatrix} i&0&0&0 \\ 0&0&0&-i \\ 0&0&-i&0 \\ 0&-i&0&0 \end{pmatrix}$.

We write elements $diag(z, w, z^{-1}, w^{-1})$ of T as (z, w), and similarly for elements of $\mathfrak{t}$ or its dual, and the roots of G with respect to T are $\{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$.

Limit of discrete series. *If $T(\mathbb{R}) = T_k$, then $M = G$,and we get discrete series or limit of discrete series representations of G. The Harish-Chandra parameter will be of the form $\lambda_0 = (k, l) \in \mathfrak{t}^*$ with k and l integers.*

- *For the first representation π_1, we choose $\lambda_0 = (2, 1)$ and $\Psi = \{2e_1, 2e_2, e_1 - e_2, e_1 + e_2\}$;*
- *π_2 is the discrete series representation with $\lambda_0 = (1, -2)$ and $\Psi = \{2e_1, -2e_2, e_1 - e_2, -e_1 - e_2\}$;*
- *for π_3 we choose the limit of discrete series representation with $\lambda_0 = (1, 0)$ and $\Psi = \{2e_1, 2e_2, e_1 - e_2, e_1 + e_2\}$;*
- *π_4 is the l.o.d.s. with $\lambda_0 = (1, 0)$ and $\Psi = \{2e_1, -2e_2, e_1 - e_2, e_1 + e_2\}$;*
- *π_5 has $\lambda_0 = (1, -1)$ and $\Psi = \{2e_1, -2e_2, e_1 - e_2, e_1 + e_2\}$;*
- *and π_6 has $\lambda_0 = (1, -1)$ and $\Psi = \{2e_1, -2e_2, e_1 - e_2, -e_1 - e_2\}$.*

Principal series. *If $T(\mathbb{R}) = T_s$, then $MA = T_s \simeq (\mathbb{R}^\times)^2$, with $M = \{\pm 1\} \times \{\pm 1\}$ and $A = \mathbb{R}^2$. So a limit of discrete series consists of a character δ of $\{\pm 1\} \times \{\pm 1\}$, i. e. $1 \otimes 1$, $1 \otimes sign$, etc. A character of A is given by a pair $\nu = (\nu_1, \nu_2)$ of complex numbers. Then if $a = (x, y) \in \mathbb{R}^2$, $\nu(a) = e^{\nu_1 x + \nu_2 y}$. For our "final" condition we require that if $\nu_i = 0$ then δ is trivial on the corresponding $\{\pm 1\}$ factor, and if $\nu_1 = \pm \nu_2$ then δ is either trivial or $sign \otimes sign$.*

- *We get the trivial representation $\pi_7 = triv$ by taking $\delta = triv$ and $\nu = (1, 2)$.*

Induced from complex Cartan. *If $T(\mathbb{R})$ is the complex Cartan subgroup T_c, then $MA \simeq GL(2, \mathbb{R})$, with $M = SL^{\pm}(2, \mathbb{R})$ and $A = \mathbb{R}_+^\times$. Limits of discrete series of M are given by pairs $\lambda_0 = (k)$ and integer and $\Psi = \{2e\}$ if $k \geq 0$, or $\Psi = \{-2e\}$ if $k \leq 0$. (This is a root system in $SL(2, \mathbb{R})$.) The parameter ν is specified by a complex number. Our data will be final provided that if $\nu = 0$ then k is odd. (Otherwise we get a sum of limit of discrete series representations; for example, $\pi_5 \oplus \pi_6$ may be obtained by choosing $k = 2$ and $\nu = 0$.)*

- *We let π_8 be given by a positive integer k and a generic ν.*

Mixed induced series. *If $T(\mathbb{R}) = T_m$ then $MA \simeq SL(2, \mathbb{R}) \times \mathbb{R}^\times$, so δ is of the form $\zeta \otimes \chi$ for some limit of discrete series ζ and a character χ of $\{\pm 1\}$. The character ν is given by a complex number. For our data to be final we require that if $\nu = 0$ then $\chi = sgn$. (Otherwise our standard module will be a sum of limits of discrete series of G; for instance, if ζ is the holomorphic discrete series with parameter (1), $\chi = triv$, and $\nu = 0$, then $X(\delta, \nu) \simeq \pi_3 \oplus \pi_4$.)*

- *Let π_9 be given by ζ the holomorphic discrete series with parameter (2), $\chi = triv$, and $\nu = 1$.*
- *For π_{10}, take ζ to be the limit of discrete series with $k = 0$, $\Psi = \{-2e\}$ (with lowest K-type -1), $\chi = sign$, and $\nu = 0$ (this representation has infinitesimal character 0).*

2.2 Final Limit Characters

These are the parameters $\gamma = (\Psi, \Gamma, \overline{\gamma})$ described in Definition 2.4 of [6]. We will use the notation $(\Psi_I, \Gamma, \lambda)$ instead. In terms of the inducing data of section 2.1, Ψ_I is the positive system Ψ, regarded as a system of imaginary roots of G (rather than M). The parameter Γ is a character of $T(\mathbb{R})$, namely a character such that δ has a lowest $(M \cap K)$-type with highest weight $\Gamma|_{T(\mathbb{R})}$, and $\Gamma|_A = exp(\nu)$. We specify Γ by a pair (γ, η) as in section 1.1, i. e., with $\gamma \in \mathfrak{t}^*$, $\eta \in X^*(T)$, and satisfying the compatibility condition (1). Let $\mathfrak{t}_c$ and $\mathfrak{t}_s$ be the $+1$ and -1 eigenspaces of θ in $\mathfrak{t}$, i. e., the complexified Lie algebras of $T(\mathbb{R})_c$ and A, respectively. Choose $\lambda \in \mathfrak{t}^*$ such that $\lambda|_{\mathfrak{t}_c} = \lambda_0$, and $\lambda|_{\mathfrak{t}_s} = \nu$. (This is the infinitesimal character of the representation.) Let ρ_n and ρ_c be one half the sums of the non-compact and compact roots in Ψ_I, respectively. Then $\gamma = \lambda + \rho_n - \rho_c$. The parameter η must be chosen accordingly. We work out the examples given in section 2.1.

Example 8 $\mathbf{G} = \mathbf{SL}(\mathbf{2},\mathbb{R})$.

- *For* π_1, $\lambda = (3)$, $\rho_n = (1)$, *and* $\rho_c = 0$, *so* $\gamma = (4) = \eta$.
- *For* π_2, $\lambda = (0)$, $\rho_n = (-1)$, *so* $\gamma = (-1) = \eta$.
- *For the trivial representation* π_3, $\lambda = (1) = \gamma$, *and* $\eta = (0)$.

Example 9 $\mathbf{G} = \mathbf{Sp}(\mathbf{4},\mathbb{R})$. *For limits of discrete series, we have* $\Psi_I = \Psi$, $\lambda = \lambda_0$, *and* $\gamma = \eta$, *so we only need to specify* γ.

- *For* π_1, $\lambda = \lambda_0 = (2, 1)$, $\rho_n = (\frac{3}{2}, \frac{3}{2})$, $\rho_c = (\frac{1}{2}, \frac{-1}{2})$, *so* $\gamma = (3, 3)$.
- *For* π_2, $\lambda = (1, -2)$, $\rho_n = (\frac{1}{2}, \frac{-3}{2})$, $\rho_c = (\frac{1}{2}, \frac{-1}{2})$, *so* $\gamma = (1, -3)$.
- *For* π_3, $\lambda = (1, 0)$, $\rho_n = (\frac{3}{2}, \frac{3}{2})$, $\rho_c = (\frac{1}{2}, \frac{-1}{2})$, *so* $\gamma = (2, 2)$.
- *For* π_4, $\lambda = (1, 0)$, $\rho_n = (\frac{3}{2}, \frac{-1}{2})$, $\rho_c = (\frac{1}{2}, \frac{-1}{2})$, *so* $\gamma = (2, 0)$.
- *For* π_5, $\lambda = (1, -1)$, $\rho_n = (\frac{3}{2}, \frac{-1}{2})$, $\rho_c = (\frac{1}{2}, \frac{-1}{2})$, *so* $\gamma = (2, -1)$.
- *For* π_6, $\lambda = (1, -1)$, $\rho_n = (\frac{1}{2}, \frac{-3}{2})$, $\rho_c = (\frac{1}{2}, \frac{-1}{2})$, *so* $\gamma = (1, -2)$.

For principal series representations, we have $\Psi_I = \emptyset$, $\lambda = \gamma = \nu$,*and* η *is only determined up to the parity of the integers.*

- *The trivial representation* π_7 *is given by* $\lambda = (1, 2) = \gamma$, *and* $\eta = (0, 0)$.

For representations induced from the complex Cartan, we must write the parameters and roots according to the imbedding of the torus of M *in our torus* T. *Notice that the Cartan involution of* T *is that of Example 3, so that* $T(\mathbb{R}) = \{(t, \bar{t}) : t \in \mathbb{C}^\times\}$. *If* $\Psi = \{2e\}$ *then* $\Psi_I = \{e_1 - e_2\}$ *(now a noncompact root), and analogously for the other choice of positive roots. The Harish-Chandra parameter* $\lambda_0 = (k)$ *becomes* $(\frac{k}{2}, -\frac{k}{2}) \in \mathfrak{t}^*$, *and* ν *becomes* $(\frac{\nu}{2}, \frac{\nu}{2})$. *So we get* $\gamma = (\frac{\nu+k+sgn(k)}{2}, \frac{\nu-k-sgn(k)}{2})$ *(in the case* $k = 0$, *the sign has to match the choice of positive root). If we want to ensure integer entries, we can write* $\eta = (k + sgn(k), 0)$ *(recall from Example 3 that only the difference of the entries is important here), if not then* $\eta = \left(\frac{k+sgn(k)}{2}, \frac{-k-sgn(k)}{2}\right)$ *is a more symmetric choice. Finally,* $\lambda = (\frac{\nu+k}{2}, \frac{\nu-k}{2})$. *So we have*

- *for* π_8, $\Psi_I = \{e_1 - e_2\}$, $\lambda = (\frac{\nu+k}{2}, \frac{\nu-k}{2})$, $\gamma = (\frac{\nu+k+1}{2}, \frac{\nu-k-1}{2})$, *and* $\eta = \left(\frac{k+1}{2}, \frac{-k-1}{2}\right)$.

Finally, we look at representations attached to the mixed Cartan T_m. The torus of the $SL(2,\mathbb{R})$ factor of M coincides with the first factor of our torus T, so $\Psi_I = \{2e_1\}$ if $\Psi = \{2e\}$. Similarly, the Harish-Chandra parameter of ζ provides the first coordinate of λ, and ν the second. The first coordinate of η is determined by λ_0, the parity of the second by χ. So we have:

- *for π_9, $\Psi_I = \{2e_1\}$, $\lambda = (2,1)$, $\gamma = (3,1)$, and $\eta = (3,0)$;*
- *for π_{10}, $\Psi_I = \{-2e_1\}$, $\lambda = (0,0)$, $\gamma = (-1,0)$, and $\eta = (-1,1)$.*

2.3 The parameters (Ψ, P, Λ) of [3].

In these parameters, Λ is a character of (the ρ-double cover of) our real torus $T(\mathbb{R})$, P a system of positive imaginary roots, and Ψ a system of positive real roots. In order to make our notation more consistent, we write Ψ_I for P and Ψ_R for Ψ. As the notation suggests, Ψ_I is the same system of roots as in the limit character parametrization. If a representation is given by final limit character $(\Psi_I, \Gamma, \lambda)$, choose a system $\Psi$of positive roots containing Ψ_I and such that the following conditions are satisfied: If $\alpha \in \Psi$ is a real or complex root then

$$-\theta\alpha \in \Psi, \text{ and} \tag{16}$$

$$\text{if } \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \text{ then Re } (\langle \alpha^\vee, \theta\lambda - \lambda \rangle) \geq 0. \tag{17}$$

For example, if λ is integral and regular, then Ψ must be such that λ is antidominant with respect to all real roots. Let $\rho_R = \rho(\Psi_R)$ and ρ_{cx} be the corresponding half sums of the real and complex roots, respectively. Because of condition (16), $2\rho_{cx}$, like $2\rho_R$, is real valued. Then our representation may be given by $(\Psi_R, \Psi_I, \Lambda)$ with Λ the character defined by the pair (λ, κ), where

$$\kappa = \eta - \rho_R - \rho_{cx} - \rho_n + \rho_c = \eta - \rho(\Psi) + 2\rho_c. \tag{18}$$

This character is clearly independent of the choice of complex roots, subject to (16), since whenever ρ'_{cx} corresponds to a second choice, $2\rho_{cx} - 2\rho'_{cx}$ is a sum of expressions of the form $2\alpha - 2\theta\alpha$ for some complex root α. Since $-\theta\alpha = {}^\vee\theta\alpha$, $\rho_{cx} - \rho'_{cx} = \xi + {}^\vee\xi$ for some $\xi \in X^*(T)$. (See condition (2).

Example 10 $\mathbf{G} = \mathbf{SL}(\mathbf{2},\mathbb{R})$. *For limits of discrete series, we have $\Psi_R = \emptyset$, and $\kappa = \lambda$. More interesting are the principal series:*

- *For the trivial representation π_3, $\Psi_I = \emptyset$, $\Psi = \Psi_R = \{-2e\}$, so $\rho(\Psi) = \rho_R = (-1)$, $\lambda = (1)$, and $\kappa = (1)$.*

Example 11 $\mathbf{G} = \mathbf{Sp}(\mathbf{4},\mathbb{R})$. *As in the last example, the parameters for limits of discrete series are very easy to determine; we have $\Psi_R = \emptyset$, Ψ_I and λ as for final limit parameters, and $\kappa = \lambda$. So we start by looking at the principal series example. Of course, $\Psi_I = \emptyset$ and $\Psi = \Psi_R$ in this case.*

- *For the trivial representation π_7 we must choose $\Psi_R = \{-2e_1, -2e_2, -(e_1 \pm e_2)\}$, so $\rho(\Psi) = \rho_R = (-2,-1)$, $\lambda = (1,2)$, and $\kappa = (2,1)$.*

With respect to the complex Cartan, the real roots are $\pm(e_1 + e_2)$, the complex roots $\pm 2e_i$, and $-\theta(2e_1) = 2e_2$.

- *Since* ν *is assumed to be generic for* π_8*, condition (17) does not apply here, and we can choose any positive system containing* $\Psi_I = \{e_1 - e_2\}$ *and satisfying (16), i.e.,* $2e_1 \in \Psi \Longleftrightarrow 2e_2 \in \Psi$*. We choose* $\Psi = \{e_1 - e_2, -e_1 - e_2, -2e_1, -2e_2\}$*, which is the system we would need to choose for the case that* ν *is a positive integer. Then* $\Psi_R = \{-e_1 - e_2\}$*, so* $\rho_R = \left(-\frac{1}{2}, -\frac{1}{2}\right)$*,* $\rho_{cx} = (-1,-1)$*,* $\rho(\Psi) = (-1,-2)$*, and* $\rho_c = 0$*, so that* $\kappa = \left(\frac{k+3}{2}, \frac{-k+3}{2}\right) \sim (k,0)$*. Notice that because of (16),* $\rho_R + \rho_{cx}$ *is of the form* (a,a) *for some half integer* a *and hence* κ *is independent of the choice of roots.*

For the mixed Cartan, the real roots are $\pm 2e_2$*, the complex roots* $\pm(e_1 \pm e_2)$*, and* $-\theta(\pm e_1 \pm e_2) = \mp e_1 \pm e_2$*.*

- *For* π_9*,* $\theta\lambda - \lambda = (0,-2)$*, so we must take* $\Psi = \{2e_1, -2e_2, e_1 - e_2, -e_1 - e_2\}$*, so* $\rho(\Psi) = (1,-2)$*,* $\Psi_R = \{-2e_2\}$ *and* $\kappa = (2,2)$*.*

- *For* π_{10}*, since* $\lambda = 0$ *we may choose for* Ψ *any root system containing* $\Psi_I = \{-2e_1\}$ *and satisfying (16) (there are four of them); for example, we can choose* $\Psi = \{-2e_1, -2e_2, e_1 - e_2, -e_1 - e_2\}$*; then* $\Psi_R = \{-2e_2\}$ *and* $\kappa = (0,3)$*.*

2.4 Theta stable data $(\mathfrak{q}, H, \delta, \nu)$

2.5 L-Parameters

The L-parameter associated to a representation is a conjugacy class by the dual group of homomorphisms from the Weil group $W_{\mathbb{R}}$ into a Cartan subgroup ${}^dT^\Gamma$ of the L-group ${}^\vee G^\Gamma$ of G. This Cartan subgroup is isomorphic either to the L-group or an E-group ${}^\vee T^\Gamma$ of our real torus $T(\mathbb{R})$, depending on whether Λ is a character or projective character of $T(\mathbb{R})$.

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