MATH 1220 Exam 2 Solutions

1. (50 pts.) Find $\frac{dy}{dx}$.

   (a) $y = \sqrt[3]{x}\tan(x) = x^{\frac{1}{3}}\tan(x)$

   $$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}\tan(x) + x^{\frac{1}{3}}\sec^2(x)$$

   (b) $y = \dfrac{\sec(x)}{\cos(3x)+3}$

   $$\frac{dy}{dx} = \frac{\tan(x)\sec(x)(\cos(3x)+3) - \sec(x)(-\sin(3x)\cdot 3)}{\cos(3x)+3)^2}$$

   (c) $x^2 + xy - y^2 + 3x = 16$

   $$2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} + 3 = 0$$
   $$\frac{dy}{dx}(x - 2y) = -2x - y - 3$$
   $$\frac{dy}{dx} = \frac{-2x - y - 3}{x - 2y}$$

   (d) $y = (x-2)^{(x-2)}$

   $$\ln(y) = (x-2)\ln(x-2)$$
   $$\frac{1}{y}\frac{dy}{dx} = \ln(x-2) + (x-2)\cdot\frac{1}{x-2} = \ln(x-2) + 1$$
   $$\frac{dy}{dx} = y(\ln(x-2)+1) = (x-2)^{(x-2)}(\ln(x-2)+1)$$

   (e) $y = \sin^{-1}(x^3)$

   $$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}\cdot 3x^2 = \frac{3x^2}{\sqrt{1-x^6}}$$

2. (10 pts.) Consider a triangle with base b and height h, which are changing with time. If A is the area of the triangle, how is $\frac{dA}{dt}$ related to $\frac{db}{dt}$ and $\frac{dh}{dt}$?

   $$A = \frac{1}{2}bh$$
   $$\frac{dA}{dt} = \frac{1}{2}\left(h\frac{db}{dt} + b\frac{dh}{dt}\right)$$

3. (10 pts.) Use a linearization of $y = e^x$ to estimate $e^{-.15}$.

   We know that $e^0 = 1$, and $\frac{d}{dx}(e^x) = e^x$, so we can use the linearization at $x = 0$:

   $$y = f(0) + f'(0)(x-0)$$
   $$= e^0 + e^0(x) = 1 + x$$
   $$e^{-.15} \approx 1 - .15 = .85$$

4. (10 pts.) Rewrite the following expression in terms of the natural logarithm, and evaluate.
$y = \dfrac{\log_x(a)}{\log_{x^2}(a)}$

$$\log_x(a) = \frac{\ln(a)}{\ln(x)}$$
$$\log_{x^2}(a) = \frac{\ln(a)}{\ln(x^2)} = \frac{\ln(a)}{2\ln(x)}$$
$$\frac{\log_x(a)}{\log_{x^2}(a)} = \frac{\frac{\ln(a)}{\ln(x)}}{\frac{\ln(a)}{2\ln(x)}} = 2$$

5. (20 pts.) For each of the following functions, decide whether it has an inverse. If yes, find it, and show that your answer is indeed the inverse function. If not, explain why.

(a) $f(x) = 4 - 3x^3$

The function has inverse $f^{-1}(x) = \sqrt[3]{\frac{4-x}{3}}$:

$$f^{-1}\left(f(x)\right) = \sqrt[3]{\frac{4-(4-3x^3)}{3}} = \sqrt[3]{x^3} = x;$$
$$f(f^{-1}(x)) = 4 - 3\left(\sqrt[3]{\frac{4-x}{3}}\right)^3 = 4 - 3\frac{4-x}{3} = x$$

(b) $f(x) = (x-3)^2$

This function does not have an inverse because it is not one-to-one: We have $f(1) = (-2)^2 = 4 = 2^2 = f(5)$.