1 Setup

Let \((G, \tau)\) and \((\check{G}, \check{\tau})\) be given. Here \(G\) is an algebraic group with dual group \(\check{G}\), \(\tau\) an outer automorphism determining an inner class of real forms, \(\delta \in \text{Aut}(G)\) a strong real form in the inner class determined by \(\tau\), chosen as in \([3]\), and \(\check{\tau}, \check{\delta}\) the corresponding objects for \(\check{G}\). (Recall: \(\delta\) is the unique involution in the chosen inner class fixing the chosen pinning; e.g., if \(\tau\) is trivial then \(\delta\) is the identity automorphism.) Form the extended groups \(G^\Gamma = G \rtimes \Gamma = G \cup G\delta\) and \(\check{G}^\Gamma = \check{G} \rtimes \Gamma = \check{G} \cup \check{G}\check{\delta}\). Recall that a strong involution of \(G\) is an element \(x \in G\delta\) such that \(x^2 \in Z(G)\), and \(\theta_x = \text{int}(x)\) is the Cartan involution of the corresponding real group \(G(\mathbb{R})\).

2 Integral L-Data

Recall from \([1]\) (see also \([2]\)) the sets of Integral L-Data parametrizing irreducible representations with integral infinitesimal character. These are septuples as follows:

\[
(S; \lambda) = (x, H, B, y^dH, dB; \lambda)
\]

(1)

where \(x\) is a strong involution of \(G\), \(H \subset B\) a Cartan and a Borel subgroup of \(G\) such that \(H\) is \(\theta_x\)-stable; and \(y^\check{\tau}, H,\) and \(\check{\tau}\) are corresponding objects on the dual side. The data determine an isomorphism

\[
\zeta^d : H \rightarrow^\check{\tau} H
\]

(2)

which also identifies \(d\mathfrak{h}\) with \(\check{\mathfrak{h}}^*\) taking the positive root system \(d\Psi^+\) corresponding to \(dB\) to the set \(\check{\Psi}^+\) of coroots of the system of positive roots \(\Psi^+\) determined by \(B\).
The parameter $\lambda$ is then an element of $^d\mathfrak{h} \simeq \mathfrak{h}$ such that $\exp(2\pi i \lambda) = y^2$, and dominant regular with respect to $\Psi^+$. The involutions $\theta_x$ and $^d\theta_y$ must be compatible in the sense that the corresponding involutions of $\mathfrak{h}$ and $^\vee \mathfrak{h}$ must satisfy $^d\theta_y = -t^x \theta_x$. We make all these identifications and replace the superscripts $d$ by $^\vee$ everywhere. Conjugacy classes by $G \times ^\vee G$ of sets of integral $L$-data correspond in a one-one fashion to irreducible admissible representations with integral regular infinitesimal character of strong real forms in the given inner class.

Since all pairs $H \subset B$ are conjugate by $G$ (and similarly on the dual side), we may fix $H, B, ^\vee H$, and $^\vee B$ and look at pairs $(x, y)$, up to conjugation by $H \times ^\vee H$ instead (see [3] for details). These pairs parametrize translation families of representations; giving $\lambda$ amounts to choosing a particular representation in the family, with infinitesimal character.

Which representation is it? The element $x$ specifies a real form $G(\mathbb{R})$, along with a conjugacy class of Cartan subgroups $H(\mathbb{R})$. To get a representation of $G(\mathbb{R})$, we need a character $\Lambda$ of (a double cover of) $H(\mathbb{R})$; the representation is then obtained by parabolic induction. We have $\lambda = d\Lambda$. If $H(\mathbb{R})$ is connected, this determines the character uniquely. Otherwise, the element $y$ determines it on the $\mathbb{Z}/2\mathbb{Z}$ factors. Details will be spelled out in the Dictionary (this is work in progress). The Atlas software produces the $x$'s and $y$'s using the ‘kbg’ command, and the compatible pairs $(x, y)$ using the ‘block’ command.

**Example 1** $SL(2, \mathbb{R})$. We have $G = SL(2, \mathbb{C}), ^\vee G = SO(3, \mathbb{C})$, which we think of as the isometry group of the form on $\mathbb{C}^3$ given by $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\delta$ and $^\vee \delta$ trivial so that we can think of $x$ and $y$ as elements of $G$ and $^\vee G$, rather than the extended groups. We choose the diagonal Cartan subgroups. Let $t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{C})$, $s = \text{diag}(-1, -1, 1) \in SO(3, \mathbb{C})$. Then up to conjugacy, a complete list of the pairs $(x, y)$ for $x$ giving the split real form $SL(2, \mathbb{R})$ are $(t, M)$, $(-t, M)$, $(w, I)$, and $(w, s)$. The elements $t$ and $-t$ give the compact Cartan which is connected, so only one $y$ (giving the split Cartan on the dual side) is matched with these parameters. For $x = w$ we get the split Cartan $\simeq \mathbb{R}^\times$ of $SL(2, \mathbb{R})$, so there are two characters with the same differential, distinguished by the two matching choices $I, s$ for $y$. If we fix $\lambda$ as an integral multiple of $\rho$, the four representations of $SL(2, \mathbb{R})$ are the two discrete series, and the two principal series at infinitesimal character $\lambda$. If $\lambda$ is an odd multiple of $\rho$, then $(w, I)$ corresponds to the non-spherical principal series and $(w, s)$ to the spherical one; if $\lambda$ is even, these are switched.
3 Non-Integral L-Data

According to [1], general (not necessarily integral) $L$-data are septuples

$$(S; \lambda) = (x, H, P, y, ^\vee H, ^\vee P; \lambda), \quad (3)$$

where $(x, H, ^\vee H)$ is as in (1) (we assume we have chosen an isomorphism $\zeta$ as in (2) and made the appropriate identifications), $y \in G^\vee \delta, y^2 \in \vee H, ^\vee \theta_y$ normalizes $^\vee H, \theta_x$ and $^\vee \theta_y$ are compatible as above, $^\vee P$ is a positive set of roots for $G_{y^2} = Cent_G(y^2)$, $P$ is the set of roots of $G$ dual to $^\vee P$, $\lambda \in h = h^*$ is such that $\exp(2\pi i \lambda) = y^2$, and $\lambda$ is dominant with respect to $P$. If $\lambda$ is also regular and $\Psi = \Delta(h, g)$ then $P = \{\alpha \in \Psi ; < \lambda, ^\vee \alpha > \in \mathbb{Z}_{>0}\}$. As in the integral case, conjugacy classes by $G \times ^\vee G$ of these sets of data parametrize irreducible representations of all strong real forms in the given inner class with regular (not necessarily integral) infinitesimal character.

As before, fix $H, B \leftrightarrow \Psi^+, ^\vee H,$ and $^\vee B \leftrightarrow ^\vee \Psi^+$. Then given $(x, y), P \subset \Psi^+$ and $^\vee P \subset ^\vee \Psi^+$ are uniquely determined. Our parameters will be triples $(x, y, \lambda)$ which specify representations, or pairs $(x, y)$ which give translation families of representations.

3.1 First Approach: Use Integral Data for $G$

Assume that $\lambda$ is real, i.e., $\lambda \in X^*(H) \otimes \mathbb{Z} \mathbb{R}$, and regular. (The assumption that $\lambda$ is real is not essential; we make it because this is the case we are most interested in, and so that we have a linear order on the elements. We could of course easily define such an order on $\mathbb{C}$.) If we require that $\lambda$ is strictly dominant with respect to $\Psi^+$ (rather than just $P$), then representations will be in one-one correspondence with triples $(x, y, \lambda)$ up to conjugation by $H \times ^\vee H$.

Remark 2 DAV: Although this does indeed give a one-one correspondence, it is a bad idea (mathematically) to require $\lambda$ to be dominant with respect to the whole root system. Think about how else to account for equivalences by Weyl group elements taking $P$ into $\Psi^*...$

The parameters $x$ are as for integral data, and hence the Atlas software computes them (using ‘kbg’). Given a fixed $x$, what are the possible $y$ and $\lambda$?

Let $(x, y_I)$ be an integral pair, i.e., a pair giving an integral $L$-datum (listed in Atlas using the ‘block’ command). Any element $y$ normalizing $^\vee H$ and compatible with $x$ (i.e., mapping to the same twisted involution as $y_I$) is of the form $y = ty_I$ for some $t \in ^\vee H$. Then

$$y^2 = ty_Ity_I = t^\vee \theta(t)y_I^2. \quad (4)$$
Now $t^\nu X(t) = \exp(X)$ for some $X \in (1 + ^\nu X)^\vee \mathfrak{h} = ^\vee \mathfrak{h}^\nu$. Consequently, the infinitesimal characters $\lambda$ allowed (still assuming $x$ fixed) are those of the form

$$\lambda = \lambda_0 + \lambda_I,$$

where $\lambda_0 \in ^\vee \mathfrak{h}^\nu$ and $\lambda_I$ integral. \hfill (5)

**Proposition 3**  Fix $x \in \mathcal{X}$ (Fokko’s one-sided parameter space $[3]$).

1. The possible infinitesimal characters of representations associated to $x$ are those of the form $\lambda = \lambda_0 + \lambda_I$, where $\lambda_0 \in ^\vee \mathfrak{h}^\nu$, $\lambda_I$ is integral, i.e., $\exp(2\pi i \lambda_I) = y_I^2$ for some integral pair $(x, y_I)$, and $\lambda$ is regular dominant.

2. Suppose $\lambda = \lambda_0 + \lambda_I$ is as above (the decomposition is not unique; choose one). Write $t_{\lambda_0} = \exp(\pi i \lambda_0)$. If $^\vee G$ has trivial center then the representations with infinitesimal character $\lambda$ associated to $x$ are given by the pairs $(x, t_{\lambda_0} y)$ such that $(x, y)$ is an integral pair (i.e., fixed $x$, vary $y$).

**Example 4**  $SL(2, \mathbb{R})$. If $x = \pm t$ then $^\vee X = -X$ for $X \in ^\vee \mathfrak{h}$, so $^\vee \mathfrak{h}^\vee = \{0\}$, and there are no non-integral infinitesimal characters (as expected since $H(\mathbb{R})$ is compact). If $x = w$ then $H(\mathbb{R}) \simeq \mathbb{R}^2$, $^\vee \mathfrak{h} = 1$, $^\vee \mathfrak{h}^\nu = \mathfrak{h}$, and all (dominant) infinitesimal characters are allowed. Take $\lambda = \nu \rho$ for some $\nu > 0$, $\lambda = \text{diag}(\nu, -\nu, 0) \in ^\vee \mathfrak{h}$. Then $t_{\lambda_0} = \text{diag}(e^{\pi i \nu}, e^{-\pi i \nu}, 1)$, so we get

$$y_1 = t_{\lambda_0} I = \text{diag}(e^{\pi i \nu}, e^{-\pi i \nu}, 1)$$

for the spherical principal series, \hfill (6)

$$y_2 = t_{\lambda_0} s = \text{diag}(-e^{\pi i \nu}, -e^{-\pi i \nu}, 1) = \text{diag}(e^{\pi i (\nu+1)}, e^{-\pi i (\nu+1)}, 1)$$

for the nonspherical series. \hfill (7)

Notice that these reduce to the elements of Example 1 if $\nu$ is an integer.

**Remark 5**  If the center of $^\vee G$ is not trivial, there are fewer representations than there are integral pairs. Working guess for the general case (this works for $SO(2, 1)$, $SO(3, 2)$ and a split torus, e.g.): Fix $\lambda$ as in part 1 of Proposition 3 and a particular decomposition $\lambda = \lambda_0 + \lambda_I$, and write $z = \exp(2\pi i \lambda_I) \in Z(\mathcal{V}G)$. Then the representations with infinitesimal character $\lambda$ which are associated to $x$ are given by the pairs $(x, t_{\lambda_0} y)$ such that $(x, y)$ is an integral pair with $y^2 = z$.

**Example 6**  $SO(2, 1)$. This is the dual picture to $SL(2, \mathbb{R})$. The principal series are given by pairs $(x, y)$ with $x = M$, and there are four choices for $y : \pm 1, \pm t$. The first two satisfy $y^2 = I$, the second two $y^2 = -I$. For a given infinitesimal character $\nu \rho$, there are two non-isomorphic principal series of $SO(2, 1)$ parametrized by

$$(x, y_1) = (M, \text{diag}(e^{\pi i \frac{\nu}{2}}, e^{-\pi i \frac{\nu}{2}}))$$

\hfill (8)
and

\[(x, y_2) = (M, \text{diag}(-e^{\pi i \frac{x}{2}}, -e^{-\pi i \frac{x}{2}})),\]  

which we can get either by multiplying $\pm I$ by $t_{\lambda_0} = \text{diag} \left( e^{\pi i \frac{x}{2}}, e^{-\pi i \frac{x}{2}} \right)$, or by multiplying $\pm t$ by $t_{\lambda_0} = \text{diag} \left( e^{\pi i \frac{y+1}{2}}, e^{-\pi i \frac{y+1}{2}} \right)$.

### 3.2 Second Approach: Reduce to a Smaller Group $E$

Idea: An infinitesimal character $\lambda$ determines (as above) the sets $P, \vee P$. Then $(X, P, \vee X, \vee P)$ is a based root datum. The corresponding group $E$ (an endoscopic group for $G$) is not necessarily a subgroup of $G$; however, the dual group $\vee E$ is the subgroup of $\vee G$ corresponding to the subrootsystem $\vee P$ of $\vee \Psi^+$. The $x$’s for $G$ may be identified with certain $x_E$ for $E$, and integral pairs $(x_E, y)$ for $E$ should then parametrize representations for $G$ with infinitesimal character $\lambda$. Details need to be worked out; in particular this identification $x \mapsto x_E$, eliminating duplication, dealing with centers, and keeping track of the correct inner class and real forms in $E$. Stay tuned...

### References

