

THE GENUINE OMEGA-REGULAR UNITARY DUAL OF THE METAPLECTIC GROUP

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ABSTRACT. We classify all genuine unitary representations of the metaplectic group whose infinitesimal character is real and at least as regular as that of the oscillator representation.

1. INTRODUCTION

In [PPS1], we formulated a conjecture on the unitary dual of the metaplectic group, which gives a classification of all genuine unitary *omega-regular* representations. (Roughly, a representation is called “omega-regular” if its infinitesimal character is real and at least as regular as that one of the oscillator representation.) In this paper, we prove the conjecture. In particular, we show that all such representations are obtained by cohomological parabolic induction from the tensor product of a one-dimensional representation and an oscillator representation. This constitutes, for the case of genuine representations of the metaplectic group, a generalization of the main result of [S2], which asserts that, for real reductive Lie groups, every irreducible unitary representation with *strongly regular* infinitesimal character is cohomologically induced from a unitary character. Recall that an infinitesimal character is strongly regular if it is at least as regular as the infinitesimal character of the trivial representation. Here we will assume that the infinitesimal character is also real

Let $Mp(2n)$ be the metaplectic group of rank n , *i. e.*, the two-fold connected cover of the symplectic group $Sp(2n, \mathbb{R})$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a theta stable parabolic subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$, and let L be the Levi subgroup of $Mp(2n)$ corresponding to \mathfrak{l} . Then L is a (quotient of a) group of the form

$$\left(\prod_{i=1}^u \tilde{U}(p_i, q_i) \right) \times Mp(2m).$$

Let \mathbb{C}_λ be a genuine (unitary) character of $\prod_{i=1}^u \tilde{U}(p_i, q_i)$, and let ω be an irreducible summand of one of the two oscillator representations of $Mp(2m)$. In [PPS1], we showed that if $\Omega := \mathbb{C}_\lambda \otimes \omega$ is in the good range with respect to \mathfrak{q} (see Definition 2 in Section 2), then the representation

$$A_{\mathfrak{q}}(\Omega) := \mathcal{R}_{\mathfrak{q}}(\Omega)$$

of $Mp(2n)$ is genuine, nonzero, irreducible, unitary and ω -regular. Our main result is the following converse of this statement.

Date: September 2009.

This material is based on work supported by NSF grants DMS 0554278 and DMS 0201944.

Theorem 1. *Let X be a genuine admissible ω -regular and irreducible unitary representation of $Mp(2n)$. Then there exist a theta stable subalgebra \mathfrak{q} of $\mathfrak{sp}(2n, \mathbb{C})$, and a representation $\Omega = \mathbb{C}_\lambda \otimes \omega$ in the good range for \mathfrak{q} , such that $X \simeq A_{\mathfrak{q}}(\Omega)$.*

In [PPS1], we proved Theorem 1 for the case $n = 2$. The proof was a case by case calculation. For each (genuine) $\tilde{U}(2)$ -type μ which is the lowest K -type of an $A_{\mathfrak{q}}(\Omega)$ representation, we showed that there exists a unique unitary and ω -regular representation of $Mp(4)$ with lowest K -type μ . For each genuine $\tilde{U}(2)$ -type μ that is not the lowest K -type of an $A_{\mathfrak{q}}(\Omega)$ representation, we showed that every ω -regular representation of $Mp(4)$ with lowest $\tilde{U}(2)$ -type μ must be non-unitary. Both proofs are based on Parthasarathy's Dirac Operator Inequality (PDOI, [P]), see Lemma 6.1 of [S1]. This scheme worked for all genuine $\tilde{U}(2)$ -types, except for the (unique) fine $\tilde{U}(2)$ -type which occurs in the genuine non-pseudospherical principal series. In this case, we explicitly computed the intertwining operator giving the invariant Hermitian form on the representation space, and showed that its signature is indefinite. It is clear that this proof can not be generalized to arbitrary n .

The main idea of the proof for the general case is similar to the reduction argument used in [S1]; however, we also need some of the non-unitarity results and non-unitarity certificates obtained in [PPS2]. (The question of the unitarity of the ω -regular principal series of $Mp(2n)$ was the motivation for [PPS2].)

We sketch the proof of Theorem 1. Let X be a genuine admissible irreducible unitary representation of $Mp(2n)$. The first step is to realize X as the Langlands quotient of a module of the form

$$\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0),$$

where $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a theta stable subalgebra with Levi subgroup $L \simeq \tilde{U}(r, s) \times Mp(2d)$, X_1 is a genuine strongly regular irreducible representation of $\tilde{U}(r, s)$, and X_0 is the Langlands quotient of a genuine ω -regular principal series representation of $Mp(2d)$.

If $X_1 \otimes X_0$ is unitary, then the results of [S1] imply that $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$ for some theta stable parabolic subalgebra \mathfrak{q}_2 and some parameter λ_2 , and the results of [PPS2] tell us that X_0 must be the even half of an oscillator representation. In this case, by a version of induction by stages, X must be of the desired form.

If $X_1 \otimes X_0$ is not unitary, then there must be an $(L \cap K)$ -type μ^L which detects non-unitarity (in the sense that the invariant Hermitian form on μ^L is not positive definite). The results of [S1], which rely heavily on PDOI, together with the calculations in [PPS2] (see Lemma 5) give us specific information about what μ^L could look like. Recall that, associated to the cohomological induction functor $\mathcal{R}_{\mathfrak{q}}$, there is the bottom layer map, which takes $(L \cap K)$ -types (in $(X_1 \otimes X_0)$) to K -types (in $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$). Let μ be the image of μ^L under this map. If μ were nonzero, then by a theorem of Vogan, μ would occur in the irreducible Langlands quotient X of $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$ and would carry the same signature as μ^L ; hence X would be not unitary. Since we are assuming that X is unitary, we deduce that μ^L must be mapped to 0. It turns out that, in this case, there exists a different theta stable parabolic subalgebra \mathfrak{q}' with Levi subgroup $L' \simeq \tilde{U}(r-1, s) \times Mp(2d+2)$, such that X is the Langlands quotient of $\mathcal{R}_{\mathfrak{q}'}(X'_1 \otimes X'_0)$. Here X'_1 is an $A_{\mathfrak{q}_3}(\lambda_3)$ module

of $\tilde{U}(r-1, s)$, and X'_0 is the odd half of an oscillator representation of $Mp(2d+2)$. As in the previous case, using induction by stages, we get the desired result.

In [PPS1], we also classify the non-genuine unitary ω -regular representations of $Mp(4)$, *i. e.*, the ω -regular part of the unitary dual of $Sp(4, \mathbb{R})$. We plan to address the generalization of this result in a future paper.

Once again, we thank David Vogan and Jeffrey Adams for posing the problem, and for their patience and encouragement along the way.

2. DEFINITIONS AND MAIN RESULT

We begin with some notation. For a metaplectic group of rank r , we denote by ω an irreducible summand of an oscillator representation of the group, and by ω^r the corresponding infinitesimal character. For any compact connected group, we often identify irreducible representations of the group with their highest weight.

For $G = Mp(2n)$, set $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and let \mathfrak{t} be the complexified Lie algebra of a compact Cartan subgroup T of G . Let $\langle \cdot, \cdot \rangle$ denote a fixed non-degenerate G -invariant, θ -invariant symmetric bilinear form on \mathfrak{g}_0 , negative definite on \mathfrak{k}_0 and positive definite on \mathfrak{p}_0 , and use the same notation for its complexification, and its various restrictions and dualizations.

Definition 1. *Let $\gamma \in i\mathfrak{t}_0^*$. Choose a positive system $\Delta^+(\gamma) \subseteq \Delta(\mathfrak{g}, \mathfrak{t})$ such that $\langle \alpha, \gamma \rangle \geq 0$ for all $\alpha \in \Delta^+(\gamma)$, and let ω^n be the representative of the infinitesimal character of the oscillator representation of G which is dominant with respect to $\Delta^+(\gamma)$. We call γ “ ω -regular” if the following regularity condition is satisfied:*

$$(2.1) \quad \langle \alpha, \gamma - \omega^n \rangle \geq 0, \quad \forall \alpha \in \Delta^+(\gamma).$$

We say that a representation of G is ω -regular if its infinitesimal character is.

Given a θ -stable parabolic subalgebra \mathfrak{q} of \mathfrak{g} , let $\mathcal{R}_{\mathfrak{q}}^i$ be the functors of cohomological parabolic induction carrying $(\mathfrak{l}, L \cap K)$ -modules to (\mathfrak{g}, K) -modules (*cf.* [V], Def. 6.3.1). The restriction of these functors to K is denoted $(\mathcal{R}_{\mathfrak{q}}^K)^i$.

Definition 2. *An $A_{\mathfrak{q}}(\Omega)$ representation is a genuine representation of G of the following form. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a theta stable parabolic subalgebra of \mathfrak{g} , with corresponding Levi subgroup L of G . Then*

$$(2.2) \quad L \cong \left[\prod_{i=1}^u \tilde{U}(p_i, q_i) \right] \times Mp(2m).$$

(More precisely, L is a quotient by a finite group of a group of the above form; the factors intersect in a group of order 2 in the center.) Let \mathbb{C}_{λ} be a genuine one-dimensional representation of $\left[\prod_{i=1}^u \tilde{U}(p_i, q_i) \right]$ and let ω be an irreducible summand of an oscillator representation of $Mp(2m)$. We define

$$A_{\mathfrak{q}}(\Omega) := \mathcal{R}_{\mathfrak{q}}(\Omega).$$

Remark 1. *Our (new) definition of $A_{\mathfrak{q}}(\Omega)$ is slightly different from the one given in [PPS1], because we do not require Ω to be in the good range for \mathfrak{q} .*

Remark 2. *In Definition 2, m is allowed to be equal to 0 or n ; in these cases, we obtain an $A_{\mathfrak{q}}(\lambda)$ representation or an oscillator representation, respectively.*

Proposition 1 (Proposition 3 of [PPS1]). *Retain the notation of Definition 2. Assume that the representation $\Omega = \mathbb{C}_\lambda \otimes \omega$ of L is in the good range for \mathfrak{q} , i. e., that the infinitesimal character γ^Ω of Ω is such that $\gamma^\Omega + \rho(\mathfrak{u})$ is strictly dominant with respect to the roots of \mathfrak{u} . Then every $A_{\mathfrak{q}}(\Omega)$ representation is genuine, irreducible, nonzero, unitary and ω -regular.*

In this paper, we prove the following converse of Proposition 1.

Theorem 2. *Let X be a genuine ω -regular irreducible representation of $Mp(2n)$ with a nonzero invariant Hermitian form $\langle \cdot, \cdot \rangle_G$. Then either*

- (1) $X \cong A_{\mathfrak{q}}(\Omega)$, for some theta stable parabolic subalgebra \mathfrak{q} of \mathfrak{g} , and some representation $\Omega = \mathbb{C}_\lambda \otimes \omega$, in the good range for \mathfrak{q} , of the Levi subgroup corresponding to \mathfrak{q} ; or
- (2) The representation X is not unitary.

We will prove Theorem 2 in the next section.

3. PROOF OF THEOREM 2

The proof of Theorem 2 relies on a series of auxiliary lemmas and propositions. We will state these results as needed along the way but, whenever a proof is too long, it will be postponed to a later section.

Fix X as in the theorem, i. e., let X be a genuine irreducible ω -regular Hermitian representation of $Mp(2n)$. Let

$$(3.1) \quad (\lambda_a, \mathfrak{q}_a, L_a, X^{L_a})$$

be the “ θ -stable parameters” associated to X via Vogan’s classification of admissible representations, so that X is the unique Langlands quotient of $\mathcal{R}_{\mathfrak{q}_a}^{S_a}(X^{L_a})$, with $S_a = \dim(\mathfrak{u}_a \cap \mathfrak{k})$ (cf. [V1]). If

$$(3.2) \quad \lambda_a = \left(\underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_t, \dots, g_t}_{r_t} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{-g_t, \dots, -g_t}_{s_t}, \dots, \underbrace{-g_1, \dots, -g_1}_{s_1} \right)$$

with $g_1 > \dots > g_t > 0$, then we can write

$$(3.3) \quad L_a \simeq \left[\prod_{i=1}^t \tilde{U}(r_i, s_i) \right] \times Mp(2d),$$

where $L_a = \text{Cent}_G(\lambda_a)$ and each factor is quasisplit.

For $a > 0$, the parameter

$$(3.4) \quad \xi = \left(\underbrace{a, a, \dots, a}_r \mid \underbrace{0, 0, \dots, 0}_d \mid \underbrace{-a, -a, \dots, -a}_s \right)$$

is a singularization of λ_a (cf. [SaV]). Here $r = \sum_{i=1}^t r_i$ and $s = \sum_{i=1}^t s_i$. Set

$$L = \text{Cent}_G(\xi) \simeq \tilde{U}(r, s) \times Mp(2d)$$

and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Then $L \supseteq L_a$, $\mathfrak{l}_a \subseteq \mathfrak{l}$, $\mathfrak{u}_a = \mathfrak{u} + (\mathfrak{l} \cap \mathfrak{u}_a)$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supseteq \mathfrak{q}_a$.

Proposition 2 below lists a few results regarding the functors $\mathcal{R}_{\mathfrak{q}}^S$ of cohomological induction and their restriction to K (see [V1] and [KV] for proofs). These results already appear in [SaV], but are stated there for the functors $\mathcal{L}_{\mathfrak{q}}^S$ instead of $\mathcal{R}_{\mathfrak{q}}^S$.

Proposition 2. *Suppose that X is an irreducible admissible (\mathfrak{g}, K) -module of $Mp(2n)$ and that μ is a highest weight of a lowest $\tilde{U}(n)$ -type of X . Define L and \mathfrak{q} as above, and let $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. Then*

- (1) *There is a unique irreducible $(\mathfrak{l}, L \cap K)$ -module X^L associated to X so that X can be realized as the unique Langlands quotient of*

$$\mathcal{R}_{\mathfrak{q}}^S(X^L).$$

- (2) *If μ^L is the highest weight of an irreducible representation V_{μ^L} of $L \cap \tilde{U}(n)$ and $\mu \equiv \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k}, \mathfrak{t})$ -dominant, then every irreducible constituent of $(\mathcal{R}_{\mathfrak{q}}^K)^S(V_{\mu^L})$ has highest weight μ . If μ is not dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$, then $(\mathcal{R}_{\mathfrak{q}}^K)^S(V_{\mu^L}) = 0$.*
- (3) *There is an injective map, the bottom layer map, of K -representations*

$$\mathcal{B}_{X^L} : (\mathcal{R}_{\mathfrak{q}}^K)^S(X^L) \rightarrow \mathcal{R}_{\mathfrak{q}}^S(X^L).$$

Moreover, there is a one-to-one correspondence (with multiplicities) between the lowest $\tilde{U}(n)$ -types of X and the lowest $(L \cap \tilde{U}(n))$ -types of X^L .

- (4) *The Hermitian dual $(X^L)^h$ of X^L is endowed with a non-zero invariant Hermitian form $\langle \cdot, \cdot \rangle^L$.*
- (5) *The bottom layer map is unitary. Therefore, on the $\tilde{U}(n)$ -types, the signature of the Hermitian form on X agrees with the signature on X^L .*
- (6) *If γ^{X^L} is a representative of the infinitesimal character of X^L , then*

$$\gamma^X = \gamma^{X^L} + \rho(\mathfrak{u})$$

is a representative of the infinitesimal character of X .

Proof. Because ξ in (3.4) is a singularization of λ_a , these facts follow from Lemma 2.7, Propositions 2.8, 2.10 and 2.12 and Theorem 2.13 in [SaV]. For example, part (1) and the second part of (3) follow from part (2) of Theorem 2.13 and Proposition 2.12. Part (6) is Proposition 6.3.11 of [V1]. \square

Lemma 1. *Retain the notation of Proposition 2. Set $L = L_1 \times L_0$ with $L_0 = Mp(2d)$ and $L_1 = \tilde{U}(r, s)$, and write $X^L \simeq X_1 \otimes X_0$, with X_i an $(\mathfrak{l}_i, L_i \cap K)$ -module ($i = 0, 1$). If X is ω -regular, then X_1 is strongly regular for L_1 and X_0 is ω -regular for L_0 .*

Proof. Let $\gamma^{X^L} = (\gamma^{X_1}, \gamma^{X_0})$ be (a representative of) the infinitesimal character of X^L , and write

$$(3.5) \quad \gamma^X = \gamma^{X^L} + \rho(\mathfrak{u})$$

as in part (6) of Proposition 2. Choose a positive system Δ^+ of roots so that γ^X is dominant with respect to Δ^+ , and choose a representative of the infinitesimal character ω^n which is dominant with respect to Δ^+ . Since X is ω -regular, we have

$$(3.6) \quad \langle \gamma^X, \alpha \rangle \geq \langle \omega^n, \alpha \rangle$$

for all $\alpha \in \Delta^+$. If we normalize our form so that $\langle \alpha, \alpha \rangle = 2$ for all short roots α , then this is equivalent to saying that

$$(3.7) \quad \langle \gamma^X, \alpha \rangle \geq 1$$

for all $\alpha \in \Delta^+$. Now choose

$$(3.8) \quad \Delta^+(\mathfrak{l}) = \Delta^+ \cap \Delta(\mathfrak{l}).$$

Since $\rho(\mathfrak{u})$ is orthogonal to the roots of \mathfrak{l} , we have

$$(3.9) \quad \langle \gamma^{X^L}, \alpha \rangle \geq 1$$

for all $\alpha \in \Delta^+(\mathfrak{l})$. Moreover, the roots of \mathfrak{l}_1 are orthogonal to γ^{X_0} , and similarly for the roots of \mathfrak{l}_0 and γ^{X_1} . Therefore, X_0 is ω -regular. For $L_1 = \tilde{U}(r, s)$, the property

$$(3.10) \quad \langle \gamma^{X_1}, \alpha \rangle \geq 1$$

for all roots $\alpha \in \Delta^+(\mathfrak{l}) \cap \Delta(\mathfrak{l}_1)$ implies that X_1 is strongly regular. \square

By virtue of Proposition 2 and Lemma 1, we can assume that our genuine irreducible ω -regular Hermitian representation X is (the unique Langlands quotient of) a representation of the form

$$\mathcal{R}_{\mathfrak{q}}^S(X_1 \otimes X_0)$$

with X_1 a genuine strongly regular $(\mathfrak{l}_1, L_1 \cap K)$ -module for $L_1 = \tilde{U}(r, s)$, and X_0 a genuine ω -regular $(\mathfrak{l}_0, L_0 \cap K)$ -module for $L_0 = Mp(2d)$. The following proposition shows that, in this setting, either X_1 is a good $A_{\mathfrak{q}_2}(\lambda_2)$ module or X is not unitary.

Proposition 3. *Let X be an irreducible admissible (\mathfrak{g}, K) -module of $Mp(2n)$. Assume that X is genuine and ω -regular; realize X as the unique Langlands quotient of $\mathcal{R}_{\mathfrak{q}}^S(X^L)$, as in Proposition 2, and write*

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad X^L \simeq X_1 \otimes X_0,$$

as in Lemma 1. Suppose that $r + s \neq 0$. Then one of the following (mutually exclusive) options occur:

- (1) There exist a theta stable parabolic subalgebra \mathfrak{q}_2 and a representation \mathbb{C}_{λ_2} of the Levi factor corresponding to \mathfrak{q}_2 , with λ_2 in the good range for \mathfrak{q}_2 , such that $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$.
- (2) The representation X is not unitary.

Remark 3. Recall that, by a result of Speh and Vogan, if a representation of the Levi factor is irreducible, unitary and in the good range for \mathfrak{q} , then the derived functor module is irreducible, unitary and non-zero (cf. [SpV], or Theorem 2.13 (f) of [SaV]).

Using Proposition 3, we will prove the following result.

Proposition 4. *In the setting of Proposition 3, assume that $d > 0$. Then one of the following (mutually exclusive) options occurs:*

- (1) X_0 is an even oscillator representation with lowest K -type (LKT)

$$\mu_0 = \pm(1/2, 1/2, \dots, 1/2).$$

- (2) There exist a subgroup $L' = \tilde{U}(r', s') \times Mp(2(d+1))$ also containing L_a , a theta stable subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \supseteq \mathfrak{q}$ and representations X'_i of L'_i ($i = 0, 1$), such that
 - (a) $\tilde{U}(r', s') \subset \tilde{U}(r, s)$ (with either $r = r' + 1$ or $s = s' + 1$),
 - (b) X'_1 is a good $A_{\mathfrak{q}_3}(\lambda_3)$,

- (c) X'_0 is an odd oscillator representation with $LKT \mu'_0 = (3/2, 1/2, \dots, 1/2)$ or $(-1/2, -1/2, \dots, -1/2, -3/2)$, and
- (d) X can be realized as the unique Langlands quotient of $\mathcal{R}_{\mathfrak{q}'}^{S'}(X'_1 \otimes X'_0)$.
- (3) X is not unitary.

Putting all these results together, we conclude every genuine irreducible ω -regular unitary representation X of $Mp(2n)$ satisfies one of the following two options:

- (1) X is (the unique Langlands quotient of) a representation of the form

$$\mathcal{R}_{\mathfrak{q}}^S(X^L)$$

with $L = \tilde{U}(r, s) \times Mp(2d)$, $X^L = X_1 \otimes X_0$ and

- X_1 a good $A_{\mathfrak{q}_2}(\lambda_2)$ module for $\tilde{U}(r, s)$
- X_0 an even oscillator representation of $Mp(2d)$ with lowest K -type $\mu_0 = \pm(1/2, 1/2, \dots, 1/2)$.

- (2) X is (the unique Langlands quotient of) a representation of the form

$$\mathcal{R}_{\mathfrak{q}'}^{S'}(X^{L'})$$

with $L' = \tilde{U}(r', s') \times Mp(2(d+1))$, $X^{L'} = X'_1 \otimes X'_0$ and

- X'_1 a good $A_{\mathfrak{q}_3}(\lambda_3)$ module for $\tilde{U}(r', s')$
- X'_0 an odd oscillator representation of $Mp(2(d+1))$ with lowest K -type $(3/2, 1/2, \dots, 1/2)$ or $(-1/2, -1/2, \dots, -1/2, -3/2)$.

In order to finish the proof of Theorem 2, we must show that (in both cases) X can also be realized as an $A_{\mathfrak{q}}(\Omega)$ representation, for some theta stable parabolic subalgebra \mathfrak{q} and some representation $\Omega = \mathbb{C}_{\lambda} \otimes \omega$ in the good range for \mathfrak{q} . For this, we need two more results.

Proposition 5. *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ -stable parabolic subalgebra. Assume that $L = N_G(\mathfrak{q})$ is a direct product of two reductive subgroups $L = L_1 \times L_0$. (Here L can be of the form $\tilde{U}(r, s) \times Mp(2d)$, as in Proposition 2, or $\tilde{U}(r', s') \times Mp(2(d+1))$, as in Proposition 4.) Suppose further that we have a representation*

$$X \simeq \mathcal{R}_{\mathfrak{q}}(A_{\mathfrak{q}'}(\lambda') \otimes \omega)$$

where ω is an irreducible summand of an oscillator representation, $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subseteq \mathfrak{l}_1$, and $A_{\mathfrak{q}'}(\lambda')$ is good for \mathfrak{q}' . Then there is a theta stable parabolic subalgebra $\mathfrak{q}_{\omega} = \mathfrak{l}_{\omega} + \mathfrak{u}_{\omega}$ of \mathfrak{g} such that

$$X \simeq \mathcal{R}_{\mathfrak{q}_{\omega}}(\mathbb{C}_{\lambda'} \otimes \omega) = A_{\mathfrak{q}_{\omega}}(\Omega)$$

with $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$. (See Definition 2.)

Proof. We will use cohomological induction functors for different groups; we distinguish them by using different subscripts on their (complexified) Lie algebras. Assume that $X \simeq \mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}(A_{\mathfrak{q}'}(\lambda') \otimes \omega)$, where $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subseteq \mathfrak{l}_1$. Set

$$\mathfrak{q}_b = \mathfrak{q}' + \mathfrak{l}_0 = (\mathfrak{l}' + \mathfrak{u}') \oplus \mathfrak{l}_0.$$

Then \mathfrak{q}_b is a theta stable parabolic subalgebra of \mathfrak{l} , and

$$(3.11) \quad A_{\mathfrak{q}'}(\lambda') \otimes \omega \simeq \mathcal{R}_{\mathfrak{q}'}^{\mathfrak{l}_1}(\mathbb{C}_{\lambda'}) \otimes \mathcal{R}_{\mathfrak{l}_0}^{\mathfrak{l}_0}(\omega) \simeq \mathcal{R}_{\mathfrak{q}_b}^{\mathfrak{l}}(\mathbb{C}_{\lambda'} \otimes \omega).$$

Therefore,

$$X \simeq \mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{R}_{\mathfrak{q}_b}^{\mathfrak{l}}(\mathbb{C}_{\lambda'} \otimes \omega)).$$

Now set $\mathfrak{q}_\omega = \mathfrak{q}_b + \mathfrak{u} (= (\mathfrak{l}' \oplus \mathfrak{l}_0) + \mathfrak{u}' + \mathfrak{u})$. Note that \mathfrak{q}_ω is a parabolic subalgebra of \mathfrak{g} . Since $X_1 \cong A_{\mathfrak{q}'}(\lambda')$, by Proposition 3.5 in [S2], we know that $\mathfrak{l}' \supseteq \mathfrak{l}_a \cap \mathfrak{l}_1$ and $\mathfrak{u}' \subseteq \mathfrak{u}_a \cap \mathfrak{l}_1$. We have

$$\mathfrak{l}_\omega = \mathfrak{l}' \oplus \mathfrak{l}_0 \subseteq \mathfrak{l} \quad \text{and} \quad \mathfrak{u} \subseteq \mathfrak{u}' + \mathfrak{u} = \mathfrak{u}_\omega,$$

and

$$\mathfrak{q}_\omega \subseteq \mathfrak{q}.$$

Hence, by induction in stages (*cf.* [V1], Proposition 6.3.6), we find that

$$\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{R}_{\mathfrak{q}_b}^{\mathfrak{l}}(\mathbb{C}_{\lambda'} \otimes \omega)) \simeq \mathcal{R}_{\mathfrak{q}_\omega}^{\mathfrak{g}}(\mathbb{C}_{\lambda'} \otimes \omega) = A_{\mathfrak{q}_\omega}(\Omega),$$

and Proposition 5 is proved. \square

Finally, we need to show that our representation Ω is indeed in the good range for \mathfrak{q}_ω .

Proposition 6. *In the same setting of Proposition 5, let X be the irreducible (Langlands quotient of the) representation $\mathcal{R}_{\mathfrak{q}_\omega}^{\mathfrak{g}}(\mathbb{C}_{\lambda'} \otimes \omega)$. Assume, moreover, that λ' is in the good range for \mathfrak{q}' . Then $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$ is in the good range for \mathfrak{q}_ω .*

This concludes the proof of Theorem 2.

The proofs of Propositions 3, 4 and 6 will be given in the next few sections.

4. PROOF OF PROPOSITION 3.

In this section, we prove the following proposition.

Proposition. *Let X be an irreducible admissible (\mathfrak{g}, K) -module of $Mp(2n)$. Assume that X is genuine and ω -regular; realize X as the unique Langlands quotient of $\mathcal{R}_{\mathfrak{q}}^S(X^L)$, as in Proposition 2, and write*

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad X^L \simeq X_1 \otimes X_0,$$

as in Lemma 1. Suppose that $r + s \neq 0$. Then one of the following (mutually exclusive) options occur:

- (1) *There exist a theta stable parabolic subalgebra \mathfrak{q}_2 and a representation \mathbb{C}_{λ_2} of the Levi factor corresponding to \mathfrak{q}_2 , with λ_2 in the good range for \mathfrak{q}_2 , such that $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$.*
- (2) *The representation X is not unitary.*

Let μ be a lowest K -type of X . By Proposition 2 (a), X can be realized as the unique Langlands quotient of $\mathcal{R}_{\mathfrak{q}}^S(X^L)$. Write $X^L \simeq X_1 \otimes X_0$, as in Lemma 1. Note that, by Proposition 2, the $(\mathfrak{l}, L \cap K)$ -module X^L is irreducible, and so are the Harish-Chandra modules X_i of L_i ($i = 0, 1$). Let μ^L be the $(L \cap K)$ -type of X^L corresponding to μ via the bottom layer map (*cf.* Proposition 2 (2)-(3)); then

$$(4.1) \quad \mu^L = \mu_1 + \mu_0$$

with $\mu_i = \mu^{L_i}$ a lowest $(L_i \cap K)$ -type of X_i . By Lemma 1, the representation X_1 of $L_1 = \tilde{U}(r, s)$ is strongly regular (and irreducible).

By Theorem 1.2 in [S1] for the case $G = SU(p, q)$, if μ_1 is not the lowest $(L_1 \cap K)$ -type of a good $A_{\mathfrak{q}}(\lambda)$ module, then there exists an $(L_1 \cap K)$ -type η_1 such that

$$(4.2) \quad \eta_1 = \mu_1 + \beta$$

for some $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$, and the form

$$(4.3) \quad \langle \cdot, \cdot \rangle^{L_1} |_{V(\mu_1) \oplus V(\eta_1)}$$

is indefinite.

Denote the coordinates of μ by

$$(4.4) \quad \mu = \left(\underbrace{a_1, \dots, a_1}_{r_1}, \dots, \underbrace{a_t, \dots, a_t}_{r_t} \mid \underbrace{c_1, \dots, c_d}_d \mid \underbrace{b_t, \dots, b_t}_{s_t}, \dots, \underbrace{b_1, \dots, b_1}_{s_1} \right)$$

according to how they break into the subgroup $L_a = \left[\prod_{i=1}^t \tilde{U}(r_i, s_i) \right] \times Mp(2d)$.

Then, if X_1 is not a good $A_{\mathfrak{q}}(\lambda)$ and $\eta = \mu + \beta = (\mu_1 + \beta) + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is also dominant, by Proposition 2 (3)-(5), η also occurs in X and, by the ‘‘bottom layer’’ argument, the Hermitian form changes sign there. Considering the types of roots β that would be needed (they are short roots), we can conclude that X is not unitary unless

$$(4.5) \quad d = 0 \quad \text{and} \quad a_t = b_t,$$

or

$$(4.6) \quad a_t = c_1 \quad \text{or} \quad c_d = b_t.$$

Remark 4. *Note that the two identities in equation (4.6) can not hold simultaneously. One can check this by looking closely at the algorithm that associates L_a to μ (cf. [V1], Section 5.3.3). We will give an alternative concrete proof in Lemma 3 (c) below. Similarly, (4.5) is not possible.*

Without loss of generality we can now assume that $d > 0$ and

$$(4.7) \quad a_t = c_1.$$

To continue with the proof of Proposition 3, we will focus on what happens around the $Mp(2d)$ factor of L_a , along with the last $\tilde{U}(r_t, s_t)$ factor. In particular, we will concentrate on the coordinates a_t and c_1 of μ . Note that, because $\tilde{U}(r_t, s_t)$ is quasisplit, it must be of the form $\tilde{U}(s_t + \varepsilon, s_t)$ with $\varepsilon = 0, \pm 1$.

We will need two lemmas.

Lemma 2. *Let X be an irreducible admissible (\mathfrak{g}, K) -module of $Mp(2n)$, and let μ be a lowest $\tilde{U}(n)$ -type of X . Let λ_a and $\mathfrak{q}_a = \mathfrak{l}_a + \mathfrak{u}_a$ be the Vogan classification parameter and the theta stable parabolic subalgebra associated to X , respectively, as in (3.1). Then μ is of the form*

$$(4.8) \quad \mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a}$$

where δ^{L_a} is a fine $(L_a \cap K)$ -type.

This is a known result due to Vogan. It follows from Proposition 2 (b)-(c) (when $\mathfrak{q} = \mathfrak{q}_a$) and from the fact that (the highest weight of) the representation μ^{L_a} in that Proposition is of the form $\lambda_a - \rho(\mathfrak{u}_a)$ plus a fine $(L_a \cap K)$ -type; see Proposition 3.2.7 of [Pa1] for a detailed proof in the case of $G = U(p, q)$.

Lemma 3. *In the setting of Lemma 2, write the parameter λ_a as in equation (3.2), and assume that the K -type μ satisfies condition (4.7). Then*

$$(a) \quad g_t = \frac{1}{2} \quad \text{and} \quad \tilde{U}(s_t + \varepsilon, s_t) \quad \text{has} \quad \varepsilon = 0 \quad \text{or} \quad 1.$$

(b) If $\varepsilon = 0$, then δ^{L_a} must be of the form

$$(4.9) \quad \delta^{L_a} = \left(\underbrace{\dots, -\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t}, \dots \right)$$

with $p > 0$.

(c) $c_d > b_t$, and $a_t > b_t$.

Proof. Recall that μ satisfies equation (4.8). Write

$$\begin{aligned} \mu &= \left(\underbrace{a_1, \dots, a_1}_{r_1}, \dots, \underbrace{a_t, \dots, a_t}_{r_t} \middle| \underbrace{c_1, \dots, c_d}_d \middle| \underbrace{b_t, \dots, b_t}_{s_t}, \dots, \underbrace{b_1, \dots, b_1}_{s_1} \right), \\ \lambda_a &= \left(\underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_t, \dots, g_t}_{r_t} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{-g_t, \dots, -g_t}_{s_t}, \dots, \underbrace{-g_1, \dots, -g_1}_{s_1} \right), \\ \delta^{L_a} &= (\dots, y | z, \dots, a | b, \dots) \end{aligned}$$

and

$$\rho(\mathbf{u}_a \cap \mathfrak{p}) - \rho(\mathbf{u}_a \cap \mathfrak{k}) = \left(\dots, \underbrace{f_i, f_i, \dots, f_i}_{s_i + \varepsilon_i} \middle| \underbrace{c, c, \dots, c}_d \middle| \dots, \underbrace{h_i, h_i, \dots, h_i}_{s_i}, \dots \right).$$

Here

$$(4.10) \quad \begin{aligned} f_i &= \sum_{j < i} (r_j - s_j) + \frac{\varepsilon_i + 1}{2} \\ h_i &= \sum_{j < i} (r_j - s_j) + \frac{\varepsilon_i - 1}{2} \\ c &= r - s. \end{aligned}$$

Then

$$\begin{aligned} a_t &= g_t + (r - s) - (r_t - s_t) + \frac{\varepsilon_t + 1}{2} + y = g_t + (r - s) + \frac{1}{2}(1 - \varepsilon) + y, \\ c_1 &= (r - s) + z, \\ c_d &= (r - s) + a, \\ b_t &= -g_t + (r - s) - (r_t - s_t) + \frac{\varepsilon_t - 1}{2} + b = -g_t + (r - s) + \frac{1}{2}(-1 - \varepsilon) + b, \end{aligned}$$

with $\varepsilon = \varepsilon_t = r_t - s_t$. We obtain:

$$(4.11) \quad a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon) + y - z$$

and

$$(4.12) \quad c_d - b_t = g_t + \frac{1}{2}(1 + \varepsilon) + a - b.$$

The coordinate g_t of λ_a is either an integer or a half-integer. We distinguish the two cases.

(1) First, assume that $g_t \in \mathbb{Z}$.

(a) If $\varepsilon = 0$, then

$$\begin{cases} a_t \in \mathbb{Z} + \frac{1}{2} + y \\ c_1 \in \mathbb{Z} + z. \end{cases}$$

In order for μ to be genuine, the coordinates of the fine $(L_a \cap K)$ -type δ^{L_a} must satisfy:

$$\begin{cases} y \in \mathbb{Z} \\ z \in \mathbb{Z} + \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ z = \pm \frac{1}{2}. \end{cases}$$

This says that the representation δ^{L_a} of $(L_a \cap K)$ is trivial on the $\tilde{U}(s_t, s_t)$ -factor of L_a , and non-trivial on the $Mp(2d)$ -factor:

$$\delta^{L_a} = \left(\dots, \underbrace{0, \dots, 0}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{0, \dots, 0}_{s_t}, \dots \right).$$

Equation (4.11) then gives:

$$a_t - c_1 = \begin{cases} g_t & \text{if } p > 0, \\ g_t + 1 & \text{if } p = 0. \end{cases}$$

We reach a contradiction, because μ is supposed to satisfy condition (4.7) and $g_t > 0$. Hence, case 1 (a) cannot hold.

(b) If $\varepsilon \neq 0$, then

$$\begin{cases} a_t \in \mathbb{Z} + y \\ c_1 \in \mathbb{Z} + z. \end{cases}$$

In order for μ to be genuine, the fine representation δ^{L_a} of $(L_a \cap K)$ should be non-trivial on both the $\tilde{U}(s_t + \varepsilon, s_t)$ -factor and the $Mp(2d)$ -factor of L_a . Again, we reach a contradiction, because if $\varepsilon \neq 0$ there are no non-trivial fine representations (on the semisimple part) of $\tilde{U}(s_t + \varepsilon, s_t)$. We conclude that case 1 (b) cannot hold, hence $g_t \notin \mathbb{Z}$.

(2) Now, assume that $g_t \in \mathbb{Z} + \frac{1}{2}$.

(a) If $\varepsilon = 0$, the fine representation δ^{L_a} of $(L_a \cap K)$ must be non-trivial on both the $\tilde{U}(s_t, s_t)$ -factor and the $Mp(2d)$ -factor of L_a . Then either

$$(4.13) \quad \delta^{L_a} = \left(\dots, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t}, \dots \right)$$

or

$$(4.14) \quad \delta^{L_a} = \left(\dots, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{s_t}, \dots \right)$$

for some $p, q \geq 0$ such that $p + q = d$. Equation (4.14) contradicts condition (4.7), because the difference $a_t - c_1$ is always positive, hence

equation (4.13) must hold. We find:

$$a_t - c_1 = 0 \iff g_t = \frac{1}{2} \quad \text{and} \quad p > 0.$$

Note that, in this case, $c_d - b_t > 0$ (by equation (4.12)).

- (b) Finally, if $\varepsilon \neq 0$ and $g_t \in \mathbb{Z} + \frac{1}{2}$, the fine representation δ^{L_a} of $(L_a \cap K)$ must be trivial on the $\tilde{U}(s_t + \varepsilon, s_t)$ -factor of L_a and non-trivial on the $Mp(2d)$ -factor of L_a :

$$(4.15) \quad \delta^{L_a} = \left(\underbrace{\dots, 0, \dots, 0}_{s_t} \left| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \right| \underbrace{0, \dots, 0}_{s_t}, \dots \right).$$

Just as above, we get that

$$a_t - c_1 = 0 \iff g_t = \frac{1}{2}, \quad \varepsilon = 1 \quad \text{and} \quad p > 0.$$

Moreover, $c_d - b_t > 0$ always in this case. This also implies that if $d = 0$ then $a_t > b_t$.

This concludes the proof of Lemma 3. \square

To continue with the proof of Proposition 3, we first look at the case in Lemma 3 when $\varepsilon = 1$ and $g_t = \frac{1}{2}$. Write $\gamma^X = (\lambda_a, \nu)$ for the infinitesimal character of X . Because γ^X is ω -regular, and $\frac{1}{2}$ occurs in λ_a , this entry can not occur in ν . Write $X^L = X_1 \otimes X_0$ as in Lemma 1. Note that $L_0 = Mp(2d)$ is the quasisplit Mp -factor of L_a (see equation (3.3)), hence X_0 is a principal series representation of $Mp(2d)$.

We need to recall a few results about genuine minimal principal series of $Mp(2d)$.

Lemma 4. (cf. [PPS2])

- (1) *Every genuine minimal principal series representation of $Mp(2d)$ has a unique lowest $\tilde{U}(d)$ -type of the form*

$$\mu_{\delta_{p,q}} = \left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \right)$$

with $p + q = d$. Here $\delta_{p,q}$ is the character of the finite subgroup M of $Mp(2d)$ used to construct the representation.

- (2) *For every pair of non-negative integers p and q with $p+q = d$, the Langlands quotients of genuine minimal principal series of $Mp(2d)$ with lowest $\tilde{U}(d)$ -type $\mu_{\delta_{p,q}}$ are parametrized by d -tuples of real numbers $\nu = (\nu_1, \dots, \nu_d) := (\nu^p | \nu^q)$, where $\nu_1 \geq \nu_2 \geq \dots \geq \nu_p \geq 0$ and $\nu_{p+1} \geq \nu_{p+2} \geq \dots \geq \nu_{p+q} \geq 0$. The infinitesimal character of the corresponding representation is then ν .*
- (3) *Irreducible genuine pseudospherical representations of $Mp(2d)$, i. e., those with lowest $\tilde{U}(d)$ -type $\mu_{\delta_{a,0}}$ or $\mu_{\delta_{0,a}}$, are uniquely determined by their infinitesimal character.*

The following lemma summarizes the non-unitarity results for genuine minimal principal series of $Mp(2d)$ that are needed for this paper.

Lemma 5. *Let $J(\delta_{p,q}, \nu)$ be the Langlands quotient of a genuine minimal principal series representation of $Mp(2d)$, with lowest $\tilde{U}(d)$ -type $\mu_{\delta_{p,q}}$ and parameter $\nu = (\nu^p | \nu^q)$ as in Lemma 4.*

- (1) *If $\nu_p > \frac{1}{2}$ or if $\nu_i - \nu_{i+1} > 1$ for some $1 \leq i \leq p-1$, then X_0 is not unitary, and the $\tilde{U}(d)$ -type*

$$\delta_1 = \left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q, -\frac{3}{2} \right)$$

detects non-unitarity.

- (2) *If $\nu_{p+q} > \frac{1}{2}$ or if $\nu_i - \nu_{i+1} > 1$ for some $p+1 \leq i \leq p+q-1$, then X_0 is not unitary, and the $\tilde{U}(d)$ -type*

$$\delta_2 = \left(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-1} \right)$$

detects non-unitarity.

- (3) *If X_0 is ω -regular, $pq \neq 0$, $\nu^p \neq \omega^p$ and $\nu^q \neq \omega^q$, then both δ_1 and δ_2 detect non-unitarity. As usual, ω^r denotes the infinitesimal character of the oscillator representation of $Mp(2r)$.*

Before proving this lemma, we finish the proof of Proposition 3.

Suppose that ε in Lemma 3 equals 1 (hence $p > 0$ and $g_t = \frac{1}{2}$), then $\frac{1}{2}$ does not occur as a coordinate of ν . Because $\nu|_{L_0}$ is ω -regular for L_0 , by part(1) of Lemma 5, we know that the representation X_0 of $L_0 = Mp(2d)$ is not unitary, and the signature of the Hermitian form changes on δ_1 . Then, if μ_1 is as in equation (4.1), the weight

$$\eta = \mu_1 + \delta_1 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$$

is K -dominant. By Proposition 2 (c) and (e), our original representation X is not unitary.

Now, let $\varepsilon = 0$ (so that we are in case (b) of Lemma 3). Recall that the fine K -type δ^{L_a} is of the form (4.9); in particular, its restriction to the subgroup $\tilde{U}(s_t, s_t)$ of L_a is given by

$$\delta^{L_a}|_{\tilde{U}(s_t, s_t)} = \left(0, \dots, 0, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t}, 0, \dots, 0 \right).$$

Hence, the representation $\mu_1|_{\tilde{U}(s_t, s_t)}$, inside $Mp(2n)$, is of the form:

$$\left(0, \dots, 0, \underbrace{a, \dots, a}_{s_t} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{-a-1, \dots, -a-1}_{s_t}, 0, \dots, 0 \right)$$

for some $a \in \frac{1}{2}\mathbb{Z}$.

By Lemma 8.8 and Lemma 6.3 (b) in [S1], if $\beta = (1, 0 \dots 0; 0, \dots 0, -1)$ is the root

$\epsilon_1 - \epsilon_{2s_t}$ in $U(s_t, s_t)$, then $\mu_1|_{U(s_t, s_t)} + \beta$ detects the signature change and survives in the bottom layer. Note that the root in $Mp(2n, \mathbb{R})$ corresponding to β is

$$\left(0, \dots, 0, \underbrace{1, 0, \dots, 0}_{s_t} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{1, 0, \dots, 0}_{s_t}, 0, \dots, 0 \right).$$

Therefore, X is not unitary. The proof of Proposition 3 is now complete.

Proof of Lemma 5. The main ingredients for the proof are Proposition 5 of [PPS2] and certain non-unitarity results contained in [ABPTV]. Note that the assumptions in part (1) of Lemma 5 are precisely conditions (1) and (3) of Proposition 5 of [PPS2]; similarly, the assumptions in part (2) coincide with conditions (2) and (4) of that proposition. Therefore, in both cases, the principal series representation $J(\delta_{p,q}, \nu)$ of $Mp(2d)$ is not unitary. (The proof of) Lemma 14.6 of [ABPTV] tells us which Weyl group representations detect non-unitarity: in case (1), there is a non-unitarity certificate on any $\tilde{U}(d)$ -type which carries a representation of $W^\delta = W(C_p) \times W(C_q)$ of the form $\sigma_R \otimes \sigma$, where σ is any representation of $W(C_q)$ and σ_R is the natural reflection representation of $W(C_R)$. An analogous statement holds in case (2). We are left with the problem of identifying a $\tilde{U}(d)$ -type with the desired properties. For this, we refer the reader to table (5.1.6) of [PPS2]. This table gives a list of petite $\tilde{U}(d)$ -types and the corresponding W^δ -representations; in particular, it tells us that δ_1 carries $\sigma_R \otimes \text{triv}$, and δ_2 carries $\text{triv} \otimes \sigma_R$.

For part (3) of the lemma, notice that if $\nu = (\nu^p | \nu^q)$ is ω -regular and $\nu^p \neq \omega^p$, then $\nu_p > \frac{1}{2}$ or $\nu_i - \nu_{i+1} > 1$ for some $1 \leq i \leq p-1$ (similarly for $\nu^q \neq \omega^q$). Therefore, the assumptions of both part (1) and part (2) of the lemma must hold, hence both δ_1 and δ_2 detect non-unitarity. \square

Lemma 6. *Suppose that we are in the setting of Lemma 1, with X ω -regular and genuine. Write the Vogan classification parameter of X as in equation (3.2).*

- (1) *If X_1 is an $A_{\mathfrak{q}'}(\lambda')$ module of $\tilde{U}(r, s)$, then the representation X has a unique lowest $\tilde{U}(n)$ -type μ of the form*

$$(4.16) \quad \mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \mu_0,$$

where μ_0 is the lowest $\tilde{U}(d)$ -type of X_0 .

- (2) *For all $i = 1, \dots, t$, if $r_i \neq s_i$, then $g_i \in \mathbb{Z} + \frac{1}{2}$. If X has a unique lowest $\tilde{U}(n)$ -type and $r_i = s_i$, then $g_i \in \mathbb{Z}$.*
- (3) *For all $i = 1, \dots, t$, if $r_i \neq s_i$, then the infinitesimal character of X contains an entry $\pm g_i$, and the infinitesimal character of X_0 does not contain an entry $\pm g_i$.*

Proof. (1) Since both X_1 and X_0 have unique lowest K -types, this follows from Proposition 2 (b)-(c).

For parts (2) and (3), recall from Section 3.1 of [Pa2] the relationship between the parameter λ_a , the infinitesimal character, the lowest K -types, and the Langlands parameters of a representation. There, the theory is laid out in detail for representations of the symplectic group. We can obtain the corresponding statements for genuine representations of $Mp(2n)$ by making slight modifications. In the case of ω -regular (hence, in particular, non-singular) representations, all limits of discrete series representations are discrete series, and all lowest K -types of a

standard module are contained in the same irreducible representation. A genuine representation π with associated parameter λ_a as in (3.2) is the Langlands subquotient of an induced representation from a cuspidal parabolic subgroup with Levi component MA isomorphic to (a quotient of) $Mp(2m) \times \widetilde{GL}(2, \mathbb{R})^s \times \widetilde{GL}(1, \mathbb{R})^d$, where $m = \bigoplus_{i=1}^t |r_i - s_i|$. If $r_i = s_i + 1$ for some i , then the entry g_i coincides with an entry of the Harish-Chandra parameter of a genuine discrete series representation of $Mp(2m)$. Similarly for $-g_i$, if $r_i = s_i - 1$. Therefore, $g_i \in \mathbb{Z} + \frac{1}{2}$ whenever $r_i \neq s_i$. This also implies in this case that g_i will be an entry in the infinitesimal character of π . The fact that X_0 does not contain an entry $\pm g_i$ follows then from the (ω^-) -regular condition.

The claim about the integrality of the entry g_i for the case $r_i = s_i$ comes from the assumption that our representation must have a unique lowest $\widetilde{U}(n)$ -type. In general, the lowest $\widetilde{U}(n)$ -types are of the form (4.8). That is,

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a},$$

where δ^{L_a} is a fine $(L_a \cap K)$ -type. Which fine $(L_a \cap K)$ -types are allowed depends in part on the integrality of the expression $\lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k})$. In order to have a unique $\widetilde{U}(n)$ -type, the entries of this expression corresponding to the factors $\widetilde{U}(r_i, s_i)$ of L_a should be in $\mathbb{Z} + \frac{1}{2}$, so that the fine K -types on these factors are trivial (see Proposition 6(2) of [Pa2]); the point here is that non-trivial such fine K -types come in pairs. Keeping in mind that genuine $\widetilde{U}(n)$ -types have half-integral entries, it is now straightforward to check that this implies $g_i \in \mathbb{Z}$ whenever

$r_i = s_i$. Consequently, we obtain that if $\mu_0 = \left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \right)$ is the lowest $\widetilde{U}(d)$ -type of X_0 , then the unique lowest $\widetilde{U}(n)$ -type of X is

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \mu_0.$$

□

5. PROOF OF PROPOSITION 4

We now give the proof of Proposition 4. For convenience of the reader, we begin by restating the result.

Proposition. *Let X be an irreducible admissible (\mathfrak{g}, K) -module of $Mp(2n)$. Assume that X is genuine and ω -regular; realize X as the unique Langlands quotient of $\mathcal{R}_{\mathfrak{q}}^S(X^L)$, as in Proposition 2, and write*

$$L = L_1 \times L_0 = \widetilde{U}(r, s) \times Mp(2d) \quad X^L \simeq X_1 \otimes X_0,$$

as in Lemma 1. Suppose that $r + s \neq 0$ and $d > 0$. Then one of the following (mutually exclusive) options occurs:

- (1) X_0 is an even oscillator representation with lowest K -type (LKT)

$$\mu_0 = \pm (1/2, 1/2, \dots, 1/2).$$

- (2) There exist a subgroup $L' = \widetilde{U}(r', s') \times Mp(2(d+1))$ also containing L_a , a theta stable subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \supseteq \mathfrak{q}$ and representations X'_i of L'_i ($i = 0, 1$), such that

- (a) $\widetilde{U}(r', s') \subset \widetilde{U}(r, s)$ (with either $r = r' + 1$ or $s = s' + 1$),

- (b) X'_1 is a good $A_{q_3}(\lambda_3)$,
 - (c) X'_0 is an odd oscillator representation with $LKT \mu'_0 = (3/2, 1/2, \dots, 1/2)$
or $(-1/2, -1/2, \dots, -1/2, -3/2)$, and
 - (d) X can be realized as the unique Langlands quotient of $\mathcal{R}_{q'}(X'_1 \otimes X'_0)$.
- (3) X is not unitary.

Recall that, by Proposition 3, either X is not unitary, or X_1 is a good $A_{q_2}(\lambda_2)$ module. Assume that the latter is the case.

To prove Proposition 4, we first show that if X_0 is non-pseudospherical then X is not unitary. So assume that $X_0 \simeq J(\delta_{p,q}, \nu)$ with $p > 0$ and $q > 0$. Write the lowest K -type μ of X as in equation (4.4) and the Vogan's classification parameter λ_a as in (3.2). By an argument similar to the proof of Lemma 3 (c), (at least) one of the following conditions must hold:

$$a_t > c_1 \quad \text{or} \quad c_d > b_t,$$

and that the $\tilde{U}(d)$ -types

$$\delta_1 = \left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q, -\frac{3}{2} \right) \quad \text{and} \quad \delta_2 = \left(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-1} \right)$$

(of Lemma 5) are obtained from μ_0 by adding or subtracting a short root. One can check that if $a_t > c_1$, then δ_2 survives in the bottom layer; if $c_d > b_t$, then δ_1 survives in the bottom layer.

We distinguish two cases. If $\nu_p \neq \omega^p$ and $\nu_q \neq \omega^q$, then (by Case 3 of Lemma 5) X_0 is not unitary and *both* δ_1 and δ_2 detect non-unitarity. Because one of the two $\tilde{U}(d)$ -types survives in the bottom layer, we conclude that X is not unitary.

Now assume that $\nu^p = \omega^p$ or $\nu^q = \omega^q$. (Note that the two options can not hold at the same time, because X_0 is ω -regular.) If $\nu^p = \omega^p$, then the entries of ν^q are all greater than or equal to $p + \frac{1}{2}$; in particular, $\nu_{p+q} > \frac{1}{2}$. Hence, we are in Case (2) of Lemma 5: X_0 is not unitary and δ_2 detects non-unitarity. Similarly, if $\nu^q = \omega^q$, then X_0 is not unitary and δ_1 detects non-unitarity. In either case, the infinitesimal character of X_0 contains an entry $\frac{1}{2}$. By Lemma 6, this implies that $g_t \geq 1$. A direct calculation, using (4.16), shows that the coordinates of μ satisfy *both* conditions $a_t > c_1$ and $c_d > b_t$. Then *both* $\tilde{U}(d)$ -types δ_1 and δ_2 survive in the bottom layer, and X is not unitary.

Now consider the pseudospherical case. Without loss of generality, we can assume that X_0 has lowest $\tilde{U}(d)$ -type $\mu_0 = (-\frac{1}{2}, \dots, -\frac{1}{2})$. If X_0 is the even antiholomorphic oscillator representation (*i. e.*, if X_0 has infinitesimal character ω^d), then we are in case (1) of Proposition 4, and we are done. So suppose not. Recall that, by Lemma 1, X_0 is ω -regular. Note that if the infinitesimal character of X_0 is not equal to ω^d , then it must satisfy one of the conditions of Lemma 5 (2). We conclude that X_0 is not unitary, and the $\tilde{U}(d)$ -type δ_2 detects non-unitarity. If δ_2 survives in the bottom layer, then X is not unitary, and we are done. If δ_2 does not survive in the bottom layer, then the lowest K -type μ has $a_t - c_1 \leq 1$. Let us compare this inequality with equation (4.11):

$$a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon_t) + y - z.$$

Here $y = 0$, because the fine K -type δ^{L_a} in (4.8) is trivial except on $Mp(2d)$ by equation (4.16), and $z = -\frac{1}{2}$. Hence we obtain:

$$(5.1) \quad a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon_t) + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1.$$

Equality holds only if $g_t = \frac{1}{2}$, and $\varepsilon_t = 1$. So $r_t - s_t = 1$.

Use the (new) singularization

$$(5.2) \quad \xi' = \left(\underbrace{a, a, \dots, a}_{r-r_t} \mid \underbrace{0, 0, \dots, 0}_{d+r_t+s_t} \mid \underbrace{-a, -a, \dots, -a}_{s-s_t} \right)$$

of λ_a to construct a (new) parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$, just as at the beginning of Section 3, and let

$$(5.3) \quad L' = L'_1 \times L'_0 \cong \tilde{U}(r - r_t, s - s_t) \times Mp(2(d + r_t + s_t))$$

be the subgroup corresponding to the Levi factor of \mathfrak{q}' . Then X may be realized as the unique Langlands quotient of a representation

$$\mathcal{R}_{\mathfrak{q}'}^{S'}(X^{L'}),$$

as in Proposition 2. Using notation analogous to the one in Lemma 1, write $X^{L'} = X'_1 \otimes X'_0$. Note that X'_0 is still ω -regular, and X'_1 is still strongly regular, as representations of L'_0 and L'_1 , respectively.

To finish the proof of our proposition, we will show that, if X is unitary, then:

- (A) $s_t = 0$ (hence $r_t = 1$);
- (B) X'_0 is an odd oscillator representation with lowest $\tilde{U}(d+1)$ -type

$$\mu'_0 = \left(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_d \right);$$

and

- (C) X'_1 is a good $A_{\mathfrak{q}_3}(\lambda_3)$ module.

We begin with the proof of (A). Assume, by contradiction, that $c := s_t > 0$, and let μ'_0 be the lowest $\tilde{U}(d + 2c + 1)$ -type of X'_0 . By Proposition 2 (2), the lowest $(L' \cap \tilde{U}(n))$ -type μ' of $X^{L'}$ satisfies

$$(5.4) \quad \mu' = \mu - 2\rho(\mathfrak{u}' \cap \mathfrak{p}).$$

Therefore, μ'_0 must be of the form

$$(5.5) \quad \mu'_0 = \left(\underbrace{a_t - r' + s', \dots, a_t - r' + s'}_{c+1}, \right. \\ \left. \underbrace{c_1 - r' + s', \dots, c_d - r' + s'}_d, \underbrace{b_t - r' + s', \dots, b_t - r' + s'}_c \right),$$

with $r' = r - r_t$ and $s' = s - s_t$. Using $g_t = \frac{1}{2}$ and $\varepsilon_t = 1$, we obtain:

$$(5.6) \quad \mu'_0 = \left(\underbrace{\frac{3}{2}, \dots, \frac{3}{2}}_{c+1}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_d, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_c \right).$$

We will show, using Parthasarathy Dirac Operator Inequality, that X'_0 is not unitary, and that it contains a $\tilde{U}(d+2c+1)$ -type η'_0 on which the signature of the Hermitian form changes, and which survives under the bottom layer map. (This will then imply that $c = 0$, and will complete the proof of part (A).)

We recall some results from Lemmas 6.1 and 6.3 of [S1]. Let $\Delta^+(\mathfrak{l}'_0)$ be a choice of positive roots for $Mp(2(d+2c+1))$ which is compatible with our fixed system of positive compact roots, and let ρ_n be the corresponding half sum of non-compact roots. Choose a Weyl group element w such that $w^{-1}(\mu'_0 - \rho_n)$ is K -dominant, and denote by $\gamma^{X'_0}$ the infinitesimal character of X'_0 . If

$$(5.7) \quad \langle \mu'_0 - \rho_n + w\rho_c, \mu'_0 - \rho_n + w\rho_c \rangle < \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle,$$

then X'_0 is not unitary. In this case, there is a non-compact root $\beta \in \Delta^+(\mathfrak{l}'_0)$ such that $\eta'_0 = \mu'_0 - \beta$ occurs in X'_0 , and detects non-unitarity (in the sense that the signature of the Hermitian form on X'_0 , restricted to $\mu'_0 \oplus \eta'_0$, is indefinite).

We choose $\Delta^+(\mathfrak{l}'_0)$ such that $\rho(\mathfrak{l}'_0) = (2c+d+1, 2c+d, \dots, c+1, -1, -2, \dots, -c)$, and

$$(5.8) \quad \rho_n = \left(\underbrace{\frac{d}{2} + c + 1, \dots, \frac{d}{2} + c + 1}_{c+d+1}, \underbrace{\frac{d}{2}, \dots, \frac{d}{2}}_c \right).$$

Then

$$(5.9) \quad \mu'_0 - \rho_n = \left(\underbrace{-\frac{d}{2} - c + \frac{1}{2}, \dots, -\frac{d}{2} - c + \frac{1}{2}}_{c+1}, \right. \\ \left. \underbrace{-\frac{d}{2} - c - \frac{1}{2}, \dots, -\frac{d}{2} - c - \frac{1}{2}}_d, \underbrace{-\frac{d}{2} - \frac{1}{2}, \dots, -\frac{d}{2} - \frac{1}{2}}_c \right).$$

The element $w\rho_c$ can be chosen to be:

$$(5.10) \quad \left(\underbrace{\frac{d}{2}, \frac{d}{2} - 1, \dots, \frac{d}{2} - c}_{c+1}, \underbrace{\frac{d}{2} - c - 1, \frac{d}{2} - c - 2, \dots, -c - \frac{d}{2}}_d, \underbrace{c + \frac{d}{2}, c + \frac{d}{2} - 1, \dots, \frac{d}{2} + 1}_c \right).$$

Hence, we obtain:

$$(5.11) \quad \mu'_0 - \rho_n + w\rho_c = \left(\underbrace{-c + \frac{1}{2}, -c - \frac{1}{2}, \dots, -2c + \frac{1}{2}}_{c+1}, \right. \\ \left. \underbrace{-2c - \frac{3}{2}, -2c - \frac{5}{2}, \dots, -2c - d - \frac{1}{2}}_d, \underbrace{c - \frac{1}{2}, c - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}}_c \right).$$

Because X'_0 is ω -regular, its infinitesimal character $\gamma^{X'_0}$ must satisfy:

$$(5.12) \quad \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle \geq \langle \omega^{d+2c+1}, \omega^{d+2c+1} \rangle.$$

Writing ω^{d+2c+1} in coordinates, we get

$$(5.13) \quad \omega^{d+2c+1} = \left(d + 2c + \frac{1}{2}, d + 2c - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right).$$

Rearranging the entries of $\mu'_0 - \rho_n + w\rho_c$ in equation (5.11) and comparing them with the entries of ω^{d+2c+1} in equation (5.13), we can see that

$$(5.14) \quad \langle \omega^{d+2c+1}, \omega^{d+2c+1} \rangle - \langle \mu'_0 - \rho_n + w\rho_c, \mu'_0 - \rho_n + w\rho_c \rangle = \\ = \left(2c + \frac{1}{2} \right)^2 - \left(c - \frac{1}{2} \right)^2 = 3c^2 + 3c.$$

This is strictly positive if $c > 0$. Hence, by (5.12), we find:

$$\langle \gamma^{X'_0}, \gamma^{X'_0} \rangle - \langle \mu'_0 - \rho_n + w\rho_c, \mu'_0 - \rho_n + w\rho_c \rangle > 0$$

whenever $c > 0$, or when $c = 0$ and the inequality (5.12) is strict. We conclude that, in these cases, X'_0 is not unitary, and there is a non-compact root $\beta \in \Delta^+(l'_0)$ such that $\eta'_0 = \mu'_0 - \beta$ (is dominant and) detects non-unitarity. If $c > 0$, then the only such roots are $\epsilon_c + \epsilon_{c+1}$, $\epsilon_{c+1} + \epsilon_{c+d+1}$, $\epsilon_{c+d} + \epsilon_{c+d+1}$, $\epsilon_{c+1} + \epsilon_{2c+d+1}$, $\epsilon_{c+d+1} + \epsilon_{2c+d+1}$ and $-\epsilon_{c+d+2} - \epsilon_{c+d+3}$. If β is one of these roots, $\eta'_0 = \mu'_0 - \beta$ survives in the bottom layer provided that $b_t - b_{t-1} \geq 1$ (in equation (4.4)).

If $c = 0$ and the inequality (5.12) is strict, then the choices for β are $\epsilon_1 + \epsilon_{d+1}$, $\epsilon_d + \epsilon_{d+1}$, and $2\epsilon_{d+1}$. In each case, $\eta'_0 = \mu'_0 - \beta$ survives in the bottom layer if $c_d - b_{t-1} \geq 2$.

Remember that, by Proposition 3 and Lemma 6, X has a unique lowest $\tilde{U}(n)$ -type, so the full Lemma 6 with equation (4.16) applies. Using equations (4.10), we get:

$$(5.15) \quad b_t - b_{t-1} = -g_t + h_t - (-g_{t-1} + h_{t-1}) \\ = -g_t + g_{t-1} + \frac{\varepsilon_{t-1} + \varepsilon_t}{2} = -g_t + g_{t-1} + \frac{\varepsilon_{t-1} + 1}{2}$$

(because $\varepsilon_t = 1$). Note that $g_t = 1/2$, $g_{t-1} \geq 1$ and $\varepsilon_{t-1} = 0$ when $g_{t-1} \in \mathbb{Z}$, by Lemma 6 (because $b_t - b_{t-1}$ must be an integer). It follows that $b_t - b_{t-1}$ is at least 1, so that $c = 0$, and our Claim (A) is proved.

Now we know that $c = s_t = 0$. A calculation similar to the one in (5.15) shows that

$$(5.16) \quad c_d - b_{t-1} = \frac{1}{2} + g_{t-1} + \frac{\varepsilon_{t-1} + 1}{2} \geq 2$$

because $g_{t-1} \geq 1$. Moreover, $g_{t-1} = 1$ implies $\varepsilon_{t-1} = 0$ again by Lemma 6.

We also know that X'_0 is an irreducible genuine representation of $Mp(2(d+1))$ with lowest $\tilde{U}(d+1)$ -type

$$\mu'_0 = \left(\frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_d \right),$$

and by the inequality in (5.12) and the above discussion, the infinitesimal character $\gamma^{X'_0}$ satisfies

$$(5.17) \quad \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle = \langle \omega^{d+1}, \omega^{d+1} \rangle.$$

Hence

$$(5.18) \quad \gamma^{X'_0} = \omega^{d+1}.$$

Notice that X'_0 is uniquely determined by its infinitesimal character: in fact, since the classification parameter of X'_0 is given by

$$\lambda'_a = \left(\frac{1}{2}, 0, \dots, 0 \right),$$

its infinitesimal character must be of the form

$$(5.19) \quad \gamma^{X'_0} = \left(\frac{1}{2} \middle| \nu \right),$$

where ν is the continuous parameter on the pseudospherical principal series of $Mp(2d)$ (*cf.* Lemma 4). By (5.18),

$$(5.20) \quad \nu = \left(d + \frac{1}{2}, d - \frac{1}{2}, \dots, \frac{3}{2} \right).$$

This implies that X'_0 is indeed the appropriate odd oscillator representation, proving Claim (B).

Finally, for (C), we need to show that X'_1 is a good $A_{\mathfrak{q}_3}(\lambda_3)$ module. As usual, write the coordinates of μ as in Equation (4.4); this time, $r_t = 1$, $s_t = 0$, $a_t - c_1 = 1$, and $c_d - b_{t-1} \geq 2$. Note that it suffices to show that $a_{t-1} - a_t \geq 1$. If this is the case, then our Claim (C) follows by the same kind of argument used in the first part of the proof of Proposition 3: if X'_1 is not a good $A_{\mathfrak{q}_3}(\lambda_3)$ module, then it is not unitary, and we can use PDOI and a bottom layer argument to show that X is also non-unitary.

We now show that $a_{t-1} - a_t \geq 1$. By contradiction, suppose that $a_{t-1} = a_t$.

Then, by Lemma 6 and some calculations similar to those in the proof of Lemma 3, we must have $g_{t-1} = 1$ and $r_{t-1} = s_{t-1} > 0$. Roughly, a jump of $\frac{3}{2}$ or more in the entries of λ_a results in a corresponding jump of 1 or more in the coordinates of μ . Note that, by part (3) of Lemma 6, we cannot have $g_{t-1} = \frac{3}{2}$ because the infinitesimal character of X_0 , given by (5.20), contains an entry $\frac{3}{2}$.

Recall the subgroup $L = L_1 \times L_0$ and our assumption that X_1 is a good $A_{q_2}(\lambda_2)$ module of $L_1 = \tilde{U}(r, s)$. Write $\mathfrak{q}_2 = \mathfrak{l}_2 \oplus \mathfrak{u}_2$. By Proposition 3.5 of [S2],

$$\mathfrak{l}_a \subseteq \mathfrak{l}_2 \oplus \mathfrak{l}_0, \text{ and } \mathfrak{u}_2 \subseteq \mathfrak{u}_a.$$

We also have

$$\mathfrak{u} \subseteq \mathfrak{u}_a.$$

The lowest $(\tilde{U}(r) \times \tilde{U}(s))$ -type μ_1 of X_1 is of the form

$$(5.21) \quad \mu_1 = (\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}))|_{\tilde{U}(r) \times \tilde{U}(s)},$$

where

$$(5.22) \quad 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \left(\underbrace{r+d+1, \dots, r+d+1}_r \mid \underbrace{r-s, \dots, r-s}_d \mid \underbrace{-s-d-1, \dots, -s-d-1}_s \right).$$

Recall from [S1] (*cf.* §8), that one choice for \mathfrak{q}_2 is given by the theta stable parabolic subalgebra of $\tilde{U}(r, s)$ determined by $\mu_1 + 2\rho_k(\mu_1)$, where $2\rho_k(\mu_1)$ is the sum of all the compact roots α of \mathfrak{l}_1 which satisfy $\langle \mu_1, \alpha \rangle > 0$. Then $\mu_1 + 2\rho_k(\mu_1)$ (embedded into \mathfrak{g}) is of the form

$$\left(\underbrace{m_1, \dots, m_1}_{p_1}, \dots, \underbrace{m_u, \dots, m_u}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{-m_u, \dots, -m_u}_{q_u}, \dots, \underbrace{-m_1, \dots, -m_1}_{q_1} \right)$$

with $m_1 > m_2 > \dots > m_u$. Then the Levi subgroup of L_1 corresponding to \mathfrak{l}_2 is

$$(5.23) \quad L_2 = \prod_{i=1}^u \tilde{U}(p_i, q_i).$$

By the shape of μ_1 , the last factor $\tilde{U}(p_u, q_u)$ of L_2 must satisfy $p_u \geq 2$, and since $L_a \subseteq L_2 \times L_0$, this implies $q_u \geq 1$ because we must have $\tilde{U}(r_{t-1}, s_{t-1}) \times \tilde{U}(r_t, s_t) \subseteq L_2$. Now consider the infinitesimal character of X_1 :

$$(5.24) \quad \gamma^{X_1} = \lambda_2 + \rho(\mathfrak{l}_2) + \rho(\mathfrak{u}_2).$$

Using (5.21) and the fact that $a_t = r - s + \frac{1}{2}$ and $b_{t-1} = r - s - \frac{5}{2}$ (by equation (5.16) with $g_{t-1} = 1$ and $\varepsilon_{t-1} = 0$), we compute that

$$\mu_1 = \left(\dots, \underbrace{-s-d-\frac{1}{2}, \dots, -s-d-\frac{1}{2}}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{r+d-\frac{3}{2}, \dots, r+d-\frac{3}{2}}_{q_u}, \dots \right)$$

and

$$(5.25) \quad 2\rho(\mathfrak{u}_2 \cap \mathfrak{p}) = \left(\dots, \underbrace{-s+q_u, \dots, -s+q_u}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{r-p_u, \dots, r-p_u}_{q_u}, \dots \right).$$

Hence

$$(5.26) \quad \lambda_2 = \mu_1 - 2\rho(\mathfrak{u}_2 \cap \mathfrak{p}) = \left(\underbrace{\dots, -d - q_u - \frac{1}{2}, \dots, -d - q_u - \frac{1}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{d + p_u - \frac{3}{2}, \dots, d + p_u - \frac{3}{2}}_{q_u}, \dots \right).$$

Note that, since λ_2 must be constant on $\tilde{U}(p_u, q_u)$ (with twist because of the embedding into \mathfrak{g}), we can conclude that $p_u = q_u + 2$. (Recall that $q_u > 0$.) Write

$$(5.27) \quad \rho(\mathfrak{u}_2) = \left(\dots, \underbrace{\frac{-r-s+p_u+q_u}{2}, \dots, \frac{-r-s+p_u+q_u}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{\frac{r+s-p_u-q_u}{2}, \dots, \frac{r+s-p_u-q_u}{2}}_{q_u}, \dots \right),$$

and choose

$$(5.28) \quad \rho(\mathfrak{l}_2) = \left(\dots, \underbrace{\frac{p_u+q_u-1}{2}, \frac{p_u+q_u-3}{2}, \dots, \frac{q_u-p_u+1}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{\frac{p_u+q_u-1}{2}, \frac{p_u+q_u-3}{2}, \dots, \frac{p_u-q_u+1}{2}}_{q_u}, \dots \right).$$

Then

$$(5.29) \quad \lambda_2 + \rho(\mathfrak{u}_2) + \rho(\mathfrak{l}_2) = \left(\dots, \underbrace{-d - \frac{r+s}{2} + p_u - 1, -d - \frac{r+s}{2} + p_u - 2, \dots, -d - \frac{r+s}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{d + \frac{r+s}{2} + p_u - 2, d + \frac{r+s}{2} + p_u - 3, \dots, d + \frac{r+s}{2} + p_u - q_u - 1, \dots}_{q_u} \right).$$

Recall that by Proposition 2 (6), we have

$$(5.30) \quad \gamma^X = \gamma^{X^L} + \rho(u).$$

Now

$$(5.31) \quad \rho(u) = \left(\underbrace{\left(d + \frac{r+s+1}{2}, \dots, d + \frac{r+s+1}{2} \right)}_r \left| \underbrace{(0, \dots, 0)}_d \right| \underbrace{\left(-d - \frac{r+s+1}{2}, \dots, -d - \frac{r+s+1}{2} \right)}_s \right),$$

hence the infinitesimal character of X is

$$(5.32) \quad \gamma^X = \gamma^{X_1} + \gamma^{X_0} + \rho(u) = \left(\dots, \underbrace{\left(p_u - \frac{1}{2}, p_u - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right)}_{p_u} \left| \frac{3}{2}, \frac{1}{2} \right| \right. \\ \left. d + \frac{1}{2}, d - \frac{1}{2}, \dots, \frac{5}{2}, \frac{3}{2} \left| \underbrace{\left(p_u - \frac{5}{2}, p_u - \frac{7}{2}, \dots, p_u - q_u - \frac{3}{2}, \dots \right)}_{q_u} \right. \right).$$

This contradicts our assumption that γ^X is nonsingular, and concludes the proof of Proposition 4.

6. PROOF OF PROPOSITION 6

Finally, we re-state and prove Proposition 6.

Proposition. *In the same setting of Proposition 5, let X be the irreducible (Langlands quotient of the) representation $\mathcal{R}_{\mathfrak{q}_\omega}^{\mathfrak{g}}(\mathbb{C}_{\lambda'} \otimes \omega)$. Assume, moreover, that λ' is in the good range for \mathfrak{q}' . Then $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$ is in the good range for \mathfrak{q}_ω .*

Proof. Recall from Proposition 5 that \mathfrak{q}_ω has Levi component $\mathfrak{l}_\omega = \mathfrak{l}' + \mathfrak{l}_0$, and nilpotent part $\mathfrak{u}_\omega = \mathfrak{u}' + \mathfrak{u}$. Since $X_1 \cong A_{\mathfrak{q}'}(\lambda')$, arguing as in the previous Proposition, we know that $\mathfrak{l}' \supseteq \mathfrak{l}_a \cap \mathfrak{l}_1$ and $\mathfrak{u}' \subseteq \mathfrak{u}_a \cap \mathfrak{l}_1$. Therefore,

$$\mathfrak{l}_\omega \supseteq \mathfrak{l}_a = (\mathfrak{l}_a \cap \mathfrak{l}_1) \oplus (\mathfrak{l}_a \cap \mathfrak{l}_0).$$

(Notice that $\mathfrak{l}_a \cap \mathfrak{l}_0 = \mathfrak{l}_0$ unless ω is an odd oscillator representation.) Moreover, if

$$\begin{aligned} \Delta^+ &= \Delta^+(\mathfrak{l}_a) \cup \Delta(\mathfrak{u}_a) \\ &= \Delta^+(\mathfrak{l}_a \cap \mathfrak{l}_1) \cup \Delta^+(\mathfrak{l}_a \cap \mathfrak{l}_0) \cup \Delta(\mathfrak{u}_a) \end{aligned}$$

is a choice of positive roots defined by λ_a , by Lemma 3.6 in [S2] (making $\mathfrak{g} = \mathfrak{l}_1$ and $\mathfrak{l} = \mathfrak{l}'$ in that lemma), $\Delta^+(\mathfrak{l}_1)$ is compatible with a choice of positive roots

$$\Delta^+(\mathfrak{l}') \cup \Delta(\mathfrak{u}'),$$

and hence

$$\rho_{\mathfrak{l}_1} = \rho_{\mathfrak{l}'} + \rho(\mathfrak{u}').$$

We also observe that

$$\Delta(\mathfrak{u}_a) = \Delta(\mathfrak{l}' \cap \mathfrak{u}_a) \cup \Delta(\mathfrak{l}_0 \cap \mathfrak{u}_a) \cup \Delta(\mathfrak{u}') \cup \Delta(\mathfrak{u}).$$

We conclude then that

$$\Delta^+ = [\Delta^+(l') \cup \Delta(u')] \cup \Delta^+(l_0) \cup \Delta(u)$$

is compatible with

$$\Delta^+(l_\omega) \cup \Delta(u_\omega).$$

Hence, the infinitesimal character γ^X of X and the infinitesimal character γ^Ω of Ω are related by

$$\gamma^X = \gamma^\Omega + \rho(u_\omega).$$

(this follows by keeping track of infinitesimal characters in the induction by stages argument in Proposition 5). Now we need to show that

$$\langle \gamma^\Omega + \rho(u_\omega), \alpha \rangle > 0$$

for all α in $\Delta(u_\omega)$, so that Ω is in the good range for \mathfrak{q}_ω . Let $d_0 (= d$ or $d + 1)$ be the rank of L_0 , so that the infinitesimal character of the oscillator representation of L_0 is ω^{d_0} .

Because of the compatibility of the root systems above we can write the infinitesimal character of X as

$$(6.1) \quad \gamma^X = \lambda' + \rho(l') + \omega^{d_0} + \rho(u') + \rho(u).$$

If $\alpha \in \Delta(u')$, then α is orthogonal to ω^{d_0} and $\rho(u)$, and

$$(6.2) \quad \langle \lambda' + \rho(l') + \rho(u'), \alpha \rangle > 0$$

because λ' is assumed to be in the good range for \mathfrak{q}' . So it remains to consider $\alpha \in \Delta(u)$.

We will compute γ^X in our usual coordinates, for the case that X_0 is an even oscillator representation, in order to keep the notation simple and familiar. The case of the odd oscillator representation will follow in exactly the same way. Let λ_a , μ and $L = L_1 \times L_0$ be as in the proof of Proposition 3, and write $L' = \prod_{i=1}^u \tilde{U}(p_i, q_i)$ as in (5.21) (it was called L_2 there). In these coordinates, the roots of \mathfrak{u} are those that are positive on

$$(6.3) \quad \xi = \left(\underbrace{a, a, \dots, a}_r \mid \underbrace{0, 0, \dots, 0}_d \mid \underbrace{-a, -a, \dots, -a}_s \right)$$

with $a > 0$. Write

$$(6.4) \quad \gamma^X = (\gamma_1, \gamma_2, \dots, \gamma_r \mid d - 1/2, d - 3/2, \dots, 1/2 \mid \gamma_{r+d+1}, \gamma_{r+d+2}, \dots, \gamma_{r+d+s}).$$

It suffices to show that

$$(6.5) \quad \gamma_i > -d - 1/2 \text{ for all } i \leq r, \text{ and } \gamma_i < d + 1/2, \text{ for all } i \geq r + d + 1.$$

Because of the ω -regular assumption and the shape of ω^d , it follows then that $\gamma_i \geq d + 1/2$ for $i \leq r$ and $\gamma_i \leq -d - 1/2$ for $i \geq r + d + 1$. In fact, it is sufficient to prove this for the entries of γ^X corresponding to the last factor $\tilde{U}(p_u, q_u)$ of L' . We essentially did the calculation of γ^X on this factor in the proof of Proposition 4; the only thing we need to change is the assumption on the entries of μ , and the

notation for the parabolic subalgebra of L_1 : algebras with subscript 2 need to be replaced by primed algebras. We can write

$$(6.6) \quad \mu = \left(\underbrace{\dots, r-s+k, \dots, r-s+k}_{p_u} \mid \underbrace{c_1, \dots, c_d}_d \mid \underbrace{r-s-l, \dots, r-s-l, \dots}_{q_u} \right)$$

for some $k, l \in \mathbb{Z} + \frac{1}{2}$ with $k, l \geq \frac{1}{2}$. (This follows, for example, from Lemma 2 or part (1) of Lemma 6.) Then since $\mu = \mu_1 + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$, and $2\rho(\mathfrak{u} \cap \mathfrak{p})$ is given by (5.22),

$$(6.7) \quad \mu_1 = \left(\underbrace{\dots, -s-d-1+k, \dots, -s-d-1+k}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{r+d+1-l, \dots, r+d+1-l, \dots}_{q_u} \right).$$

Using (5.25), we have

$$\begin{aligned} \lambda' &= \mu_1 - 2\rho(\mathfrak{u}' \cap \mathfrak{p}) \\ &= \left(\underbrace{\dots, -q_u-d-1+k, \dots, -q_u-d-1+k}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{p_u+d+1-l, \dots, p_u+d+1-l, \dots}_{q_u} \right). \end{aligned}$$

Since λ' is constant on $\tilde{U}(p_u, q_u)$, we must have $p_u + d + 1 - l = q_u + d + 1 - k$, so that

$$(6.8) \quad l = k + p_u - q_u.$$

Then we obtain

$$\begin{aligned} \gamma^X &= \lambda' + \rho(\mathfrak{u}') + \rho(\mathfrak{l}') + \rho(\mathfrak{u}) \\ &= \left(\underbrace{\dots, p_u-1+k, p_u-2+k, \dots, k+1, k}_{p_u} \mid d - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \mid \underbrace{q_u-k, q_u-k-1, \dots, -k+1, \dots}_{q_u} \right). \end{aligned}$$

If $q_u = 0$ then we are done by (6.5) because $k > 0$. If $p_u = 0$, then $q_u - k = -l < 0$, so we are done as well. So we may assume p_u and q_u are both nonzero. By ω -regularity, we must have that $k \geq d + \frac{1}{2}$, so $-k + 1 \leq -d + \frac{1}{2}$. By ω -regularity, it is less or equal to $-d - \frac{1}{2}$. Since the remaining entries form a sequence of half-integers decreasing by 1 and ending with $-k + 1$, by ω -regularity, they must all be less than or equal to $-d - \frac{1}{2}$, so we are done.

Notice that this argument also applies in the case $d = 0$. This concludes the proof of Proposition 6. \square

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