Hermitian forms for $Sp(4,\mathbb{R})$*

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1 Notation for $Sp(4,\mathbb{R})$

Basic references are Vogan's notes on $Sp(4,\mathbb{R})$: [3] (branching) and [4] (Hermitian forms). In fact these notes are largely a rewriting of [4] in more explicit atlas terms.

$G = Sp(4,\mathbb{R})$ with all the usual notation, including `atlas` stuff.

Write (x,y) for the usual coordinates, in which $\rho = (2,1)$. The *software* coordinates (assuming we define G as `sc`) are fundamental weight coordinates. Write $[a,b]$ for these coordinates. The changes of coordinates are

$$\begin{aligned}(x,y) &\to [x-y,y]\\ (a+b,b) &\leftarrow [a,b]\end{aligned} \tag{1.1}$$

For example $\rho = (2,1) = [1,1]$.

2 Standard Modules

We always write H for the once-and-for-all fixed Cartan of G, and $X^* = X^*(H), X_* = X_*(H)$. We may also have use for $H^\vee$, $X^{\vee,*} = X^*(H^\vee)$ and $X_*^\vee = X_*(H^\vee)$. We also fixed once and for all a set of positive roots.

In `atlas` language a *parameter* is a triple (x,λ,ν) where:

(1) x is a `kgb` element, set $\theta = \theta_x$;

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(2) $\lambda \in (X^* + \rho)/(1-\theta)X^*$;

(3) $\nu \in (X^* \otimes_{\mathbb{Z}} \mathbb{Q})^{-\theta} = \mathfrak{a}(\mathbb{Q})^*$.

This defines a standard module $I(x, \lambda, \nu)$. The infinitesimal character is

$$\gamma = \frac{1}{2}(1+\theta)\lambda + \frac{1}{2}(1-\theta)\nu \in X^* \otimes \mathbb{Q}. \tag{2.1}$$

For example $\lambda = \nu$ gives $\gamma = \lambda$ (note that γ is integral).

Recall θ defines positive imaginary and real roots, and ρ_r, ρ_i and ρ_{cx}.

Some conditions are:

(a) The parameter is *standard* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all positive imaginary roots.

(b) The parameter is *final* if for all positive real roots, $\langle \nu, \alpha^\vee \rangle = 0$ implies $\langle \lambda + \rho_r, \alpha^\vee \rangle$ is even. Equivalently: $\langle \lambda, \alpha^\vee \rangle$ is *odd* for all real-simple roots (simple roots in the subsystem of real roots).

(c) We say $(x, \lambda, \nu) \equiv (x, w(\lambda + \rho_r) - \rho_r, w\nu)$ for $w \in W_r$. Using this we can assume $\langle \nu, \alpha \rangle \geq 0$ for all positive real roots.

(d) If α is simple and θ_x-complex we say $(x, \lambda, \nu) \equiv (s_\alpha \times x, s_\alpha \lambda, s_\alpha \nu)$. This allows us to move x to any fiber on the same Cartan.

(e) For a standard parameter, $I(x, \lambda, \nu)$ is zero if and only if $\langle \lambda, \alpha^\vee \rangle = 0$ for some imaginary-simple root which is compact.

A nonzero final standard module $I(x, \lambda, \nu)$ has a unique irreducible quotient $J(x, \lambda, \nu)$.

We generally write the subscript K to indicate restriction to K; so here we have $I_K(x, \lambda, \nu)$ and $J_K(x, \lambda, \nu)$.

Recall each fiber has a distinguished basepoint, and each conjugacy class of fibers has a canonical fiber. In the output of `KGB`, the canonical fiber is labelled `#`, and the basepoints have entry `(0,...,0)` preceding the `#`.

Using (c) we can write every final standard parameter in the form (x, λ, ν) where x is in a canonical fiber.

2.1 Standard Modules for $Sp(4, \mathbb{R})$

There are 11 `kgb` elements numbered $0 - 10$. I'll call them $x_0, \ldots, x_{10}$. Here is the output of `KGB`.

```
 0:  0  [n,n]    1   2     4   5  (0,0)#0 e
 1:  0  [n,n]    0   3     4   6  (1,0)#0 e
```

2:	0	[c,n]	2	0	*	5	(0,1)#0	e	
3:	0	[c,n]	3	1	*	6	(1,1)#0	e	
4:	1	[r,C]	4	9	*	*	(0,0)	1	1
5:	1	[C,r]	7	5	*	*	(0,0)	2	2
6:	1	[C,r]	8	6	*	*	(1,0)	2	2
7:	2	[C,n]	5	8	*	10	(0,0)#2	1,2,1	
8:	2	[C,n]	6	7	*	10	(0,1)#2	1,2,1	
9:	2	[n,C]	9	4	10	*	(0,0)#1	2,1,2	
10:	3	[r,r]	10	10	*	*	(0,0)#3	2,1,2,1	

The basepoints in the canonical fibers are x_0, x_7, x_9, x_{10}.

Cartan 0 This is the compact Cartan subgroup, so $\nu = 0$, and $\lambda \in X^* \simeq \mathbb{Z}^2$. The standard modules are ($a, b \in \mathbb{Z}$, $a \geq b \geq 0$):

$$\begin{array}{ll} I(x_0, (a,b)) & \text{(large DS)} \\ I(x_1, (a,b)) & \text{(large DS)} \\ I(x_2, (a,b)) & \text{(holomorphic DS)} \\ I(x_3, (a,b)) & \text{(antiholomorphic DS)} \end{array}$$

These are always final. The standard modules are always nonzero in the first two cases, and if and only if $a > b$ in the last two.

Cartan 1 This is the $\mathbb{C}^*$ Cartan subgroup. The canonical fiber has $\theta = w = 2,1,2$, so $\theta(x,y) = (-y,-x)$. Then $(1-\theta)X^* = \{(c,c) \mid c \in \mathbb{Z}\}$ so $\lambda = (a,b) \mod (c,c)$ with $a, b, c \in \mathbb{Z}$. (This is isomorphic to $\mathbb{Z}$ by the map $(a,b) \to a - b$). On the other hand $(X^* \otimes \mathbb{Q})^{-\theta} = \{(x,x) \mid x \in \mathbb{Q}\}$.

The root $(1,-1)$ is imaginary, and $(1,1)$ is real. The standard limit modules are

(2.1.2)(a) $$I(x_9, (a,b) \operatorname{mod}(c,c), (x,x)) \quad (a \geq b, x \geq 0)$$

with infinitesimal character

$$\gamma = (x + \frac{a-b}{2}, x - \frac{a-b}{2})$$

These standard modules are always nonzero.

In this case $M \simeq GL(2,\mathbb{R})$. The standard module given by a, b, x is the one associated to the representation on M with restriction to $SL(2,\mathbb{R})^{\pm}$ the discrete series with infinitesimal character $a-b=0,1,2,\dots$, and such that $\mathrm{diag}(e^t,e^t)$ acts by the scalar e^{2tx}. (These may be not quite the inducing data since we are working with the "Λ-parameters"; see [2] or [1].)

The final condition is

$$x=0\Rightarrow a-b\in 2\mathbb{Z}+1.$$

Suppose $r\ge s\ge 0$. Provided $r,s\in\mathbb{Z}$ there are two standard modules with infinitesimal character (r,s):

$$\text{(2.1.2)(b)}\qquad I(x_9,(r-s,0),\tfrac{1}{2}(r+s,r+s))\to\gamma=(r,s)$$
$$\text{(2.1.2)(c)}\qquad I(x_9,(r+s,0),\tfrac{1}{2}(r-s,r-s))\to\gamma=(r,-s)$$

In these coordinates, the final condition is:

$$\text{If } x=0 \text{ then } I(x_9,(c,0),(0,0)) \text{ is final } \Leftrightarrow c\in 2\mathbb{Z}+1.$$

In (b) γ is dominant, whereas in (c) it is not if $s>0$. Using the cross action of the complex root $(0,2)$ (note that $\theta_{x_4}(x,y)=(y,x)$):

$$\text{(2.1.3)}\quad I(x_9,(r+s,0),\tfrac{1}{2}(r-s,r-s))=I(x_4,(r+s,0),\tfrac{1}{2}(r-s,-s+r))$$

It is convenient to change variables, and renormalize by $\frac{1}{2}$ as follows. Define

$$\text{(2.1.4)(a)}\qquad I(x_9,c,x)=I(x_9,(c,0),\tfrac{1}{2}(x,x))\quad(c\in\mathbb{Z}_{\ge 0},x\in\mathbb{R}_{\ge 0}).$$

This has infinitesimal character

$$\text{(2.1.4)(b)}\qquad \gamma=\tfrac{1}{2}(x+c,x-c).$$

Cartan 2 This is the $\mathbb{R}^*\times S^1$ Cartan subgroup. The canonical involution is $1,2,1$, i.e. $\theta(a,b)=(-a,b)$. Therefore $(1-\theta)X^*=(2\mathbb{Z},0)$, and $\lambda=(\overline{a},b)\in\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$. On the other hand $\nu=(x,0)$ with $x\in\mathbb{Q}$. The root $(2,0)$ is real, and $(0,2)$ is imaginary. So the standard modules are

$$I(x_7,(\overline{a},b),(x,0))$$
$$I(x_8,(\overline{a},b),(x,0))$$

with $b \in \mathbb{Z}_{\geq 0}, x \in \mathbb{Q}_{\geq 0}$ and $\overline{a} \in \mathbb{Z}/2\mathbb{Z}$. The infinitesimal character is

$$\gamma = (x, b)$$

The final condition is

$$x = 0 \Rightarrow \overline{a} = 1.$$

These standard modules are always nonzero.

The standard modules are those given by the holomorphic discrete series of $SL(2,\mathbb{R})$ with infinitesimal character $b = 0, 1, 2, \ldots$, and the character of $\mathbb{R}^*$ $t \rightarrow |t|^x \mathrm{sgn}(x)^a$. The standard modules $I(x_8, (\overline{a}, b), (x, 0))$ are the same, with antiholomorphic in place of holomorphic.

If $x \geq b$ then γ is dominant. If $x < b$ it makes sense to apply (d) of Section 2. Using the fact that for x_5, x_6, $\theta(x, y) = (x, -y)$, with the obvious notation we have

$$I(x_7, (\overline{a}, b), (x, 0)) = I(x_5, (b, \overline{a}), (0, x))$$
$$I(x_8, (\overline{a}, b), (x, 0)) = I(x_6, (b, \overline{a}), (0, x))$$

Cartan 3 This is the split Cartan subgroup, with $x = x_{10}$, $\theta = -1$, $(1 - \theta)X^* = 2\mathbb{Z} \times 2\mathbb{Z}$, $\lambda = (\overline{a}, \overline{b}) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\nu = (x, y) \in \mathbb{Q}^2$. The standard modules are

$$I(x_{10}, (\overline{a}, \overline{b}), (x, y)) \quad (x \geq y \geq 0)$$

with infinitesimal character (x, y). The *final* condition is

$$y = 0 \Rightarrow \overline{b} = 1$$
$$x - y = 0 \Rightarrow \overline{a} - \overline{b} = 1$$

These standard modules are always nonzero.

3 `nblock`

The `nblock` command is essential to these computations. The translation between `nblock` coordinates and human ones is tricky. The main point of this section is to explain the `nblock` command, and give a translation for $Sp(4,\mathbb{R})$ between `nblock` and human coordinates.

Here is an overview of the `nblock` command, which (among other things) allows the user to input a general standard module. Recall a standard module is determined by a parameter (x, λ, ν). See Section 2.

The user first inputs a Cartan, from the `atlas` list `0..n`. The software then gives a list of `kgb` elements x in the distinguished fiber for this Cartan. In particular this specifies a specific Cartan involution θ of H.

The user chooses one of the elements x.

Next the user defines λ. The user inputs $(\lambda-\rho)_{in} \in X^*$ in the "software" coordinates. Suppose G is semisimple. If it is entered as `sc` these are fundamental weight coordinates, while `ad` gives simple root coordinates. The software computes

$$\lambda = (\lambda - \rho)_{in} + \rho.$$

Only the image of λ in $\rho + X^*/(1-\theta)X^*$ matters.

Next the user defines $\nu \in (X^* \otimes_{\mathbb{Z}} \mathbb{Q})^{-\theta} = \mathfrak{a}(\mathbb{Q})^*$. The user inputs an arbitrary element $\nu_{in} \in X^* \otimes_{\mathbb{Z}} \mathbb{Q}$ (in software coordinates), and the software defines

$$\nu = \frac{1}{2}(1-\theta)\nu_{in} \in \mathfrak{a}(\mathbb{Q})^*$$

Let

$$\lambda_0 = \frac{1}{2}(1+\theta)\lambda.$$

Then (cf. (2.1)) the infinitesimal character is computed as

$$\gamma = \lambda_0 + \nu = \frac{1}{2}(1+\theta)\lambda + \frac{1}{2}(1-\theta)\nu_{in}.$$

NB: Only $\nu = (1-\theta)\nu_{in}$ matters, and for the infinitesimal character only $\lambda_0 = (1+\theta)\lambda$ matters. However, λ has extra (torsion) information not contained in λ_0. For example, if the Cartan is split $\lambda_0 = 0$ but λ is an element of $\rho + X^*/2X^*$.

If γ is dominant the triple (x, λ, ν) defines a standard module. If the infinitesimal character is singular it may be reducible or 0. If γ is not dominant on the integral roots, the software uses complex cross actions to replace this triple with (x', λ', ν') such that $\gamma' = \lambda_0' + \nu'$ is dominant, and conjugate to γ.

4 Standard Modules for $Sp(4,\mathbb{R})$ using `nblock`

We show how to construct the standard modules of Section 2.1 using `nblock`. This is primarily just a change of coordinates.

In each case the user will enter a Cartan, `kgb` element, $(\lambda-\rho)_{in} = [\widetilde{a}, \widetilde{b}]$, and $\nu_{in} = [\widetilde{x}, \widetilde{y}]$.

(a) Cartan #0

This case is clear. If $0 \le k \le 3$:

$$I(x_k, (a, b)) = I(x_k, [\widetilde{a}, \widetilde{b}], [\widetilde{x}, \widetilde{y}])$$

provided $a = \widetilde{a} + \widetilde{b} + 2, b = \widetilde{b} + 1$ ($\widetilde{x}, \widetilde{y}$ are irrelevant).

(b) Cartan #1

This is the $\mathbb{C}^*$ Cartan, with $M \simeq GL(2, \mathbb{R})$.

As in Section 2.1, there is only one kgb element `x9` in the distinguished fiber, $\theta = 2, 1, 2$, $\theta(x, y) = (-y, -x)$. In fundamental weight coordinates $\theta[x, y] = [x, -x - y]$.

The coroot $[1, 0] = (1, -1)$ is imaginary, so the software requires $\widetilde{a} \ge -1$. Since $(1 - \theta)X^* = [0, \mathbb{Z}]$, $\widetilde{b}$ has no effect (the Cartan is connected).

If $\nu = [\widetilde{x}, \widetilde{y}]$ then $\frac{1}{2}(1 - \theta)\nu = [0, \widetilde{y} + \frac{\widetilde{x}}{2}]$. It is convenient to take $\widetilde{x} = 0$, so $\frac{1}{2}(1 - \theta)\nu = \nu$.

So the module

$$I(x_9, (a, b)\mathrm{mod}(c, c), (x, x)) \quad (a - b \ge 0, x \ge 0)$$

of Section 2.1 is given in these coordinates by

$$I(x_9, [\widetilde{a}, 0]\mathrm{mod}(0, \mathbb{Z})), [0, \widetilde{x}]) \quad (\widetilde{a} \ge -1, \widetilde{x} \ge 0)$$

with $\widetilde{a} = a - b - 1 \ge -1$ and $\widetilde{x} = x$. In particular the infinitesimal characters match up:

$$(x + \frac{a - b}{2}, x - \frac{a - b}{2}) = (\widetilde{x} + \frac{\widetilde{a} + 1}{2}, \widetilde{x} - \frac{\widetilde{a} + 1}{2}).$$

In terms of the `nblock` interaction, here is what happens (dropping the tildes). Choose x_9,

$$(\lambda - \rho)_{in} = [a, b] = (a + b, b)$$
$$\nu = \nu_{in} = [0, x] = (x, x)$$

If $x \ge \frac{a+1}{2}$ then the software computes:

$$\lambda = [a + 1, b + 1] = (a + b + 2, b + 1)$$
$$\lambda_0 = [a + 1, -\frac{a + 1}{2}] = (\frac{a + 1}{2}, -\frac{a + 1}{2})$$
$$\gamma = [a + 1, x - \frac{a + 1}{2}] = (x + \frac{a + 1}{2}, x - \frac{a + 1}{2})$$

and γ is dominant.

If $x < \frac{a+1}{2}$, γ is not dominant, so (provided that $x - \frac{a+1}{2}$ is an integer) the software conjugates λ, ν by s_2. The new Cartan involution is $\theta(x, y) = (y, x)$, and we get:

$$\begin{aligned}
\lambda &= [a + 2b + 3, -b - 1] = (a + b + 2, -b - 1) \\
\lambda_0 &= [0, \frac{a+1}{2}] = (\frac{a+1}{2}, \frac{a+1}{2}) \\
\nu &= [2x, -x] = (x, -x) \\
\gamma &= [2x, -x + \frac{a+1}{2}] = (x + \frac{a+1}{2}, -x + \frac{a+1}{2})
\end{aligned}$$

(c) Cartan #2

This is the Cartan $S^1 \times \mathbb{R}^*$, with $M = SL(2, \mathbb{R}) \times \mathbb{R}^*$. As in Section 2.1 the distinguished fiber has $\theta = 1, 2, 1$, so $\theta(x, y) = (-x, y)$. In fundamental weight coordinates $\theta[a, b] = [-a - 2b, b]$. The coroot $[0, 1] = (0, 1)$ is imaginary, so we have the condition $\widetilde{b} \geq -1$. Since $(1 - \theta)X^* = [2\mathbb{Z}, 0]$, only the image of $a \in Z/2\mathbb{Z}$ matters, so we could take $\widetilde{a} = 0, 1$.

Since $\nu_{in} = [\widetilde{x}, \widetilde{y}]$ gives $\nu = \frac{1}{2}(1 - \theta)\nu_{in} = [\widetilde{x} + \widetilde{y}, 0]$, we may as well take

$$\nu_{in} = [\widetilde{x}, 0] = (\widetilde{x}, 0)$$

in which case $\nu = \nu_{in}$.

The translation is, with $k = 7, 8$,

$$I(x_k, (\overline{a}, b), (x, 0)) \quad (b, x \geq 0)$$

is given in the new coordinates by

$$I(x_k, [\widetilde{a}, \widetilde{b}], [\widetilde{x}, 0]) \quad (\widetilde{b} \geq -1, \widetilde{x} \geq 0)$$

provided $\widetilde{a} + \widetilde{b} \equiv \overline{a} \pmod 2$, $\widetilde{b} + 1 = b$ and $\widetilde{x} = x$.

In terms of the software interaction (dropping the tildes), if $x \geq b + 1$:

$$\begin{aligned}
\nu &= [x, 0] = (x, 0) \\
\lambda &= [a + 1, b + 1] = (a + b + 2, b + 1) \\
\lambda_0 &= [-b - 1, b + 1] = (0, b + 1) \\
\gamma &= [x - b - 1, b + 1] = (x, b + 1)
\end{aligned}$$

and γ is dominant.

If $x < b+1$ γ is not dominant, so we conjugate everything by s_1. The new Cartan involution is $s_2 : (x, y) = (x, -y)$.

$$\begin{aligned} \lambda &= [-a-1, a+b+2] = (b+1, a+b+2) \\ \lambda_0 &= [b+1, 0] = (b+1, 0) \\ \nu &= [-x, x] = (0, x) \\ \gamma &= [b+1-x, x] = (b+1, x) \end{aligned}$$

(d) Split Cartan

$$I(x_{10}, (\overline{a}, \overline{b}), (x, y))$$

is the same as

$$I(x_{10}, [\widetilde{a}, \widetilde{b}], [\widetilde{x}, \widetilde{y}])$$

provided $\overline{a} = \widetilde{a} + \widetilde{b} \pmod 2$, $\overline{b} = \widetilde{b} + 1 \pmod 2$, and $\widetilde{x} + \widetilde{y} = x, \widetilde{y} = y$.

5 Standard K-Representations

Suppose $I(x, \lambda, 0)$ is a nonzero, final standard limit representation, with $\nu = 0$. As in Section 2 its restriction to K is $I_K(x, \lambda, 0)$. This is an important object so we give it a name:

Definition 5.1 $I_K(x, \lambda)$ *is the restriction of the nonzero, final standard limit module* $I(x, \lambda, 0)$ *to* K*.*

These are a basis of the Grothendieck group of K. Here is a list of these modules for $Sp(4, \mathbb{R})$. The last column gives the highest weight of the lowest K-types.

(5.2)

$$\begin{array}{lll} I_K(x_0, (a,b)) & a \ge b \ge 0 & (a+1, -b) \\ I_K(x_1, (a,b)) & a \ge b \ge 0 & (b, -a-1) \\ I_K(x_2, (a,b)) & a > b \ge 0 & (a+1, b+2) \\ I_K(x_3, (a,b)) & a > b \ge 0 & (-b-2, -a-1) \\ I_K(x_9, (c,0)) & c > 0 \text{ odd} & (\frac{c+1}{2}, -\frac{c+1}{2}) \\ I_K(x_7, (\overline{1}, b)) & b \ge 0 & (b+1, 1) \\ I_K(x_8, (\overline{1}, b)) & b \ge 0 & (-1, -b-1) \\ I_K(x_{10}, (\overline{0}, \overline{1})) & & (0,0) \end{array}$$

We will need some Hecht-Schmid identities for some non-final parameters. For example $I(x_9,(c,0),(0,0))$ is not final if c is even. To see this using the software, use the `Ktypeform` command. In this example $\lambda=(2,0)$, $\lambda-\rho=(0,-1)$ which equals $[1,-1]$ in fundamental weight coordinates.

```
real: Ktypeform
Choose KGB element: 9
2rho = [ 2, 2 ]
Give lambda-rho: 1 -1
Representation [ 4, 4 ]@(0,0)#1 is not final, as witnessed by coroot [1,2].
```

This is because

(5.3)(a) $$I(x_9,c,0)=J(x_0,\tfrac{1}{2}(c,c))+J(x_1,\tfrac{1}{2}(c,c))$$

and so

(5.3)(b) $$I_K(x_9,(c,0))=I_K(x_0,\tfrac{1}{2}(c,c))+I_K(x_1,\tfrac{1}{2}(c,c))$$

with lowest K-types

(5.3)(c) $$(\frac{c}{2}+1,-\frac{c}{2}),(\frac{c}{2},-\frac{c}{2}-1)$$

Similarly if $b\ge 0$:

(5.4)(a) $$\begin{aligned} I_K(x_7,\overline{0},b)=&I_K(x_0,(b,0))+I_K(x_2,(b,0))\\ I_K(x_8,\overline{0},b)=&I_K(x_1,(b,0))+I_K(x_3,(b,0))\end{aligned}$$

with lowest K-types

(5.4)(b) $$\begin{aligned} &I_K(x_7,(\overline{0},b)) \quad (b+1,0),(b+1,2)\\ &I_K(x_8,(\overline{0},b)) \quad (0,-b-1),(-2,-b-1)\end{aligned}$$

without the second term if $b=0$.

Question: What is the best way to see this using the software?

6 Reducibility: Real Roots (parity condition)

Suppose $I(x, \lambda, \nu)$ is a standard module. A real root α satisfies the *parity* condition if

(6.1)(a) $$\lambda(m_\alpha) = -(-1)^{\langle \nu+\rho_r, \alpha^\vee \rangle}$$

or equivalently

(6.1)(b) $$\langle \lambda + \nu + \rho_r, \alpha^\vee \rangle \in 2\mathbb{Z} + 1.$$

This holds if and only $I(x, \lambda, \nu)$ has some reducibility accounted for by $SL(2, \mathbb{R})_\alpha$.

Note Here is a mnemonic for remembering this condition. The trivial representation of $SL(2, \mathbb{R})$ is given by the holomorphic character ρ: $\rho(z) = z$ ($z \in \mathbb{C}^*$). The character ρ is given by $(\lambda, \nu) = (\rho, \rho)$, and note that $\rho = \rho_r$. So (b) says $\langle 3\rho, \alpha^\vee \rangle \in 2\mathbb{Z}+1$ which is true; this standard module is reducible.

Note that *no real root satisfies the parity condition* if and only if

(6.1)(c) $$\langle \lambda + \nu, \alpha^\vee \rangle \in 2\mathbb{Z} + 1 \text{ for all real-simple roots } \alpha.$$

In particular

(6.1)(d) $$I(x, \lambda, \nu) \text{ irreducible } \Rightarrow (6.1)(c) \text{ holds.}$$

An important special case is $\nu = 0$:
(6.1)(e)

$$I(x, \lambda, 0) \text{ irreducible } \Rightarrow \langle \lambda, \alpha^\vee \rangle \in 2\mathbb{Z} + 1 \text{ for all real-simple roots } \alpha.$$

This is the *final* condition of Section 2. Also see the help file for the `Ktypeform` command.

Cartan 1 There is a single real root $\alpha = (1, 1)$, and the parity condition for $I(x_9, (a, b) \pmod{(c, c)}, (x, x))$ is:

$$a + b + 2x + 1 \in 2\mathbb{Z} + 1$$

Cartan 2 The real root is $(2, 0)$, and the parity condition for $I(x_k, (\overline{a}, b), (x, 0))$, $k = 7, 8$, is

$$\overline{a} + x + 1 \in 2\mathbb{Z} + 1$$

Cartan 3 All roots are real. The parity conditions on $I(x_{10}, (\overline{a}, \overline{b}), (x, y))$ are

$$(6.2) \qquad \langle \lambda + \nu + \rho_r, \alpha^\vee \rangle \in 2\mathbb{Z} + 1.$$

$$\begin{aligned} \alpha_1 = (1,-1): &\quad \overline{a} - \overline{b} + x - y + 1 \in 2\mathbb{Z}+1 \\ \alpha_2 = (0,2): &\quad \overline{b} + y + 1 \in 2\mathbb{Z}+1 \\ (1,1): &\quad \overline{a} + \overline{b} + x + y + 3 \in 2\mathbb{Z}+1 \\ (2,0): &\quad \overline{a} + x + 2 \in 2\mathbb{Z}+1 \end{aligned}$$

For example for $I(x_{10}, (2,1), (2,1))$, α_1, α_2 satisfy the parity condition. On the other hand for $I(x_{10}, (0,0), (2,1))$, no root satisfies the parity condition.

We collect these conditions in a table, using the alternate parametrization of (2.1.2)(b) for Cartan 1.

Parity for $Sp(4,\mathbb{R})$

Cartan	Module	root	parity condition
0	no real roots		
1	$I(x_9, (r \pm s, 0), \frac{1}{2}(r \mp s, r \mp s)$	$(1,1)$	$2r+1 \in 2\mathbb{Z}+1$
2	$I(x_k, (\overline{a}, b), (x, 0))$	$(2,0)$	$\overline{a} + x + 1 \in 2\mathbb{Z}+1$
3	$I(x_{10}, (\overline{a}, \overline{b}), (x, y))$	$(1,-1)$	$\overline{a} - \overline{b} + x - y + 1 \in 2\mathbb{Z}+1$
3		$(0,2)$	$\overline{b} + y + 1 \in 2\mathbb{Z}+1$
3		$(2,0)$	$\overline{a} + x + 2 \in 2\mathbb{Z}+1$
3		$(1,1)$	$\overline{a} + \overline{b} + x + y + 3 \in 2\mathbb{Z}+1$

On Cartan 1 recall α is integral if $2r \in \mathbb{Z}$.

It might be better to write the conditions on the split Cartan (the last four entries) as follows:

$(1,-1)$	$(\overline{a} + x) - (\overline{b} + y) \in 2\mathbb{Z}$
$(0,2)$	$\overline{b} + y \in 2\mathbb{Z}$
$(2,0)$	$(\overline{a} + x) \in 2\mathbb{Z}+1$
$(1,1)$	$(\overline{a} + x) + (\overline{b} + y) \in 2\mathbb{Z}$

7 Reducibility: Complex Roots

Suppose $I(x, \lambda, \nu)$ is standard, with regular infinitesimal character. An integral complex root α contributes to reducibility of $I(x, \lambda, \nu)$ if and only if $\theta_x(\alpha) < 0$; equivalently $\alpha \in \tau(I)$), or α is of type `C-`.

For $Sp(4,\mathbb{R})$ only Cartans $1,2$ have complex roots.

Cartan 1 Consider $I(x_9,(r\pm s,0),\frac{1}{2}(r\mp s,r\mp s))$ with $r,s\ge 0$, $r\pm s\in\mathbb{Z}$. Assume $\gamma=(r,\mp s)$ is regular, i.e. $r>s>0$.

Consider the positive complex root $\alpha=(2,0)$. Assume this is integral, i.e. $r\in\mathbb{Z}$. Then $\theta(\alpha)=(0,-2)$, and this is positive or negative depending on the sign. Thus:

$$I(x_9,(r-s,0),\tfrac{1}{2}(r+s,r+s))\to\alpha \text{ is type } \texttt{C}-$$
$$I(x_9,(r+s,0),\tfrac{1}{2}(r-s,r-s))\to\alpha \text{ is type } \texttt{C}+$$

Cartan 2 Consider $I(x_k,(\overline{a},b),(x,0))$ with $k=7,8$, and $x,b\ge 0$. so $\gamma=(x,b)$.

Assume γ is regular, i.e. $x\ne b$, and $x,b\ne 0$, and $\alpha=(1,1)$ is integral, i.e. $x+b\in\mathbb{Z}$. Recall $\theta(a,b)=(-a,b)$. Thus $\alpha>0$ is complex, and $\theta(\alpha)=(-1,1)>0$ if $x<b$, and <0 otherwise. Therefore

$$\alpha \text{ is type } \texttt{C}- \Leftrightarrow x>b.$$

8 Integral blocks of $Sp(4,\mathbb{R})$

Here is the big block:

```
 0( 0,6):  0  [i1,i1]   1   2   ( 4, *)  ( 5, *)    0  e
 1( 1,6):  0  [i1,i1]   0   3   ( 4, *)  ( 6, *)    0  e
 2( 2,6):  0  [ic,i1]   2   0   ( *, *)  ( 5, *)    0  e
 3( 3,6):  0  [ic,i1]   3   1   ( *, *)  ( 6, *)    0  e
 4( 4,5):  1  [r1,C+]   4   9   ( 0, 1)  ( *, *)    1  1
 5( 5,4):  1  [C+,r1]   7   5   ( *, *)  ( 0, 2)    2  2
 6( 6,4):  1  [C+,r1]   8   6   ( *, *)  ( 1, 3)    2  2
 7( 7,3):  2  [C-,i1]   5   8   ( *, *)  (10, *)    2  1,2,1
 8( 8,3):  2  [C-,i1]   6   7   ( *, *)  (10, *)    2  1,2,1
 9( 9,2):  2  [i2,C-]   9   4   (10,11)  ( *, *)    1  2,1,2
10(10,0):  3  [r2,r1]  11  10   ( 9, *)  ( 7, 8)    3  2,1,2,1
11(10,1):  3  [r2,rn]  10  11   ( 9, *)  ( *, *)    3  2,1,2,1
```

Fix an integral regular infinitesimal character (a,b) with $a>b>0$. Here are the parameters realized using only one fiber on each Cartan, using the cross action to replace x_4 with x_9, and $x_{4,5}$ with $x_{7,8}$:

number	Cartan	length	x	λ	ν
0	0	0	0	(a,b)	$*$
1	0	0	1	(a,b)	$*$
2	0	0	2	(a,b)	$*$
3	0	0	3	(a,b)	$*$
4	1	1	9	$(a+b,0)$	$(\frac{1}{2}(a-b),\frac{1}{2}(a-b))$
5	2	1	7	$(\overline{b},a)$	$(b,0)$
6	2	1	8	$(\overline{b},a)$	$(b,0)$
7	2	2	7	$(\overline{a},b)$	$(a,0)$
8	2	2	8	$(\overline{a},b)$	$(a,0)$
9	1	2	9	$(a-b,0)$	$(\frac{1}{2}(a+b),\frac{1}{2}(a+b))$
10	3	3	10	$(\overline{a},\overline{b})$	(a,b)
11	3	3	10	$(\overline{a}+1,\overline{b}+1)$	(a,b)

Note that `10` is the trivial representation, given by $\lambda=\nu=\rho$.

Alternatively, using all x_i, so the x-values match up with the output of `block`:

number	Cartan	length	x	λ	ν
0	0	0	0	(a,b)	$*$
1	0	0	1	(a,b)	$*$
2	0	0	2	(a,b)	$*$
3	0	0	3	(a,b)	$*$
4	1	1	4	$(a+b,0)$	$(\frac{1}{2}(a-b),-\frac{1}{2}(a-b))$
5	2	1	5	$(a,\overline{b})$	$(0,b)$
6	2	1	6	$(a,\overline{b})$	$(0,b)$
7	2	2	7	$(\overline{a},b)$	$(a,0)$
8	2	2	8	$(\overline{a},b)$	$(a,0)$
9	1	2	9	$(a-b,0)$	$(\frac{1}{2}(a+b),\frac{1}{2}(a+b))$
10	3	3	10	$(\overline{a},\overline{b})$	(a,b)
11	3	3	10	$(\overline{a}+1,\overline{b}+1)$	(a,b)

Here is the block dual to $SO(4,1)$:

```
0( 5,2):  1  [C+,rn]  2  0   (*,*)  (*,*)    2  2
1( 6,2):  1  [C+,rn]  3  1   (*,*)  (*,*)    2  2
2( 7,1):  2  [C-,i1]  0  3   (*,*)  (4,*)    2  1,2,1
3( 8,1):  2  [C-,i1]  1  2   (*,*)  (4,*)    2  1,2,1
4(10,0):  3  [rn,r1]  4  4   (*,*)  (2,3)    3  2,1,2,1
```

number	Cartan	length	x	λ	ν
0	2	1	7	$(\overline{b}+1,a)$	$(b,0)$
1	2	1	8	$(\overline{b}+1,a)$	$(b,0)$
2	2	2	7	$(\overline{a}+1,b)$	$(a,0)$
3	2	2	8	$(\overline{a}+1,b)$	$(a,0)$
4	3	3	10	$(\overline{a}+1,\overline{b})$	(a,b)

The principal series $I(x_{10},(\overline{a},\overline{b}+1),(a,b))$ is irreducible, dual to $SO(5,0)$.

9 Orientation Numbers for $Sp(4,\mathbb{R})$

See [4, Section 5].

Definition 9.1 *Suppose $\gamma=(x,\lambda,\nu)$ is a parameter and α is a nonintegral real root. Define*

(9.2) $$t_\alpha(\gamma)=\langle\nu-(\lambda-\rho_r),\alpha^\vee\rangle \pmod{2\mathbb{Z}}$$

Since α is not integral $t_\alpha\notin\mathbb{Z}$; think of it as an element of $(0,1)\cup(1,2)$.

We say a real nonintegral root is oriented (with respect to γ) if $0<t_\alpha(\gamma)<1$.

Definition 9.3 *Suppose $\gamma=(x,\lambda,\nu)$ is a parameter. The orientation number $\ell_0(\gamma)$ is the sum of*

the number of pairs of complex nonintegral positive roots $\{\alpha,-\theta\alpha\}$

and

the number of real nonintegral oriented roots that are positive on γ.

Here are the orientation numbers for $Sp(4,\mathbb{R})$.
On Cartan 0 all roots are imaginary and $\ell_0=0$.

9.1 Cartan 1

Consider $(x,\lambda,\nu)=(x_9,c,x)=(x_9,(c,0),\frac{1}{2}(x,x))$, with infinitesimal character $\gamma=\frac{1}{2}(x+c,x-c)$. Here $c\in\mathbb{Z}_{\geq 0}$ and $x\in\mathbb{R}_{\geq 0}$. Recall $(1,1)$ is the unique positive real root, and $(2,0),(0,2)$ are complex.

The values of the roots are $\frac{1}{2}(x \pm c), x, c$. Assume γ is not integral, i.e. c, x are not integers of the same parity.

Case 1

Assume $x \in \mathbb{Z}$, but not of the same parity as c. The integral root system is of type $D_2 = \{(\pm 1, \pm 1)\}$. and the nonintegral roots are $(2,0),(0,2)$ (complex),

Table 9.1.1		
c, x	complex pair $(2,0),(0,2)$	ℓ_0
$c > x$		0
$c < x$	Y	1

Case 2

Assume $x \notin \mathbb{Z}$. The only positive integral root is $(1,-1)$, the integral root system is type A_1, and the nonintegral roots are $(2,0),(0,2)$ (complex) and $(1,1)$ (real).

Compute

$$\nu - (\lambda - \rho_r) = \frac{1}{2}(x, x) - (c, 0) + (\frac{1}{2}, \frac{1}{2}) = (-c + \frac{x-1}{2}, \frac{x-1}{2})$$

and

$$t_\alpha = -c + x + 1$$

Therefore α is oriented if and only if $1 < x - c < 2 \pmod{2\mathbb{Z}}$, or equivalently

$$0 < c - x < 1 \pmod{2\mathbb{Z}}$$

So:

Table 9.1.2				
c, x	range $\pmod{2\mathbb{Z}}$	complex pair $(2,0),(0,2)$	$(1,1)$	ℓ_0
$c > x$	$0 < c - x < 1 \pmod{2\mathbb{Z}}$		Y	1
	$1 < c - x < 2 \pmod{2\mathbb{Z}}$			0
$c < x$	$0 < c - x < 1 \pmod{2\mathbb{Z}}$	Y	Y	2
	$1 < c - x < 2 \pmod{2\mathbb{Z}}$	Y		1

Note that if we write the infinitesimal character as (γ_1, γ_2) then the table becomes

Table 9.1.3				
c, x	range (mod $2\mathbb{Z}$)	complex pair $(2,0),(0,2)$	$(1,1)$	ℓ_0
$\gamma_2 < 0$	$0 < \gamma_2 < \frac{1}{2} \pmod{\mathbb{Z}}$			0
	$\frac{1}{2} < \gamma_2 < 1 \pmod{\mathbb{Z}}$		Y	1
$\gamma_2 > 0$	$0 < \gamma_2 < \frac{1}{2} \pmod{\mathbb{Z}}$	Y		1
	$\frac{1}{2} < \gamma_2 < 1 \pmod{\mathbb{Z}}$	Y	Y	2

which agrees with [4, ?]

9.2 Cartan 2

Consider $(x, \lambda, \nu) = (x_k, (\overline{a}, b), (x, 0)$ with infinitesimal character (x, b). If $x \in \mathbb{Z}$ this is integral and $\ell_0 = 0$, so assume $x \notin \mathbb{Z}$.

Compute

$$\nu - (\lambda - \rho_r) = (x, 0) - (\overline{a}, b) + (1, 0) = (x - \overline{a} + 1, -b)$$

and for α the real root $(2, 0)$, $t_\alpha = x - \overline{a} + 1$. So α is oriented if and only if

$$1 < x - \overline{a} < 2 \pmod{2\mathbb{Z}}$$

Table 9.2.1					
x, b	$\overline{a}$	range (mod $2\mathbb{Z}$)	complex pair $(1, \pm 1)$	$(2,0)$	ℓ_0
$x < b$	0	$0 < x < 1$			0
$x < b$	0	$1 < x < 2$		Y	1
$x < b$	1	$0 < x < 1$		Y	1
$x < b$	1	$1 < x < 2$			0
$x > b$	0	$0 < x < 1$	Y		1
$x > b$	0	$1 < x < 2$	Y	Y	2
$x > b$	1	$0 < x < 1$	Y	Y	2
$x > b$	1	$1 < x < 2$	Y		1

9.3 Cartan 3

Consider $I(x_{10}, (\overline{a}, \overline{b}), (x, y))$ with $x > y > 0$. The nonintegral cases are as follows (not including the case with no integral roots). The third column gives the type of the integral root system, and the last column gives a consequence of the conditions in the first column.

Case	x,y	type	integral roots	
1	$x\in\mathbb{Z}, y\notin\mathbb{Z}$	A_1	$(2,0)$	
2	$x\notin\mathbb{Z}, y\in\mathbb{Z}$	A_1	$(0,2)$	
3	$x+y\in\mathbb{Z}; x-y\notin\mathbb{Z}$	A_1	$(1,1)$	$x,y\notin\mathbb{Z}+\frac{1}{2}$
4	$x-y\in\mathbb{Z}; x+y\notin\mathbb{Z}$	A_1	$(1,-1)$	$x,y\notin\mathbb{Z}+\frac{1}{2}$
5	$x\pm y\in\mathbb{Z}; x,y\notin\mathbb{Z}$	D_2	$(1,\pm 1)$	$x,y\in\mathbb{Z}+\frac{1}{2}$

We have $\rho=\rho_r=(2,1)$ and

$$\nu-(\lambda-\rho_r)=(x-a+2,y-b+1)\equiv(x-a,y-b+1). \tag{9.3.4}$$

Cartan 3, Case 1: $x\in\mathbb{Z}, y\notin\mathbb{Z}$

The nonintegral roots are $(1,\pm1)$ and $(0,2)$. They are oriented if:

$$\begin{aligned}(1,1)&: 1<x+y-(\overline{a}+\overline{b})<2 \pmod{2\mathbb{Z}}\\ (1,-1)&: 1<x-y-(\overline{a}+\overline{b})<2 \pmod{2\mathbb{Z}}\\ (0,2)&: 1<y-\overline{b}<2 \pmod{2\mathbb{Z}}\end{aligned} \tag{9.3.5}$$

It is easy to see that exactly one of $(1,\pm1)$ is an oriented root.

Table 9.3.1

$\overline{b}$	range (mod $2\mathbb{Z}$)	$(1,1)$ xor $(1,-1)$	$(0,2)$	ℓ_0
0	$0<y<1$	Y		1
	$1<y<2$	Y	Y	2
1	$0<y<1$	Y	Y	2
	$1<y<2$	Y		1

Cartan 3, Case 2: $x\notin\mathbb{Z}, y\in\mathbb{Z}$

The nonintegral roots are $(1,\pm1)$ and $(2,0)$. Now both $(1,\pm1)$ are oriented, or neither are. The condition on $(2,0)$ is

$$(2,0): 0<x-\overline{a}<1 \pmod{2\mathbb{Z}} \tag{9.3.6}$$

Table 9.3.2						
$\overline{a}$	$\overline{a}+\overline{b}$	$x \pmod{2\mathbb{Z}}$	$x+y \pmod{2\mathbb{Z}}$	$(1,1)$ and $(1,-1)$	$(2,0)$	ℓ_0
0	0	$0<x<1$	$0<x+y<1$		Y	1
			$1<x+y<2$	Y	Y	3
		$1<x<2$	$0<x+y<1$			0
			$1<x+y<2$	Y		2
1	0	$0<x<1$	$0<x+y<1$			0
			$1<x+y<2$	Y		2
		$1<x<2$	$0<x+y<1$		Y	1
			$1<x+y<2$	Y	Y	3
0	1	$0<x<1$	$0<x+y<1$	Y	Y	3
			$1<x+y<2$		Y	1
		$1<x<2$	$0<x+y<1$	Y		2
			$1<x+y<2$			0
1	1	$0<x<1$	$0<x+y<1$	Y		2
			$1<x+y<2$			0
		$1<x<2$	$0<x+y<1$	Y	Y	3
			$1<x+y<2$		Y	1

Cartan 3, Case 3: $x+y\in\mathbb{Z}, x-y\notin\mathbb{Z}$

An example is $(x,y)=(\frac{3}{4},\frac{1}{4})$. The nonintegral roots are $(2,0),(0,2)$ and $(1,-1)$. The oriented conditions are:

(9.3.7)(a)
$$\begin{aligned}&(2,0): 0<x-\overline{a}<1 \pmod{2\mathbb{Z}}\\&(0,2): 1<y-\overline{b}<2 \pmod{2\mathbb{Z}}\\&(1,-1): 1<(x-\overline{a})-(y-\overline{b})<2 \pmod{2\mathbb{Z}}\end{aligned}$$

Recall $x+y\in\mathbb{Z}$. For example, suppose $x+y$ is even, so $x=-y \pmod{2\mathbb{Z}}$, so replace x with $-y$ everywhere:

(9.3.7)(b)
$$\begin{aligned}&(2,0): 0<-y-\overline{a}<1 \pmod{2\mathbb{Z}}\\&(0,2): 1<y-\overline{b}<2 \pmod{2\mathbb{Z}}\\&(1,-1): 1<-2y-\overline{a}+\overline{b}<2 \pmod{2\mathbb{Z}}.\end{aligned}$$

After a few manipulations, including dividing the last line by 2, this is equivalent to

$$(9.3.7)(c) \qquad \begin{aligned} (2,0) &: 1 < y + \overline{a} < 2 \pmod{2\mathbb{Z}} \\ (0,2) &: 1 < y - \overline{b} < 2 \pmod{2\mathbb{Z}} \\ (1,-1) &: 0 < y + \tfrac{1}{2}(\overline{a} - \overline{b}) < \tfrac{1}{2} \pmod{\mathbb{Z}}. \end{aligned}$$

If $x + y$ is odd replace x with $-y + 1 \pmod{2\mathbb{Z}}$, and something similar happens. Here is the answer.

If $x + y \in 2\mathbb{Z}, x - y \notin \mathbb{Z}$:

Table 9.3.3

$(\overline{a}, \overline{b})$	range (mod $2\mathbb{Z}$)	$(2,0)$	$(0,2)$	$(1,-1)$	ℓ_0
$(0,0)$	$0 < y < \frac{1}{2}$			Y	1
	$\frac{1}{2} < y < 1$				0
	$1 < y < \frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2} < y < 2$	Y	Y		2
$(1,0)$	$0 < y < \frac{1}{2}$	Y			1
	$\frac{1}{2} < y < 1$	Y		Y	2
	$1 < y < \frac{3}{2}$		Y		1
	$\frac{3}{2} < y < 2$		Y	Y	2
$(0,1)$	$0 < y < \frac{1}{2}$		Y		1
	$\frac{1}{2} < y < 1$		Y	Y	2
	$1 < y < \frac{3}{2}$	Y			1
	$\frac{3}{2} < y < 2$	Y		Y	2
$(1,1)$	$0 < y < \frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2} < y < 1$	Y	Y		2
	$1 < y < \frac{3}{2}$			Y	1
	$\frac{3}{2} < y < 2$				0

If $x + y \in 2\mathbb{Z} + 1, x - y \notin \mathbb{Z}$:

Table 9.3.4					
$(\overline{a},\overline{b})$	range (mod $2\mathbb{Z}$)	$(2,0)$	$(0,2)$	$(1,-1)$	ℓ_0
$(0,0)$	$0<y<\frac{1}{2}$	Y			1
	$\frac{1}{2}<y<1$	Y		Y	2
	$1<y<\frac{3}{2}$		Y		1
	$\frac{3}{2}<y<2$		Y	Y	2
$(1,0)$	$0<y<\frac{1}{2}$			Y	1
	$\frac{1}{2}<y<1$				0
	$1<y<\frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2}<y<2$	Y	Y		2
$(0,1)$	$0<y<\frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2}<y<1$	Y	Y		2
	$1<y<\frac{3}{2}$			Y	1
	$\frac{3}{2}<y<2$				0
$(1,1)$	$0<y<\frac{1}{2}$		Y		1
	$\frac{1}{2}<y<1$		Y	Y	2
	$1<y<\frac{3}{2}$	Y			1
	$\frac{3}{2}<y<2$	Y		Y	2

Cartan 3, Case 4: $x+y\notin\mathbb{Z}, x-y\in\mathbb{Z}$

An example is $(x,y)=(\frac{4}{3},\frac{1}{3})$.

The nonintegral roots are $(2,0),(0,2)$ and $(1,1)$. The oriented conditions are

(9.3.8)(a)
$$\begin{aligned}&(2,0): 0<x-\overline{a}<1 \pmod{2\mathbb{Z}}\\&(0,2): 1<y-\overline{b}<2 \pmod{2\mathbb{Z}}\\&(1,1): 1<(x-\overline{a})+(y-\overline{b})<2 \pmod{2\mathbb{Z}}\end{aligned}$$

Suppose $x-y=n\in\mathbb{Z}$. As in Case 3, (a) becomes

(9.3.8)(b)
$$\begin{aligned}&(2,0): 0<y+n-\overline{a}<1 \pmod{2\mathbb{Z}}\\&(0,2): 1<y-\overline{b}<2 \pmod{2\mathbb{Z}}\\&(1,1): \tfrac{1}{2}<y+\tfrac{1}{2}(n-\overline{a}-\overline{b})<1 \pmod{\mathbb{Z}}.\end{aligned}$$

If $x-y\in 2\mathbb{Z}, x+y\notin\mathbb{Z}$:

Table 9.3.5					
$(\overline{a},\overline{b})$	range (mod $2\mathbb{Z}$)	$(2,0)$	$(0,2)$	$(1,1)$	ℓ_0
$(0,0)$	$0<y<\frac{1}{2}$	Y			1
	$\frac{1}{2}<y<1$	Y		Y	2
	$1<y<\frac{3}{2}$		Y		1
	$\frac{3}{2}<y<2$		Y	Y	2
$(1,0)$	$0<y<\frac{1}{2}$			Y	1
	$\frac{1}{2}<y<1$				0
	$1<y<\frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2}<y<2$	Y	Y		2
$(0,1)$	$0<y<\frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2}<y<1$	Y	Y		2
	$1<y<\frac{3}{2}$			Y	1
	$\frac{3}{2}<y<2$				0
$(1,1)$	$0<y<\frac{1}{2}$		Y		1
	$\frac{1}{2}<y<1$		Y	Y	2
	$1<y<\frac{3}{2}$	Y			1
	$\frac{3}{2}<y<2$	Y		Y	2

If $x-y\in 2\mathbb{Z}+1, x+y\notin\mathbb{Z}$:

Table 9.3.6					
$(\overline{a},\overline{b})$	range (mod $2\mathbb{Z}$)	$(2,0)$	$(0,2)$	$(1,1)$	ℓ_0
$(0,0)$	$0<y<\frac{1}{2}$			Y	1
	$\frac{1}{2}<y<1$				0
	$1<y<\frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2}<y<2$	Y	Y		2
$(1,0)$	$0<y<\frac{1}{2}$	Y			1
	$\frac{1}{2}<y<1$	Y		Y	2
	$1<y<\frac{3}{2}$		Y		1
	$\frac{3}{2}<y<2$		Y	Y	2
$(0,1)$	$0<y<\frac{1}{2}$		Y		1
	$\frac{1}{2}<y<1$		Y	Y	2
	$1<y<\frac{3}{2}$	Y			1
	$\frac{3}{2}<y<2$	Y		Y	2
$(1,1)$	$0<y<\frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2}<y<1$	Y	Y		2
	$1<y<\frac{3}{2}$			Y	1
	$\frac{3}{2}<y<2$				0

Cartan 3**, Case 5:** $x \pm y \in \mathbb{Z}, x, y \notin \mathbb{Z}$

In other words $x, y \in \mathbb{Z} + \frac{1}{2}$. An example is $(x, y) = (\frac{3}{2}, \frac{1}{2})$.

The nonintegral roots are $(2,0), (0,2)$. The oriented conditions are

(9.3.9)(a)
$$\begin{aligned}(2,0) &: 0 < x - \overline{a} < 1 \pmod{2\mathbb{Z}}\\ (0,2) &: 1 < y - \overline{b} < 2 \pmod{2\mathbb{Z}}\end{aligned}$$

$x \pm y \in \mathbb{Z}; x, y \notin \mathbb{Z}$

Table 9.3.7

$(\overline{a}, \overline{b})$	$x \pmod{2\mathbb{Z}}$	$y \pmod{2\mathbb{Z}}$	oriented	ℓ_0
$(\overline{0}, \overline{0})$	$0 < x < 1$	$0 < y < 1$	$(2,0)$	1
	$0 < x < 1$	$1 < y < 2$	$(2,0), (0,2)$	2
	$1 < x < 2$	$0 < y < 1$		0
	$1 < x < 2$	$1 < y < 2$	$(0,2)$	1
$(\overline{1}, \overline{0})$	$0 < x < 1$	$0 < y < 1$		0
	$0 < x < 1$	$1 < y < 2$	$(0,2)$	1
	$1 < x < 2$	$0 < y < 1$	$(2,0)$	1
	$1 < x < 2$	$1 < y < 2$	$(2,0), (0,2)$	2
$(\overline{0}, \overline{1})$	$0 < x < 1$	$0 < y < 1$	$(2,0), (0,2)$	2
	$0 < x < 1$	$1 < y < 2$	$(2,0)$	1
	$1 < x < 2$	$0 < y < 1$	$(0,2)$	1
	$1 < x < 2$	$1 < y < 2$		0
$(\overline{1}, \overline{1})$	$0 < x < 1$	$0 < y < 1$	$(0,2)$	1
	$0 < x < 1$	$1 < y < 2$		0
	$1 < x < 2$	$0 < y < 1$	$(2,0), (0,2)$	2
	$1 < x < 2$	$1 < y < 2$	$(2,0)$	1

10 Composition Series: Integral Infinitesimal Character

Assume the infinitesimal character γ is regular, so $\gamma = (a, b)$ with $a > b > 0$.

Cartan 1 There are two standard modules $I(x_9, (a \mp b, 0), \frac{1}{2}(a \pm b, a \pm b))$; necessarily $a \mp b \in \mathbb{Z}$ (depending on which of the two modules we consider).

By the preceding sections the real root $(1,1)$ gives reducibility if and only if $a \in \mathbb{Z}$, and the complex root $(2,0)$ gives reducibility only of $I(x_9, (a-b, 0), \frac{1}{2}(a+b, a+b))$, again only if $a \in \mathbb{Z}$. So we may assume $a, b \in \mathbb{Z}$.

First consider $I(x_9, (a+b, 0), \frac{1}{2}(a-b, a-b))$. Then (cf. (2.1.3))

$$I(x_9, (a+b, 0), \frac{1}{2}(a-b, a-b)) = I(x_4, (a+b, 0), \frac{1}{2}(a-b, -a+b));$$

this is the module `#4` in the output of `block`. Using `klbasis` we see this standard module has three composition factors, including the two large discrete series representations at this infinitesimal character.

Remark 10.1 *Recall that in order to compute the composition factors of the standard modules using* `klbasis`*, we need to invert the matrix obtained from the values of the KLV polynomials at* 1 *with the signs coming from the lengths, as in (11.1).*

So

(10.2)(a)

$$\begin{aligned} I(x_9, (a+b, 0), &\frac{1}{2}(a-b, a-b)) = \\ &J(x_9, (a+b, 0), \frac{1}{2}(a-b, a-b)) \\ &+ J(x_0, (a, b)) + J(x_1, (a, b)). \end{aligned}$$

Next, $I(x_9, (a-b, 0), \frac{1}{2}(a+b, a+b))$ is block element `9`, so `klbasis` says J has irreducible modules `9,0,1,4,5,6` as composition factors (with multiplicity one).

Then:

(10.2)(b)

$$\begin{aligned} I(x_9, (a-b, 0), &\tfrac{1}{2}(a+b, a+b)) = \\ &J(x_9, (a-b, 0), \tfrac{1}{2}(a+b, a+b)) \\ &+ J(x_0, (a, b)) + J(x_1, (a, b)) \\ &+ J(x_9, (a+b, 0), \tfrac{1}{2}(a-b, a-b)) \\ &+ J(x_7, (\overline{b}, a), (b, 0)) \\ &+ J(x_8, (\overline{b}, a), (b, 0)) \end{aligned}$$

Cartan 2

The standard modules are $I(x_k,(\epsilon,b),(a,0))$ and $I(x_k,(\epsilon,a),(b,0))$ with $k=7,8$ and $\epsilon=0,1$. We continue to assume $a>b>0$.

The real root $(2,0)$ gives reducibility $I(x_k,(\overline{a},b),(a,0))$ and $I(x_k,(\overline{b},a),(b,0))$ provided $a,b\in\mathbb{Z}$. The complex root $(1,1)$ contributes to reducibility of $I(x_k,(\epsilon,b),(a,0))$, again only if $a,b\in\mathbb{Z}$. So the infinitesimal character is necessarily integral. These representations are in one of two blocks, depending on ϵ.

Using `klbasis` we compute the following composition series.

Here are the formulas in the big block.
$I(x_7,(\overline{b},a),(b,0))$ (standard module `5`):

(10.2)(c) $$I(x_7,(\overline{b},a),(b,0))=J(x_7,(\overline{b},a),(b,0))+J(x_0,(a,b))+J(x_2,(a,b)).$$

$I(x_8,(\overline{b},a),(b,0))$ (standard module `6`):
(10.2)(d)
$$I(x_8,(\overline{b},a),(b,0))=J(x_8,(\overline{b},a),(b,0))+J(x_1,(a,b))+J(x_3,(a,b)).$$

$I(x_7,(\overline{a},b),(a,0))$ (standard module `7`):

(10.2)(e) $$\begin{aligned}I(x_7,(\overline{a},b),(a,0))&=J(x_7,(\overline{a},b),(a,0))+J(x_0,(a,b))\\&\quad+J(x_9,(a+b,0),\frac{1}{2}(a-b,a-b))+J(x_7,(\overline{b},a),(b,0))\end{aligned}$$

$I(x_8,(\overline{a},b),(a,0))$ (standard module `8`):

(10.2)(f) $$\begin{aligned}I(x_8,(\overline{a},b),(a,0))&=J(x_8,(\overline{a},b),(a,0))+J(x_1,(a,b))\\&\quad+J(x_9,(a+b,0),\frac{1}{2}(a-b,a-b))+J(x_8,(\overline{b},a),(b,0))\end{aligned}$$

Here are the formulas in the block dual to $SO(4,1)$.

By the preceding $I(x_k,(\overline{b}+1,a),(b,0))$ with $k=7,8$ are both irreducible. These are standard modules `0,1` from this block.
$I(x_7,(\overline{a}+1,b),(a,0))$ (standard module `2`):

(10.2)(g) $$\begin{aligned}I(x_7,(\overline{a}+1,b),(a,0))=&J(x_7,(\overline{a}+1,b),(a,0))\\&+J(x_7,(\overline{b}+1,a),(b,0))\end{aligned}$$

$I(x_8, (\overline{a}+1, b), (a, 0))$ (standard module `3`):

(10.2)(h)
$$\begin{aligned} I(x_8, (\overline{a}+1, b), (a, 0)) =& J(x_8, (\overline{a}+1, b), (a, 0)) \\ &+ J(x_8, (\overline{b}+1, a), (b, 0)) \end{aligned}$$

Cartan 3

These are the modules $I(x_{10}, (\delta, \epsilon), (a, b))$ with $\delta, \epsilon = 0, 1$ and $a > b > 0$.

First consider the representations in the big block, so $a, b \in \mathbb{Z}$.

$I(x_{10}, (\overline{a}+1, \overline{b}+1), (a, b))$ This is block element `11`, which has irreducible modules `11,0,1,2,3,4,5,6,9`, all of multiplicity one.

(10.2)(i)
$$\begin{aligned} I(x_{10}, (\overline{a}+1, \overline{b}+1), (a, b)) = & J(x_{10}, (\overline{a}+1, \overline{b}+1), (a, b)) \\ &+ J(x_0, (a, b)) + J(x_1, (a, b)) + J(x_2, (a, b)) + J(x_3, (a, b)) \\ &+ J(x_9, (a+b, 0), \frac{1}{2}(a-b, a-b)) \\ &+ J(x_7, (\overline{b}, a), (b, 0)) + J(x_8, (\overline{b}, a), (b, 0)) \\ &+ J(x_9, (a-b, 0), \frac{1}{2}(a+b, a+b)) \end{aligned}$$

$I(x_{10}, (\overline{a}, \overline{b}), (a, b))$ This is block element `10`, which has irreducible modules `10,0,1,4,5,6,7,8,9`, all of multiplicity one except `4` has multiplicity 2. See Section 15.

(10.2)(j)
$$\begin{aligned} I(x_{10}, (\overline{a}, \overline{b}), (a, b)) = & J(x_{10}, (\overline{a}, \overline{b}), (a, b)) \\ &+ J(x_0, (a, b)) + J(x_1, (a, b)) \\ &+ 2 \times J(x_9, (a+b, 0), \frac{1}{2}(a-b, a-b)) \\ &+ J(x_7, (\overline{b}, a), (b, 0)) + J(x_8, (\overline{b}, a), (b, 0)) \\ &+ J(x_7, (\overline{a}, b), (a, 0)) + J(x_8, (\overline{a}, b), (a, 0)) \\ &+ J(x_9, (a-b, 0), \frac{1}{2}(a+b, a+b)) \end{aligned}$$

Next, consider the integral block dual to $SO(4, 1)$. There is one representation on the split Cartan, number `4`.

$$
\begin{aligned}
I(x_{10},(\overline{a}+1,\overline{b}),(a,b)) = {} & J(x_{10},(\overline{a}+1,\overline{b}),(a,b)) \\
& + J(x_7,(\overline{b}+1,a),(b,0)) + J(x_8,(\overline{b}+1,a),(b,0)) \\
& + J(x_7,(\overline{a}+1,b),(a,0)) + J(x_8,(\overline{a}+1,b),(a,0))
\end{aligned} \tag{10.2)(k}
$$

The principal series $I(x_{10},(\overline{a},\overline{b}+1),(a,b))$ is irreducible (dual to $SO(5,0)$).

11 Character Formulas: Integral Infinitesimal Character

Assume the infinitesimal character γ is regular, so $\gamma = (a,b)$ with $a > b > 0$.

Suppose $\mu = (x,\lambda,\nu)$. Use `nblock` to compute the block of $I(\mu)$, and `klbasis` to compute $P_{\mu,\delta}$ for all δ in the block. Then:

$$
J(\mu) = \sum (-1)^{\ell(\mu)-\ell(\delta)} P_{\delta,\mu}(1) I(\delta) \tag{11.1}
$$

Here are character formulas; the numbering is parallel to that of (10.2)(a-k).

Cartan 1

$$
\begin{aligned}
& J(x_9,(a+b,0),\tfrac{1}{2}(a-b,a-b)) = \\
& \qquad I(x_9,(a+b,0),\tfrac{1}{2}(a-b,a-b)) \\
& \qquad - I(x_0,(a,b)) - I(x_1,(a,b)).
\end{aligned} \tag{11.2)(a}
$$

$$
\begin{aligned}
& J(x_9,(a-b,0),\tfrac{1}{2}(a+b,a+b)) = \\
& \quad I(x_9,(a-b,0),\tfrac{1}{2}(a+b,a+b)) \\
& \quad + I(x_0,(a,b)) + I(x_1,(a,b)) + I(x_2,(a,b)) + I(x_3,(a,b)) \\
& \quad - I(x_9,(a+b,0),\tfrac{1}{2}(a-b,a-b)) \\
& \quad - I(x_7,(\overline{b},a),(b,0)) - I(x_8,(\overline{b},a),(b,0))
\end{aligned} \tag{11.2)(b}
$$

Cartan 2

Here are the formulas on the big block.

(11.2)(c) $J(x_7,(\overline{b},a),(b,0)) = I(x_7,(\overline{b},a),(b,0)) - I(x_0,(a,b)) - I(x_2,(a,b))$

(11.2)(d) $J(x_8,(\overline{b},a),(b,0)) = I(x_8,(\overline{b},a),(b,0)) - I(x_1,(a,b)) - I(x_3,(a,b))$

(11.2)(e)
$$\begin{aligned} J(x_7,&(\overline{a},b),(a,0)) = I(x_7,(\overline{a},b),(a,0) \\ &+ I(x_0,(a,b)) + I(x_1,(a,b)) + I(x_2,(a,b)) \\ &- I(x_9,(a+b,0),\frac{1}{2}(a-b,a-b)) - I(x_7,(\overline{b},a),(b,0)) \end{aligned}$$

(11.2)(f)
$$\begin{aligned} J(x_8,&(\overline{a},b),(a,0)) = I(x_8,(\overline{a},b),(a,0) \\ &+ I(x_0,(a,b)) + I(x_1,(a,b)) + I(x_3,(a,b)) \\ &- I(x_9,(a+b,0),\frac{1}{2}(a-b,a-b)) - I(x_8,(\overline{b},a),(b,0)) \end{aligned}$$

Here are the formulas on the block dual to $SO(4,1)$:

(11.2)(g)
$$J(x_7,(\overline{a}+1,b),(a,0)) = I(x_7,(\overline{a}+1,b),(a,0)) - I(x_7,(\overline{b}+1,a),(b,0))$$

(11.2)(h)
$$J(x_8,(\overline{a}+1,b),(a,0)) = I(x_8,(\overline{a}+1,b),(a,0)) - I(x_8,(\overline{b}+1,a),(b,0))$$

Cartan 3

First consider the big block, with integral infinitesimal character.

(11.2)(i)
$$\begin{aligned} J(x_{10},(\overline{a}+1,\overline{b}+1),(a,b)) &= I(x_{10},(\overline{a}+1,\overline{b}+1),(a,b)) - I(x_2,(a,b)) - I(x_3,(a,b)) \\ &- I(x_9,(a-b,0),\frac{1}{2}(a+b,a+b)) \end{aligned}$$

See Section 15.

$$
\begin{aligned}
J(x_{10},(\overline{a},\overline{b}),(a,b)) = {} & I(x_{10},(\overline{a},\overline{b}),(a,b)) \\
& - I(x_0,(a,b)) - I(x_1,(a,b)) - I(x_2,(a,b)) - I(x_3,(a,b)) \\
& + I(x_9,(a+b,0),\frac{1}{2}(a-b,a-b)) \\
& + I(x_7,(\overline{b},a),(b,0)) + I(x_8,(\overline{b},a),(b,0)) \\
& - I(x_7,(\overline{a},b),(a,0)) - I(x_8,(\overline{a},b),(a,0)) \\
& - I(x_9,(a-b,0),\frac{1}{2}(a+b,a+b))
\end{aligned}
\tag{11.2)(j}
$$

On the block dual to $SO(4,1)$ there is a single formula on the split Cartan:

$$
\begin{aligned}
J(x_{10},(\overline{a}+1,\overline{b}),(a,b)) = {} & I(x_{10},(\overline{a}+1,\overline{b}),(a,b)) - \\
& J(x_7,(\overline{a}+1,b),(a,0)) - J(x_8,(\overline{a}+1,b),(a,0))
\end{aligned}
\tag{11.2)(k}
$$

12 Composition Series and Character Formulas: Non-Integral Infinitesimal Character

Write $\gamma = (a,b)$ with $a > b > 0$. Let $\Delta(\gamma)$ be the integral root system: $\Delta(\gamma) = \{\alpha \mid \langle\gamma,\alpha^\vee\rangle \in \mathbb{Z}\}$. If this is empty then the standard module is irreducible. Here are the remaining cases.

12.1 $\Delta(\gamma) = \{(\pm1,\pm1)\}$ (type D_2)

If $a,b \in \mathbb{Z}+\frac{1}{2}$ then $\Delta(\gamma) = \{(\pm1,\pm1)\}$ is of type D_2.

On the compact Cartan subgroup or Cartan 2 at least one of the roots $2e_i$ is imaginary, hence integral, so this case doesn't arise. So we're on the $\mathbb{C}^*$ or split Cartan.

Recall ((2.1.4)(a)) $I(x_9,c,x)$ has infinitesimal character $\gamma = \frac{1}{2}(x+c,x-c)$, with $c \in \mathbb{Z}$, which gives $D(\gamma)$ of type D_2 in the following cases:

$$
I(x_9,c,x) \quad (x,c \geq 0,\ \text{integers of opposite parity})
\tag{12.1.1}
$$

For example the representations with infinitesimal character $(\frac{3}{2}, \frac{1}{2})$ are

$$I(x_9, 1, 2) = I(x_9, (1, 0), (1, 1))$$
$$I(x_9, 2, 1) = I(x_9, (2, 0), (\tfrac{1}{2}, \tfrac{1}{2}))$$

These are irreducible:

Suppose $x > c$. For example take $c = 1, x = 2$, i.e. $I(x_9, 1, 2) = I(x_9, (1, 0), (1, 1))$. Using `nblock` (see Section 4) this is $\lambda - \rho = (-1, -1) = [0, -1]$ and $\nu = (1, 1) = [0, 1]$. The infinitesimal character is $\gamma = [1, \frac{1}{2}] = (\frac{3}{2}, \frac{1}{2})$.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: 0 -1
denominator for nu: 1
numerator for nu: 0 1
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [2,1]/2, lambda = [1,0]/1
Subsystem on dual side is of type A1.A1, with roots 4,6.
Given parameters define element 0 of the following block:
0(0,2):  0  [i2,rn]  0  0   (1,2)  (*,*)  *( 9,  [1,0]= rho+   [0,-1])   1  1
1(1,0):  1  [r2,rn]  2  1   (0,*)  (*,*)  *(10,  [1,0]= rho+   [0,-1])   0  e
2(1,1):  1  [r2,rn]  1  2   (0,*)  (*,*)  *(10, [1,-1]= rho+   [0,-2])   0  e
KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1
```

The KLV polynomial information tells us that this module (module 0) is irreducible.

The two other standard modules here, on the split Cartan subgroup, are $I(x_{10}, (\overline{1}, \overline{0}), (\frac{3}{2}, \frac{1}{2}))$ and $I(x_{10}, (\overline{0}, \overline{1}), (\frac{3}{2}, \frac{1}{2}))$ Using the `nblock` command for these two (principal series) representations, we see that they have composition series

$$I(x_{10}, (\overline{1}, \overline{0}), (\tfrac{3}{2}, \tfrac{1}{2})) = J(x_{10}, (\overline{1}, \overline{0}), (\tfrac{3}{2}, \tfrac{1}{2})) + J(x_9, 1, 2)$$
$$I(x_{10}, (\overline{0}, \overline{1}), (\tfrac{3}{2}, \tfrac{1}{2})) = J(x_{10}, (\overline{0}, \overline{1}), (\tfrac{3}{2}, \tfrac{1}{2})) + J(x_9, 1, 2)$$

and character formulas

$$J(x_{10},(\overline{1},\overline{0}),(\tfrac{3}{2},\tfrac{1}{2}))=I(x_{10},(\overline{1},\overline{0}),(\tfrac{3}{2},\tfrac{1}{2}))-I(x_9,1,2)$$
$$J(x_{10},(\overline{0},\overline{1}),(\tfrac{3}{2},\tfrac{1}{2}))=I(x_{10},(\overline{0},\overline{1}),(\tfrac{3}{2},\tfrac{1}{2}))-I(x_9,1,2)$$

More generally for $x>c>0$, x,c integers of opposite parity, the standard modules are

(12.1.2)(a)
$$\boxed{\begin{array}{l} I(x_9,c,x)\quad (x>c\geq 0,\ \text{integers of opposite parity})\\ I(x_{10},(\overline{0},\overline{c}),\tfrac{1}{2}(x+c,x-c))\\ I(x_{10},(\overline{1},\overline{1+c}),\tfrac{1}{2}(x+c,x-c))\end{array}}$$

and the character formulas are
(12.1.2)(b)
$$\boxed{\begin{array}{c} J(x_{10},(\overline{0},\overline{c}),\tfrac{1}{2}(x+c,x-c))=I(x_{10},(\overline{0},\overline{c}),\tfrac{1}{2}(x+c,x-c))-I(x_9,c,x)\\ J(x_{10},(\overline{1},\overline{1+c}),\tfrac{1}{2}(x+c,x-c))=I(x_{10},(\overline{1},\overline{1+c}),\tfrac{1}{2}(x+c,x-c))-I(x_9,c,x)\end{array}}$$

Remark 12.1.3 *If we choose $x>c=0$ with x an odd integer, say $x=1$, we get*

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: -1 -1
denominator for nu: 2
numerator for nu: 0 1
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [0,1]/2, lambda = [0,0]/1
Subsystem on dual side is of type A1.A1, with roots 4,6.
Given parameters define element 0 of the following block:
0(0,2):  0  [i2,rn]  0  0   (1,2)  (*,*)  *( 9,  [0,0]= rho+  [-1,-1])  1
1(1,0):  1  [r2,rn]  2  1   (0,*)  (*,*)   (10,  [0,0]= rho+  [-1,-1])  e
2(1,1):  1  [r2,rn]  1  2   (0,*)  (*,*)   (10, [0,-1]= rho+  [-1,-2])  e
(cumulated) KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1
```

In this case, the block contains only the module $I(x_9, 0, x)$; the parameters for $I(x_{10}, (\overline{0}, \overline{0}), \frac{1}{2}(x, x))$ and $I(x_{10}, (\overline{1}, \overline{1}), \frac{1}{2}(x, x))$ are not final; so the irreducible modules $J(x_{10}, (\overline{0}, \overline{0}), \frac{1}{2}(x, x))$ and $J(x_{10}, (\overline{1}, \overline{1}), \frac{1}{2}(x, x))$ are zero. The full principal series representations associated to these parameters are actually irreducible, and both equivalent to $J(x_9, 0, x)$.

If $0 \leq x < c$, we get the same character formulas. For example $x = 1, c = 2$ gives $I(x_9, 2, 1) = I(x_9, (2, 0), \frac{1}{2}(1, 1))$ with $\gamma = (\frac{3}{2}, -\frac{1}{2})$. In `nblock` coordinates this is $\lambda = (2, 0)$, $\lambda - \rho = (0, -1) = [1, -1]$, $\nu = (\frac{1}{2}, \frac{1}{2}) = [0, \frac{1}{2}]$.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: 1 -1
denominator for nu: 2
numerator for nu: 0 1
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [4,-1]/2, lambda = [2,0]/1
Subsystem on dual side is of type A1.A1, with roots 4,6.
Given parameters define element 0 of the following block:
0(0,2):  0  [i2,rn]  0  0   (1,2)  (*,*)  *( 9, [2,-2]= rho+   [1,-3])  1
1(1,0):  1  [r2,rn]  2  1   (0,*)  (*,*)  *(10, [2,-2]= rho+   [1,-3])  e
2(1,1):  1  [r2,rn]  1  2   (0,*)  (*,*)  *(10, [2,-1]= rho+   [1,-2])  e
KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1
```

The infinitesimal character is not dominant on the coroot $(0, 1)$; however, it is dominant on the integral coroots. The standard modules and character formulas are exactly as in (12.1.2)(a) and (12.1.2)(b). Using

$$I(x_{10}(\overline{a}, \overline{b}), (x, y)) \cong I(x_{10}(\overline{a}, \overline{b}), (x, -y)), \tag{12.1.4}$$

we write these formulas

(12.1.5)(a)

$$\boxed{\begin{array}{l} I(x_9,c,x) \quad (c>0, x\ge 0,\ \text{integers of opposite parity}) \\ I(x_{10},(\overline{0},\overline{c}),\frac{1}{2}(x+c,-x+c)) \\ I(x_{10},(\overline{1},\overline{1+c}),\frac{1}{2}(x+c,-x+c)) \end{array}}$$

and the character formulas
(12.1.5)(b)

$$\boxed{\begin{array}{c} J(x_{10},(\overline{0},\overline{c}),\frac{1}{2}(x+c,-x+c)) = I(x_{10},(\overline{0},\overline{c}),\frac{1}{2}(x+c,-x+c)) - I(x_9,c,x) \\ J(x_{10},(\overline{1},\overline{1+c}),\frac{1}{2}(x+c,-x+c)) = I(x_{10},(\overline{1},\overline{1+c}),\frac{1}{2}(x+c,-x+c)) - I(x_9,c,x) \end{array}}$$

In all remaining cases $\Delta(\gamma)$ is of type A_1.

12.2 $\Delta(\gamma) = \{\pm(1,-1)\}$ or $\{\pm(1,1)\}$

As in (12.1) we are on the $\mathbb{C}^*$ or split Cartan. On the $\mathbb{C}^*$ Cartan take

$$I(x_9,c,x) = I(x_9,(c,0),\frac{1}{2}(x,x)) \quad (x \notin \mathbb{Z}, x\ge 0, c>0).$$

with

$$\gamma = \frac{1}{2}(x+c,x-c)$$

for which $(1,-1)$ is the unique positive integral root (and is imaginary). (If $x<c$ then we can conjugate to the fiber of `x4` to make the infinitesimal character dominant; then $(1,1)$ will be the unique positive integral root.)

The standard module is irreducible.

Here is an example; compare Section 12.1.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: 0 -1
```

```
denominator for nu: 8
numerator for nu: 0 5
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [8,1]/8, lambda = [1,0]/1
Subsystem on dual side is of type A1, with roots 4.
Given parameters define element 0 of the following block:
0(0,2):  0  [i2]  0   (1,2)  *( 9,  [1,0]= rho+   [0,-1])   1  1
1(1,0):  1  [r2]  2   (0,*)  *(10,  [1,0]= rho+   [0,-1])   0  e
2(1,1):  1  [r2]  1   (0,*)  *(10, [1,-1]= rho+   [0,-2])   0  e
KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1
```

Recall the situation is $x \notin \mathbb{Z}, x, c > 0$. Compare (12.1.5)(a) and (12.1.5)(b) , which is the same except that $x \in \mathbb{Z}$ of opposite parity to c. The standard modules are

(12.2.6)(a)

$$\boxed{\begin{array}{l} I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x)) \quad (x \notin \mathbb{Z}, x > c) \\ I(x_{10}, (\overline{0}, \overline{c}), \frac{1}{2}(x + c, x - c)) \\ I(x_{10}, (\overline{1}, \overline{1 + c}), \frac{1}{2}(x + c, x - c)) \end{array}}$$

and the character formulas are as in (12.1.5)(b)

(12.2.6)(b)

$$\boxed{\begin{array}{c} J(x_{10}, (\overline{0}, \overline{c}), \frac{1}{2}(x + c, x - c)) = I(x_{10}, (\overline{0}, \overline{c}), \frac{1}{2}(x + c, x - c)) - I(x_9, c, x) \\ J(x_{10}, (\overline{1}, \overline{1 + c}), \frac{1}{2}(x + c, x - c)) = I(x_{10}, (\overline{1}, \overline{1 + c}), \frac{1}{2}(x + c, x - c)) - I(x_9, c, x) \end{array}}$$

As in (12.1.5)(a) and (12.1.5)(b), if $x < c$, it may be convenient to use the formula (12.1.4) to write

(12.2.7)

$$\boxed{\begin{array}{c} J(x_{10}, (\overline{0}, \overline{c}), \frac{1}{2}(x + c, -x + c)) = I(x_{10}, (\overline{0}, \overline{c}), \frac{1}{2}(x + c, -x + c)) - I(x_9, c, x) \\ J(x_{10}, (\overline{1}, \overline{1 + c}), \frac{1}{2}(x + c, -x + c)) = I(x_{10}, (\overline{1}, \overline{1 + c}), \frac{1}{2}(x + c, -x + c)) - I(x_9, c, x) \end{array}}$$

instead.

12.3 $\Delta(\gamma) = \{\pm(0, 2)\}$

Consider

$$I(x_k, (\overline{a}, b), (x, 0)) \quad (k = 7, 8; x \notin \mathbb{Z}; x > b)$$

with $\gamma = (x, b)$ dominant. This standard module is irreducible. On the split Cartan we have

(12.3.8) $$I(x_{10}, (\overline{a}, \overline{b}), (x, b).$$

The character formula is (recall $x > b$):

(12.3.9)

$$\boxed{J(x_{10}, (\overline{a}, \overline{b}), (x, b)) = I(x_{10}, (\overline{a}, \overline{b}), (x, b)) - I(x_7, (\overline{a}, b), (x, 0)) - I(x_8, (\overline{a}, b), (x, 0))}$$

Remark 12.3.10 *If we choose $b = 0$,* `nblock` *yields:*

```
real: nblock
choose Cartan class (one of 0,1,2,3): 2
Choose a KGB element from the following canonical fiber:
 7:  2  [C,n]    5   8     *  10  (0,0)#2 1,2,1
 8:  2  [C,n]    6   7     *  10  (0,1)#2 1,2,1
KGB number: 7
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[0,1] (>=-1)
Give lambda-rho: -1 -1
denominator for nu: 3
numerator for nu: 0 2
Name an output file (return for stdout, ? to abandon):
x = 7, gamma = [2,0]/3, lambda = [0,0]/1
Subsystem on dual side is of type A1, with roots 5.
Given parameters define element 1 of the following block:
0(0,1):  0  [i1]  1   (2,*)  *( 8,  [0,0]= rho+  [-1,-1])  2
1(1,1):  0  [i1]  0   (2,*)  *( 7,  [0,0]= rho+  [-1,-1])  2
2(2,0):  1  [r1]  2   (1,0)   (10,  [0,0]= rho+  [-1,-1])  e
(cumulated) KL polynomials (-1)^{l(1)-l(x)}*P_{x,1}:
1: 1
```

The block contains only the two modules which are related by a cross action through the imaginary root. Once again, the parameter attached to Cartan 3 is not final; the corresponding principal series representation is the direct sum of the two modules 0 and 1.

12.4 $\Delta(\gamma) = \{\pm(2,0)\}$

Consider

$$I(x_k, (\overline{a}, b), (x, 0)) \quad (k = 7, 8; x \notin \mathbb{Z}; x < b).$$

Now $\gamma = (x, b)$ is not dominant, but conjugate to (b, x); this is dominant, and gives integral roots $\pm(2,0)$. Probably it is better to think of this as

$$I(x_\ell, (b, \overline{a}), (0, x)) \quad (\ell = 5, 6; x \notin \mathbb{Z}; x < b). \tag{12.4.11}$$

which makes it clear that $(2,0)$ is an imaginary root.

These standard modules are irreducible. There is also a standard module on the split Cartan:

$$I(x_{10}, (\overline{a}, \overline{b}), (x, b)) \tag{12.4.12}$$

The character formula is precisely as in (12.3.9) (recall that now $x < b$):

(12.4.13)

$$\boxed{J(x_{10}, (\overline{a}, \overline{b}), (x, b)) = I(x_{10}, (\overline{a}, \overline{b}), (x, b)) - I(x_7, (\overline{a}, b), (x, 0)) - I(x_8, (\overline{a}, b), (x, 0))}$$

However it is probably clearer to use (12.4.11) (and rule (c) in Section 2 for conjugation by an element of W_r) and write this as (for $x < b$):

(12.4.14)

$$\boxed{\begin{aligned} &J(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) \\ &\quad = I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) - I(x_5, (b, \overline{a}), (0, x)) - I(x_6, (b, \overline{a}), (0, x)) \end{aligned}}$$

13 c-invariant Hermitian Forms

Every irreducible representation J has a distinguished c-invariant Hermitian form. We think of this as a virtual K-representation with coefficients in $\mathbb{W} = \mathbb{Z}[s]$ where $s^2 = 1$. A term $(p + qs)\mu$ means that the K-type μ has multiplicity $p + q$, and occurs p times with a plus sign, and q with a minus. In other words the μ-isotypic space is $V_\mu \otimes V[\mu]$ where V_μ is the space of μ equipped with a positive definite form, and $V[\mu] = \mathbb{C}^{p+q}$ with a form of signature (p, q).

Definition 13.1 *(a) $J(x,\lambda,\nu)_c$ is the irreducible representation $J(x,\lambda,\nu)$, equipped with its canonical c-invariant Hermitian form, normalized to be positive on the lowest K-types.*
(b) Write $J_K(x,\lambda,\nu)_c$ for the restriction of $J(x,\lambda,\nu)$ to K. I won't always make this distinction.
(c) Suppose $I_K(x,\lambda)$ is a nonzero standard final limit K-representation. Let $I_K(x,\lambda)_c$ denote this module equipped with the canonical c-invariant Hermitian form which is positive on the (unique) lowest K-type.

The $I_K(x,\lambda)_c$ form a basis of the Grothendieck group of K-representations, equipped with a c-invariant Hermitian form.

We want to compute formulas of the form

(13.2)(a)
$$J_K(x,\lambda,\nu)_c = \sum_{x',\lambda'} a(x',\lambda') I_K(x',\lambda')_c$$

where the sum is over nonzero final standard limit K-data, and $a(x',\lambda') \in \mathbb{W}$. This is an identity in the Grothendieck group of K-modules with a c-invariant form.

This proceeds in two steps. We first write $J(x,\lambda,\nu)_c$ as a linear combination of $I(x',\lambda',\nu')_c$ with coefficients in $\mathbb{W}$. Recall (11.1) there is a character formula

(13.2)(b)
$$J(x,\lambda,\nu) = \sum_{(x',\lambda',\nu')} (-1)^{\ell(x,\lambda,\nu)-\ell(x',\lambda',\nu')} P_{((x',\lambda',\nu'),(x,\lambda,\nu))}(1) I(x',\lambda',\nu')$$

This is an identity in the Grothendieck group of $(\mathfrak{g},K)$-modules and the sum is over the block containing $J(x,\lambda,\nu)$.

If $I(x,\lambda,\nu)$ is a standard module, $\mathrm{gr}I(x,\lambda,\nu)$ has a distinguished, nondegenerate, c-invariant form, obtained by deforming ν in the outward direction so it becomes irreducible. We denote this $I(x,\lambda,\nu)_c$. Note that $I(x,\lambda,\nu)_c$ is lower semi-continuous in ν. As usual write $I_K(x,\lambda,\nu)_c$ to denote the restriction to K.

In the equal rank case there is a simple generalization of (b):

(13.2)(c)
$$J(x,\lambda,\nu)_c = \sum_{(x',\lambda',\nu')} \epsilon(s) P_{((x',\lambda',\nu'),(x,\lambda,\nu))}(s) I(x',\lambda',\nu')_c$$

where

$$\epsilon(s) = s^{\frac{\ell_0(x,\lambda,\nu)-\ell_0(x',\lambda',\nu')}{2}}(-1)^{\ell(x,\lambda,\nu)-\ell(x',\lambda',\nu')}. \tag{13.2)(d}$$

This is an identity in the Grothendieck group of $(\mathfrak{g}, K)$-modules equipped with a c-invariant form. The sum is over the block containing $J(x, \lambda, \nu)$; taking $s = 1$ gives (b). The integers ℓ_0 are the orientation numbers of Section 9.

The second step is to write $I_K(x', \lambda', \nu')_c$ in terms of $I_K(x'', \lambda'')_c$. This is by *deforming* ν *to* 0, which we defer to Section 14.

Here is how to compute (13.2)(c). Use `nblock` to define $J(x, \lambda, \nu)$ and compute its block; this is at possibly nonintegral or singular infinitesimal character γ. Each parameter in the output of `nblock` may have an asterisk, indicating which of the terms are nonzero at γ. The output also includes a computation of the $P_{\delta,\mu}$ for $\mu = (x, \lambda, \nu)$.

Note that the character formula (b) gives the c-invariant form formula (c) provided $P_{*,*}$ is constant for all terms occcuring, and all orientation numbers are 0. All orientation numbers are 0 for integral infinitesimal character.

13.1 c-Invariant Forms: Integral Infinitesimal Character

Since the orientation numbers are all 0 the character formulas of Section 11 hold as stated unless some $P_{\mu,\delta}$ is not a constant. The only case in which this happens is formula (11.2)(i) in which the terms `2,3` have a coefficient of q.

Therefore formulas (11.2)(a-h,j,k) are all valid as formulas for the c-invariant form, except that (11.2)(i) should read:

$$\begin{aligned} J(x_{10}, (\overline{a}+1, \overline{b}+1), (a,b))_c = & I(x_{10}, (\overline{a}, \overline{b}), (a,b))_c \\ & - sI(x_2, (a,b))_c - sI(x_3, (a,b))_c \\ & - I(x_9, (a-b, 0), \frac{1}{2}(a+b, a+b))_c \end{aligned} \tag{13.1.5}$$

13.2 c-Invariant Forms: Nonintegral Infinitesimal Character

In the case of nonintegral infinitesimal character, the integral root system is type D_2 or A_1, and all the P polynomials are constant. Therefore the only way for a c-invariant form formula to differ from the character formula is from a difference of orientation numbers being odd.

So we have to consider character formulas (12.1.2)(b), (12.1.5)(b), (12.2.6)(b), (12.3.9), and (12.4.13) and see which of them require a correction due to the orientation numbers.

Cases (12.1.2)(b), (12.1.5)(b): $I(x_9, c, x)$ with x, c integers of opposite parity.

Combining (12.1.2)(b) and (12.1.5)(b) gives, for $x, y \in \mathbb{Z} + \frac{1}{2}$, $x > y > 0$,

(13.2.6)
$$J(x_{10}, (\overline{a}, \overline{b}), (x, y)) = I(x_{10}, (\overline{a}, \overline{b}), (x, y)) - \begin{cases} I(x_9, x-y, x+y) & x+y = \overline{a}+\overline{b}+1 \pmod{2\mathbb{Z}} \\ I(x_9, x+y, x-y) & x+y = \overline{a}+\overline{b} \pmod{2\mathbb{Z}} \end{cases}$$

The orientation numbers for $I(x_{10}, (*, *), \frac{1}{2}(x+c, x-c))$ are given in Table 9.3.7, and for $I(x_9, c, x)$ in Table 9.1.1.

A little monkeying around shows the following.

If $x + y = \overline{a} + \overline{b} + 1 \pmod{2\mathbb{Z}}$ then

$$\ell_0(x_{10}, (\overline{a}, \overline{b}), (x, y)) = 1, \ell_0(x_9, x-y, x+y) = 1$$

so there is no contribution from the orientation numbers, and (12.1.2)(b) holds as a formula for c-invariant forms as follows.

Assume $x + y = a + b + 1 \pmod{2\mathbb{Z}}$, with $x, y \in \mathbb{Z} + \frac{1}{2}$, $x > y > 0$. Then (13.2.7)

$$\boxed{J(x_{10}, (\overline{a}, \overline{b}), (x, y))_c = I(x_{10}, (\overline{a}, \overline{b}), (x, y))_c - I(x_9, x-y, x+y)_c.}$$

Now suppose $x + y = \overline{a} + \overline{b} \pmod{2\mathbb{Z}}$. Then

$$\ell_0(x_9, x+y, x-y) = 0.$$

and on the other hand

$$\ell_0(x_{10}, (\overline{a}, \overline{b}), (x, y)) = \begin{cases} 0 & 0 < y - \overline{b} < 1 \\ 2 & 1 < y - \overline{b} < 2 \end{cases}$$

So this gives the first case of a nontrivial orientation number.

So: $x+y=\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) implies

$$(13.2.8)\qquad \boxed{\begin{aligned}J(x_{10},(\overline{a},\overline{b}),(x,y))_c &= I(x_{10},(\overline{a},\overline{b}),(x,y))_c\\ &\quad-\begin{cases}I(x_9,x+y,x-y)_c & 0<y-\overline{b}<1\\ sI(x_9,x+y,x-y)_c & 1<y-\overline{b}<2\end{cases}\end{aligned}}$$

Case (12.2.6)(b): $x\notin\mathbb{Z}$. This is very similar to cases (12.1.2)(b), (12.1.5)(b); see (13.2.7) and (13.2.8).

Suppose $x-y\in\mathbb{Z}, x+y\notin\mathbb{Z}$. If $x-y=\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) then

$$(13.2.9)(a)\qquad J(x_{10},(\overline{a},\overline{b}),(x,y))=I(x_{10},(\overline{a},\overline{b}),(x,y))-I(x_9,x-y,x+y)$$

If $x-y\neq\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) then $I(x_{10},(\overline{a},\overline{b}),(x,y))$ is irreducible.

Suppose $x+y\in\mathbb{Z}, x-y\notin\mathbb{Z}$. If $x+y=\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) then

$$(13.2.9)(b)\qquad J(x_{10},(\overline{a},\overline{b}),(x,y))=I(x_{10},(\overline{a},\overline{b}),(x,y))-I(x_9,x+y,x-y)$$

If $x+y\neq\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) then $I(x_{10},(\overline{a},\overline{b}),(x,y))$ is irreducible.

In (a) all terms have $\ell_0=1$ if $0<y<\frac{1}{2}$ (mod $\mathbb{Z}$), and $\ell_0=2$ if $\frac{1}{2}<y<1$ (mod $\mathbb{Z}$), so this holds as a formula for c-invariant forms. In other words (13.2.7) still holds here: $x-y\in\mathbb{Z}, x+y\notin\mathbb{Z}, x-y=\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) implies

$$(13.2.10)\qquad \boxed{J(x_{10},(\overline{a},\overline{b}),(x,y))_c=I(x_{10},(\overline{a},\overline{b}),(x,y))_c-I(x_9,x-y,x+y)_c}$$

Similarly (13.2.8) still holds here: $x+y\in\mathbb{Z}, x-y\notin\mathbb{Z}, x+y=\overline{a}+\overline{b}$ (mod $2\mathbb{Z}$) implies

$$(13.2.11)\qquad \boxed{\begin{aligned}J(x_{10},(\overline{a},\overline{b}),(x,y))_c &= I(x_{10},(\overline{a},\overline{b}),(x,y))_c\\ &\quad-\begin{cases}I(x_9,x+y,x-y)_c & 0<y-\overline{b}<1\\ sI(x_9,x+y,x-y)_c & 1<y-\overline{b}<2\end{cases}\end{aligned}}$$

Case (12.3.9):

Now $x > y = b > 0$ with $b \in \mathbb{Z}$. The character formula in this case is:
(13.2.12)
$J(x_{10}, (\overline{a}, \overline{b}), (x, b)) = I(x_{10}, (\overline{a}, \overline{b}), (x, b)) - I(x_7, (\overline{a}, b), (x, 0)) - I(x_8, (\overline{a}, b), (x, 0))$

The orientation numbers ℓ_0 of all terms are the same, either 1 or 2, so this holds as a formula for c-invariant forms:
(13.2.13)
$$\boxed{J(x_{10}, (\overline{a}, \overline{b}), (x, b))_c = I(x_{10}, (\overline{a}, \overline{b}), (x, b))_c - I(x_7, (\overline{a}, b), (x, 0))_c - I(x_8, (\overline{a}, b), (x, 0))_c}.$$

Case (12.4.13):

This is analogous to the last case, with $x \notin \mathbb{Z}$ and $b \in \mathbb{Z}$, except that now $x < b$. The character formula is

$$\begin{aligned} J(x_{10},&(\overline{b+1}, \overline{a+1}), (b, x)) \\ &= I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) - I(x_7, (\overline{a}, b), (x, 0)) - I(x_8, (\overline{a}, b), (x, 0)). \end{aligned}$$

If $1 < x - a < 2$ (mod $2\mathbb{Z}$) then the orientation numbers of all terms are $\ell_0 = 1$; if $0 < x - a < 1$ (mod $2\mathbb{Z}$) then $I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x))$ has orientation number 2, while the other terms have $\ell_0 = 0$. So we have
(13.2.14)
$$\boxed{\begin{aligned} &J(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x))_c = I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x))_c \\ &\qquad \begin{cases} -sI(x_7, (\overline{a}, b), (x, 0))_c - sI(x_8, (\overline{a}, b), (x, 0))_c & 0 < x - \overline{a} < 1 \\ -I(x_7, (\overline{a}, b), (x, 0))_c - I(x_8, (\overline{a}, b), (x, 0))_c & 1 < x - \overline{a} < 2 \end{cases} \end{aligned}}$$

14 Deforming ν to 0

In the previous section we reduced the computation of $J(x, \lambda, \nu)_c$ to computing $I(x', \lambda', \nu')_c$ (see (13.2)(c)). In this section we discuss how to write

$$I(x, \lambda, \nu)_c = \sum_{x', \lambda'} b(x', \lambda') I_K(x', \lambda')_c \tag{14.1}$$

for $b(x', \lambda') \in \mathbb{W}$, and the sum is over nonzero final standard limit K-data. This proceeds by deformation to $\nu = 0$, and by induction, which requires using (13.2)(c) along the way.

Fix (x, λ, ν) and consider $I(x, \lambda) = I(x, \lambda, 0)$. Assume for the moment that $I(x, \lambda)$ is final (and nonzero), so $I_K(x, \lambda)$ is a final limit standard K-representation.

After deforming ν if necessary, we can assume $I(x, \lambda, t_k\nu)$ is reducible for finitely many $0 < t_1 < \cdots < t_n \leq 1$. This implies

$$I(x, \lambda, t\nu)_c = I_K(x, \lambda)_c \quad (t < t_1)$$

Assume we've computed $I(x, \lambda, t\nu)_c$ for $t_{k-1} \leq t < t_k$.

Write $\gamma = (x, \lambda, t_k\nu)$.

First compute the composition factors $J(\gamma')$ of $I(\gamma)$, and the polynomials $Q(\gamma', \gamma)$. (Recall these are the polynomials satisfying $I(\gamma) = \sum_{\gamma'} Q(\gamma', \gamma)(1)J(\gamma')$. Currently `nblock` computes $P(\gamma', \gamma)$. See Section 15.)

The c-invariant form changes sign on the odd levels of the Jantzen filtration. What this amounts to is the following.

For each γ' with $Q(\gamma', \gamma) \neq 0$ write

$$Q(\gamma', \gamma)(q) = \sum_{n=0}^{\ell(\gamma)-\ell(\gamma')} a_n(\gamma', \gamma) q^{\frac{\ell(\gamma)-\ell(\gamma')-n}{2}} \tag{14.10)(a}$$

Note that $a_n(\gamma', \gamma) = 0$ unless $\ell(\gamma) - \ell(\gamma') = n \pmod{2\mathbb{Z}}$, so

$$Q(\gamma', \gamma)(q) = \sum_{\substack{n=0 \\ n\equiv\ell(\gamma)-\ell(\gamma')}}^{\ell(\gamma)-\ell(\gamma')} a_n(\gamma', \gamma) q^{\frac{\ell(\gamma)-\ell(\gamma')-n}{2}} \tag{14.10)(b}$$

Then a_n is the multiplicity of $J(\gamma')$ in level n of the Jantzen filtration. Note that γ' can occur in an odd level of the Jantzen filtration of $I(\gamma)$ only if $\ell(\gamma) - \ell(\gamma')$ is odd. Therefore

$$\begin{aligned} I(x,\lambda, t_k\nu)_c &= I(x, \lambda, t_{k-1}\nu)_c \\ &+ (1-s) \sum_{\substack{\gamma' \\ \ell(\gamma')-\ell(\gamma) \text{ odd}}} \sum_{n \text{ odd}} s^{\frac{\ell(\gamma)-\ell(\gamma')-n}{2}} s^{\frac{\ell_0(\gamma)-\ell_0(\gamma')}{2}} a_n(\gamma', \gamma) J(\gamma')_c \end{aligned} \tag{14.11}$$

By (14.10)(b) the inner sum, (after pulling out the ℓ_0 term), is just $Q(\gamma', \gamma)(s)$. So:

$$\boxed{\begin{aligned} I(x, \lambda, t_k\nu)_c &= I(x, \lambda, t_{k-1}\nu)_c \\ &+ (1-s) \sum_{\substack{\gamma' \\ \ell(\gamma')-\ell(\gamma) \text{ odd}}} s^{\frac{\ell_0(\gamma)-\ell_0(\gamma')}{2}} Q(\gamma', \gamma)(s) J(\gamma')_c \end{aligned}} \tag{14.12}$$

Here is the big block for $Sp(4,\mathbb{R})$.

length 0

Of course there is nothing to do here:

$$J(x_k,(a,b))_c = I_K(x_k,(a,b))_c \quad (0 \le k \le 3) \tag{14.13}$$

length 1, Cartan 1

Consider $I(x_9,(a+b,0),\frac{1}{2}(a-b,a-b))$. It is easier to use the other coordinates $I(x_9,c,x) = I(x_9,(c,0),\frac{1}{2}(x,x))$ (2.1.4)(a). Notice that length 1 implies $c > x$.

The real root $(1,1)$ gives reducibility if and only if $x = c \pmod 2$. The complex root $(2,0)$ gives reducibility if and only if $x = c \pmod 2$ and $x \ge c$.

Here is how to pass back and forth. If $c > x$:

$$I(x_9,c,x) = I(x_9,(a+b,0),\frac{1}{2}(a-b,a-b)) \quad (a = \frac{1}{2}(c+x), b = \frac{1}{2}(c-x))$$

and these have length 1, while if $c \le x$:

$$I(x_9,c,x) = I(x_9,(a-b,0),\frac{1}{2}(a+b,a+b)) \quad (a = \frac{1}{2}(c+x), b = \frac{1}{2}(x-c))$$

of length 2.

Suppose c is even. Then

(14.14)(a) $$I(x_9,c,x)_c = I_K(x_9,c)_c \quad (0 \le x < 2)$$

(This module is not final, so we can write it as a sum of two limits of discrete series...but ignore this for this calculation.) If $c > 2$ then, taking $x = 2$, (10.2)(a) gives

(14.14)(b) $$I(x_9,c,2) = J(x_9,c,2) + J(x_0,\tfrac{1}{2}(c+2,c-2)) + J(x_1,\tfrac{1}{2}(c+2,c-2))$$

Apply (14.12) (or (14.11)). Since the reducibility points are at integral infinitesimal character the orientation numbers are 0, and for $2 \le x < 4$:

(14.14)(c)

$$\begin{aligned} I(x_9,c,x)_c = {} & I_K(x_9,c)_c \\ & + (1-s)[I_K(x_0,\tfrac{1}{2}(c+2,c-2))_c + I_K(x_1,\tfrac{1}{2}(c+2,c-2))_c] \end{aligned}$$

Similarly if $c > 4$ then, taking $x = 4$, (10.2)(a) gives

(14.14)(d) $$I(x_9,c,4) = J(x_9,c,4) + J(x_0,\tfrac{1}{2}(c+4,c-4)) + J(x_1,\tfrac{1}{2}(c+4,c-4))$$

and by (14.12) for ($4 \le x < 6$):
(14.14)(e)

$$\begin{aligned}\mathcal{I}(x_9, c, x)_c = I_K(x_9, c)_c &\\ &+ (1-s)[I_K(x_0, \tfrac{1}{2}(c+2, c-2))_c + I_K(x_1, \tfrac{1}{2}(c+2, c-2))_c] \\ &+ (1-s)[I_K(x_0, \tfrac{1}{2}(c+4, c-4))_c + I_K(x_1, \tfrac{1}{2}(c+4, c-4))_c]\end{aligned}$$

By induction on x we see for c even, $x < c$:
(14.14)(f)

$$\boxed{\begin{aligned}I(x_9, c, x)_c = I_K(x_9, c)_c &\\ &+ (1-s)\sum_{k=1}^{[\frac{x}{2}]}[I_K(x_0, \tfrac{1}{2}(c+2k, c-2k))_c + I_K(x_1, \tfrac{1}{2}(c+2k, c-2k))_c]\end{aligned}}$$

Similarly for c odd, $x < c$:

$$\text{(14.14)(g)}\quad \boxed{\begin{aligned}I(x_9, c, x)_c = I_K(x_9, c)_c &\\ &+ (1-s)\sum_{k=1}^{\frac{x+1}{2}}[I_K(x_0, \tfrac{1}{2}(c+(2k-1), c-(2k-1)))_c \\ &\qquad + I_K(x_1, \tfrac{1}{2}(c+(2k-1), c-(2k-1)))_c]\end{aligned}}$$

In particular at ρ take $c = 3, x = 1$:
(14.14)(h)

$$\boxed{I(x_9, 3, 1)_c = I_K(x_9, 3)_c + (1-s)[I_K(x_0, (2,1))_c + I_K(x_1, (2,1))_c]}$$

length 1, Cartan 2

Consider the representations

$$I(x_k, (\delta, a), (x, 0)) \quad (k = 7, 8; a \in \mathbb{Z}_{\ge 0})$$

which have length 1 if $x < a$. Such a representation is reducible if and only if $x = \delta \pmod 2$.

Take $k = 7$ and $\delta = 0$. There will be reducibility at $x = 0, 2, \dots,$. We start with

$$\text{(14.15)(a)}\qquad I(x_7, (\overline{0}, a), (0,0)) = I_K(x_7, (\overline{0}, a)).$$

For $0 \le x < 2 \le a$, we get

$$I(x_7, (\overline{0}, a), (x, 0))_c = I_K(x_7, (\overline{0}, a))_c.$$

If $2 < a$ then taking $x = 2$ in (11.2)(c) gives

$$I(x_7, (\overline{0}, a), (2,0)) = J(x_7, (\overline{0}, a), (2,0)) + J(x_0, (a,2)) + J(x_2, (a,2))$$

so by (14.12), if $(2 \le x < 4)$:

$$I(x_7, (\overline{0}, a), (x,0))_c = I_K(x_7, (\overline{0}, a), (x,0))_c + (1-s)[J(x_0, (a,2)) + J(x_2, (a,2))_c]$$

Repeating this as in the previous case we conclude, for $x < a$:

(14.15)(b)

$$\boxed{I(x_7, (\overline{0},a), (x,0))_c = I_K(x_7, (\overline{0}, a))_c + (1-s)\sum_{k=1}^{[\frac{x}{2}]}[I_K(x_0, (a,2k))_c + I_K(x_2, (a,2k))_c]}$$

Similarly using (10.2)(d)

(14.15)(c)

$$\boxed{I(x_8, (\overline{0},a), (x,0))_c = I_K(x_8, (\overline{0}, a))_c + (1-s)\sum_{k=1}^{[\frac{x}{2}]}[I_K(x_1, (a,2k))_c + I_K(x_3, (a,2k))_c]}$$

The cases $k = 7, 8$, $\delta = \overline{1}$ are similar. The results are:

(14.15)(d)

$$\boxed{I(x_7, (\overline{1},a), (x,0))_c = I_K(x_7, (\overline{1}, a))_c + (1-s)\sum_{k=1}^{[\frac{x+1}{2}]}[I_K(x_0, (a,2k-1))_c + I_K(x_2, (a,2k-1))_c]}$$

(14.15)(e)

$$\boxed{I(x_8, (\overline{1},a), (x,0))_c = I_K(x_8, (\overline{1}, a))_c + (1-s)\sum_{k=1}^{[\frac{x+1}{2}]}[I_K(x_1, (a,2k-1))_c + I_K(x_3, (a,2k-1))_c]}$$

Consider the case of infinitesimal character $\rho = (2,1)$. Formulas (14.15)(b-e) specialize as follows.

Take $a = 2, x = 1$ in (14.15)(b,c):

(14.16)(a) $$I(x_7, (\overline{0}, 2), (1,0))_c = I_K(x_7, (\overline{0}, 2))_c$$

(14.16)(b) $$I(x_8, (\overline{0}, 2), (1,0))_c = I_K(x_8, (\overline{0}, 2))_c$$

Take $a = 2, x = 1$ in (14.15)(d,e):

(14.16)(c)
$$\boxed{I(x_7, (\overline{1}, 2), (1,0))_c = I_K(x_7, (\overline{1}, 2))_c + (1-s)[J(x_0, (2,1))_c + J(x_2, (2,1))_c]}$$

(14.16)(d)
$$\boxed{I(x_8, (\overline{1}, 2), (1,0))_c = I_K(x_8, (\overline{1}, 2))_c + (1-s)[J(x_1, (2,1))_c + J(x_3, (2,1))_c]}$$

This completes the length 1 portion of our program. From now on we'll only include some special cases of a and b.

length 2, Cartan 1

These are the modules $I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x))$ with $c \le x$. This is reducible due to the real root $(1,1)$ if $x = c \pmod 2$. It is reducible due to the complex root $(2,0)$ provided this is integral, which is again the condition $x = c \pmod 2$. So if c is odd this is reducible at $x = 1, 3, \dots, c, c+2, \dots$.

Take $c = 1$. Then $x = 3$ gives infinitesimal character $(\frac{x+c}{2}, \frac{x-c}{2}) = (2,1) = \rho$.

Starting at $x = 0$:

(14.17)(a) $$I(x_9, 1, x) = I_K(x_9, 1) \quad (x < 1).$$

Reducibility at $x = 1$:

(14.17)(b) $$I(x_9, 1, 1) = J(x_9, 1, 1) + J(x_0, (1,0)) + J(x_1, (1,0))$$

The last two terms are limits of large discrete series. By (14.12) (since we're at integral infinitesimal character all orientation numbers are 0):

(14.17)(c) $I(x_9, 1, 1)_c = I_K(x_9, 1)_c + (1-s)[I_K(x_0, (1,0))_c + I_K(x_1, (1,0))_c]$

Reducibility at $x = 3$; the infinitesimal character is $\frac{1}{2}(3+1, 3-1) = (2,1) = \rho$, so this is directly from (10.2)(b):

$$\begin{aligned} I(x_9, 1, 3) &= J(x_9, 1, 3)^2 \\ &\quad + J(x_0, (2,1))^0 + J(x_1, (2,1))^0 \\ &\quad + J(x_9, 3, 1)^1 \\ &\quad + J(x_7, (\overline{1}, 2), (1,0))^1 + J(x_8, (\overline{1}, 2), (1,0))^1 \end{aligned} \tag{14.17(d)}$$

with lengths denoted by superscripts. Therefore (no orientation numbers here; and recall that only modules with odd length difference occur in the sum (14.12)):

(14.17)(e)

$$\begin{aligned} I(x_9, 1, 3)_c &= I(x_9, 1, 1)_c \\ &\quad + (1-s)[J(x_9, 3, 1)_c + J(x_7, (\overline{1}, 2), (1,0))_c + J(x_8, (\overline{1}, 2), (1,0))_c] \end{aligned}$$

We'll plug in (c) for the first term.

Now for the first time we need to express the c-invariant form on an irreducible (each of the three in the last line) in terms of c-invariant forms on standards, using Section 13, which goes back to the character formulas of Section 11 in this case. Thus by (11.2)(a)

$$J(x_9, 3, 1)_c = I(x_9, 3, 1)_c - J(x_0, (2,1))_c - J(x_1, (2,1))_c \tag{14.17(f)}$$

and plug in (14.14)(h) for $I(x_9, 3, 1)$ to give:

$$\begin{aligned} J(x_9, 3, 1)_c &= I_K(x_9, 3)_c \\ &\quad + (1-s)[I_K(x_0, (2,1))_c + I_K(x_1, (2,1))_c] \\ &\quad - I_K(x_0, (2,1))_c - I_K(x_1, (2,1))_c \end{aligned} \tag{14.17(g)}$$

which simplifies to

$$\boxed{J(x_9, 3, 1)_c = I_K(x_9, 3)_c - s[I_K(x_0, (2,1))_c + I_K(x_1, (2,1))_c]} \tag{14.17(h)}$$

Similarly (11.2)(c) says:

(14.17)(i)

$$J(x_7, (\overline{1}, 2), (1,0))_c = I(x_7, (\overline{1}, 2), (1,0))_c - I(x_0, (2,1))_c - I(x_2, (2,1))_c.$$

Use (14.16)(c) to expand

(14.17)(j)

$$I(x_7, (\overline{1}, 2), (1,0))_c = I_K(x_7, (\overline{1}, 2))_c + (1-s)[J(x_0, (2,1))_c + J(x_2, (2,1))_c]$$

and so
(14.17)(k)

$$\boxed{J(x_7,(\overline{1},2),(1,0))_c = I_K(x_7,(\overline{1},2))_c - s[I_K(x_0,(2,1))_c + I_K(x_2,(2,1))_c]}$$

Finally (11.2)(d) says:
(14.17)(l)

$$J(x_8,(\overline{1},2),(1,0))_c = I(x_8,(\overline{1},2),(1,0))_c - I(x_1,(2,1))_c - I(x_3,(2,1))_c.$$

and using (14.16)(d) to expand
(14.17)(m)

$$I(x_8,(\overline{1},2),(1,0))_c = I_K(x_8,(\overline{1},2))_c + (1-s)[J(x_1,(2,1))_c + J(x_3,(2,1))_c]$$

gives
(14.17)(n)

$$\boxed{J(x_8,(\overline{1},2),(1,0))_c = I_K(x_8,(\overline{1},2))_c - s[I_K(x_1,(2,1))_c + I_K(x_3,(2,1))_c].}$$

Plugging (c), (h), (k) and (n) into (e) gives:
(14.17)(o)

$$\begin{aligned} I(x_9,1,3)_c = I_K(x_9,1)_c &+ (1-s)[I_K(x_0,(1,0))_c + I_K(x_1,(1,0))_c] \\ &+ (1-s)\{I_K(x_9,3)_c - s[I_K(x_0,(2,1))_c + I_K(x_1,(2,1))_c]\} \\ &+ (1-s)\{I_K(x_7,(\overline{1},2))_c - s[I_K(x_0,(2,1))_c + I_K(x_2,(2,1))_c]\} \\ &+ (1-s)\{I_K(x_8,(\overline{1},2))_c - s[I_K(x_1,(2,1))_c + I_K(x_3,(2,1))_c]\}. \end{aligned}$$

Grouping terms finally gives:
(14.17)(p)

$$\boxed{\begin{aligned} I(x_9,1,3)_c = I_K(x_9,1)_c &+ (1-s)[I_K(x_0,(1,0))_c + I_K(x_1,(1,0))_c] \\ &+ (1-s)\{2\times I_K(x_0,(2,1)_c + 2\times I_K(x_1,(2,1)_c + I_K(x_2,(2,1)_c + I_K(x_3,(2,1))_c\} \\ &+ (1-s)\{I_K(x_9,3)_c + I_K(x_7,\overline{1},2))_c + I_K(x_8,\overline{1},2))_c\}. \end{aligned}}$$

length 2, Cartan 2 We compute $I(x_7,(\overline{2},1),(2,0))_c$ and $I(x_8,(\overline{2},1),(2,0))_c$.

For $k = 7,8$, let's deform $I(x_k,(\overline{2},1),(2,0))$ to $I(x_k,(\overline{2},1),(0,0))$. By the parity condition $I(x_k,(\overline{2},1),(x,0))$ is reducible only if $x = \overline{2}$ (mod $2\mathbb{Z}$). We're taking $0 \le x \le 2$, so this occurs only at the endpoints $x = 0, x = 2$.

At $x = 0$ this has to do with the fact that $I(x_k,(\overline{2},1),(0,0))$ is not final. Ignoring this for the moment, consider $I(x_k,(\overline{2},1),(2,0))$, at infinitesimal character $(2,1)$.

The composition series at $(2,1)$ are given by (10.2)(e,f) (with lengths given by superscripts)

(14.18)(a)
$$\begin{aligned} I(x_7,(\overline{2},1),(2,0))^2 &= J(x_7,(\overline{2},1),(2,0))^2 + J(x_0,(2,1))^0 \\ &\quad + J(x_9,3,1)^1 + J(x_7,(\overline{1},2),(1,0))^1 \\ I(x_8,(\overline{2},1),(2,0)) &= J(x_8,(\overline{2},1),(2,0))^2 + J(x_1,(2,1))^0 \\ &\quad + J(x_9,3,1)^1 + J(x_8,(\overline{1},2),(1,0))^1 \end{aligned}$$

Considering terms of odd length, with the orientation numbers being all 0, (14.12) gives

(14.18)(b)
$$\begin{aligned} I(x_7,(\overline{2},1),(2,0))_c &= I(x_7,(\overline{2},1),(0,0))_c \\ &\quad + (1-s)[J(x_9,3,1)_c + J(x_7,(\overline{1},2),(1,0))_c] \\ I(x_8,(\overline{2},1),(2,0))_c &= I(x_8,(\overline{2},1),(0,0))_c \\ &\quad + (1-s)[J(x_9,3,1)_c + J(x_8,(\overline{1},2),(1,0))_c] \end{aligned}$$

Here we are using that $I(x_k,(\overline{2},1),(2-\epsilon,0))_c = I(x_k,(\overline{2},1),(0,0))_c$, since there is no reducibility for $0 < x < 2$. We know $J(x_9,3,1)_c$, $J(x_7,(\overline{1},2),(1,0))_c$, $J(x_8,(\overline{1},2),(1,0))_c$ from (14.17)(h),(k) and (n), respectively. Also use (5.4)(a) to eliminate the terms $I(x_k,(\overline{2},1),(0,0))_c$ with $k=7,8$. Plugging these in gives:

(14.18)(c)
$$\begin{aligned} I(x_7,(\overline{2},1),(2,0))_c &= I(x_0,(1,0))_c + I(x_2,(1,0))_c \\ &\quad + (1-s)\big[\{I_K(x_9,3)_c - s[I_K(x_0,(2,1))_c + I_K(x_1,(2,1))_c]\} \\ &\qquad + \{I_K(x_7,(\overline{1},2))_c - s[I_K(x_0,(2,1)) + I_K(x_2,(2,1))]\}\big] \end{aligned}$$

and this can be rewritten (recall that $(1-s)(-s) = 1-s$)

(14.18)(d)
$$\boxed{\begin{aligned} I(x_7,(\overline{2},1),(2,0))_c &= I(x_0,(1,0))_c + I(x_2,(1,0))_c \\ &\quad + (1-s)\big[I_K(x_9,3)_c + I_K(x_7,(\overline{1},2))_c \\ &\quad + 2\times I(x_0,(2,1))_c + I(x_1,(2,1))_c + I(x_2,(2,1))_c\big]. \end{aligned}}$$

Similarly

(14.18)(e)
$$\boxed{\begin{aligned} I(x_8,(\overline{2},1),(2,0))_c &= I(x_1,(1,0))_c + I(x_3,(1,0))_c \\ &\quad + (1-s)\big[I_K(x_9,3)_c + I_K(x_8,(\overline{1},2))_c \\ &\quad + I(x_0,(2,1))_c + 2\times I(x_1,(2,1))_c + I(x_3,(2,1))_c\big]. \end{aligned}}$$

15 Digression: computing the Q polynomials

In the setting of the `atlas` software, suppose we have a block, with elements `0-n`. The `klbasis` command gives the polynomials $P_{\gamma,\gamma'}(q)$ on the block.

For example consider this part of the output for the big block of $Sp(4,\mathbb{R})$:

```
10:  0: 1
     1: 1
     2: 1
     3: 1
     4: 1
     5: 1
     6: 1
     7: 1
     8: 1
     9: 1
    10: 1
```

This says the polynomials $P_{0,10}(q), P_{1,10}(q), \ldots, P_{10,10}(q)$ are all 1. Recall this means

$$J(10) = \sum_k (-1)^{\ell(10)-\ell(k)} P_{k,10}(q) I(k).$$

Taking lengths into account (from the `block` command) gives

$$J(10) = I(10)-I(9)-I(8)-I(7)+I(6)+I(5)+I(4)-I(3)-I(2)-I(1)-I(0)$$

This is the character formula for the trivial representation. See (11.2)(j).

On the other hand we're interested in the Q polynomials, which satisfy

$$I(n) = \sum_k Q_{k,n}(1) J(k)$$

Here's how to get these from `klbasis`.

Make G the dual group, and $G^\vee$ the group. Then

$$Q_{k,n}(q) = P_{n^\vee,k^\vee}(q)$$

where $k \to k^\vee$ is the duality map, given by `dualmap`.

For example take $G = SO(3,2)$, $G^\vee = Sp(4,\mathbb{R})$. Consider this part of the output of `klbasis`:

```
7:  0: 2
    1: 1
    2: 1
    3: 1
    4: 1
    7: 1
```

Also `dualmap` gives

```
[10, 11, 9, 7, 8, 5, 6, 4, 0, 1, 2, 3]
```

The first line says $P_{0,7}(q) = 2$ for $SO(3,2)$, and `dualmap` says $0^\vee = 10, 7^\vee = 4$, so

$$2 = P^{SO}_{0,7}(q) = Q^{Sp}_{4,10}(q),$$

which says $J(4)$ has multiplicity 2 in $I(10)$ for $Sp(4,\mathbb{R})$. See (10.2)(j).

16 Second Digression: An alternative version of the calculation

Recall we used (14.12) to write the form on a standard module $I(x,\lambda,\nu)$ in terms of forms on a standard module with smaller ν parameter, and forms on more tempered irreducible modules $J(\gamma')$:

(16.1)
$$\begin{aligned} I(x,\lambda,t_k\nu)_c &= I(x,\lambda,t_{k-1}\nu)_c \\ &\quad + (1-s)\sum_{\substack{\gamma' \\ \ell(\gamma')-\ell(\gamma)\ \text{odd}}} s^{\frac{\ell_0(\gamma)-\ell_0(\gamma')}{2}} Q_{\gamma',\gamma}(s) J(\gamma')_c \end{aligned}$$

We proceeded by induction, assuming we'd already computed the terms $J(\gamma')_c$.

Here is an equivalent formulation, which combines the two steps, and is easier from a computational point of view. We restate this in a self-contained, single step.

Inductive algorithm

Suppose we are given $I(\gamma) = I(x,\lambda,\nu)$. Typically this is at singular, and possibly nonintegral, infinitesimal character. For small $\epsilon > 0$, $I(x,\lambda,(1+\epsilon)\nu)$

is irreducible. By (14.12) we have

$$I(\gamma)_c = I(x,\lambda,(1-\epsilon)\nu)_c + (1-s)\sum_{\substack{\gamma'<\gamma\\ \ell(\gamma)-\ell(\gamma')\text{ odd}}} s^{\frac{\ell_0(\gamma)-\ell_0(\gamma')}{2}} Q_{\gamma',\gamma}(s)J(\gamma')_c \tag{16.2)(a}$$

By (13.2)(c), for each γ', write

$$J(\gamma')_c = \sum_{\delta'\le\gamma'} (-1)^{\ell(\gamma')-\ell(\delta')} s^{\frac{\ell_0(\gamma')-\ell_0(\delta')}{2}} P_{\delta',\gamma'}(s) I(\delta')_c \tag{16.2)(b}$$

This formula is computed at (possibly singular, non-integral) infinitesimal character. Plug it in to give

$$I(\gamma)_c = I(x,\lambda,(1-\epsilon)\nu)_c + (1-s)\sum_{\substack{\delta'\le\gamma'<\gamma\\ \ell(\gamma)-\ell(\gamma')\text{ odd}}} (-1)^{\ell(\gamma')-\ell(\delta')} s^{\frac{\ell_0(\gamma)-\ell_0(\delta')}{2}} P_{\delta',\gamma'}(s)Q_{\gamma',\gamma}(s)I(\delta')_c \tag{16.2)(c}$$

or, spelling it out more explicitly:
(16.2)(d)

$$\boxed{\begin{aligned} &I(\gamma)_c = I(x,\lambda,(1-\epsilon)\nu)_c\\ &+(1-s)\sum_{\substack{\delta'\\ \delta'<\gamma}} s^{\frac{\ell_0(\gamma)-\ell_0(\delta')}{2}}\Big[\sum_{\substack{\gamma'\\ \delta'\le\gamma'<\gamma\\ \ell(\gamma)-\ell(\gamma')\text{ odd}}} (-1)^{\ell(\gamma')-\ell(\delta')} P_{\delta',\gamma'}(s)Q_{\gamma',\gamma}(s)\Big] I(\delta')_c \end{aligned}}$$

17 Invariant Forms

Suppose J is a representation of $Sp(4,\mathbb{R})$ with a central character, and μ is a K-type of π. Identify μ with its highest weight (r,s).

The element τ defining $Sp(4,\mathbb{R})$ is $\mathrm{diag}(i,i,-i,-i)$, which is central in K, and has square $-I$. It acts in μ by the scalar i^{r+s}. Note that if $-I$ acts in J by $\epsilon=\pm 1$ then $(i^{r+s})^2=\epsilon$, i.e.

$$r+s \text{ is } \begin{cases} \text{even} & \epsilon = 1\\ \text{odd} & \epsilon=-1. \end{cases}$$

Now assume J is irreducible and $\mu=(r,s)$ is a lowest K-type of J. Write $\langle\,,\,\rangle$ for the *invariant* form on J, and $\langle\,,\,\rangle_c$ for the c-invariant form as usual.

Lemma 17.1 *Suppose we have a formula as in* (13.2)(a)

$$J(x,\lambda,\nu)_c=\sum_{x',\lambda'}a(x',\lambda')I_K(x',\lambda')_c. \tag{17.2}$$

Suppose $-I$ acts by ϵ in $J(x,\lambda,\nu)$ and choose $\zeta^2=\epsilon$. Write the (unique) lowest K-type of $I_K(x',\lambda')$ as $(r(x',\lambda'),s(x',\lambda'))$. Define

$$\delta(x',\lambda')=\begin{cases}1 & \zeta i^{r(x',\lambda')+s(x',\lambda')}=1\\ s & \zeta i^{r(x',\lambda')+s(x',\lambda')}=-1.\end{cases}$$

Then an invariant form on J is given by

$$J(x,\lambda,\nu)_0=\sum_{x',\lambda'}\delta(x',\lambda')a(x',\lambda')I_K(x',\lambda')_0. \tag{17.3}$$

where $I_K(x',\lambda')_0$ is the unique positive definite invariant form. There is one other invariant form, $-$ this one.

In particular an invariant form on J is definite if and only if, for all x',λ' appearing in (17.2),

(17.4)(a) $\qquad \delta(x',\lambda')a(x',\lambda')\in\mathbb{Z}$ *for all* (x',λ')

or

(17.4)(b) $\qquad \delta(x',\lambda')a(x',\lambda')\in s\mathbb{Z}$ *for all* (x',λ')

17.1 Some Invariant Forms on Irreducibles for $Sp(4,\mathbb{R})$

We have some formulas for c-invariant forms on some irreducible representations: (14.17)(h,k,n) on $J(x_9,3,1)$, $J(x_7,(\overline{1},2),(1,0))$ and $J(x_8,(\overline{1},2),(1,0))$, all at ρ. Let's convert these to invariant forms.

These representations all have trivial central character. Therefore, we may take $\zeta=1$ in Lemma 17.1. If (r,s) is a lowest K-type then the sign in the lemma is $(-1)^{\frac{r+s}{2}}$.

First consider $J(x_8,(\overline{1},2),(1,0))$.

Equation (14.17)(n) says

(17.1.5)(a)

$$\boxed{J(x_8,(\overline{1},2),(1,0))_c=I_K(x_8,(\overline{1},2))_c-s[I_K(x_1,(2,1))_c+I_K(x_3,(2,1))_c]}$$

By Section 5 the lowest K-types on the right hand side are $(-1,-3),(1,-3)$ and $(-3,-3)$, respectively, which have $(-1)^{\frac{r+s}{2}} = 1,-1,-1$, respectively. Therefore
(17.1.5)(b)
$$\boxed{J(x_8,(\overline{1},2),(1,0))_0 = I_K(x_8,(\overline{1},2))_0 - [I_K(x_1,(2,1))_0 + I_K(x_3,(2,1))_0]}.$$

This representation is unitary. This is representation `6` from the output of `block`.

Of course, using (14.17)(k), $J(x_7,(\overline{1},2),(1,0))_0$ is essentially the same:
(17.1.5)(c)
$$\boxed{J(x_7,(\overline{1},2),(1,0))_0 = I_K(x_7,(\overline{1},2))_0 - [I_K(x_0,(2,1))_0 + I_K(x_2,(2,1))_0]}.$$

and this is unitary. This is representation `5` from the output of `block`.

Finally (14.17)(h) says

$$J(x_9,3,1)_c = I_K(x_9,3) - s[I_K(x_0,(2,1)) + I_K(x_1,(2,1))] \tag{17.1.5)(d}$$

and the invariant formula is

$$\boxed{J(x_9,3,1)_0 = I_K(x_9,3)_0 - [I_K(x_0,(2,1))_0 + I_K(x_1,(2,1))_0]} \tag{17.1.5)(e}$$

which is unitary. This is representation `4` from the output of `block`.

Remark 17.1.6 *Using the* `blocku` *command we can confirm the unitarity. These representations are indeed $A_{\mathfrak{q}}(\lambda)$ modules attached to theta stable parabolic subalgebras with Levi factors $L = U(1) \times SL(2,\mathbb{R})$ and $L = U(1,1)$, respectively.*

18 c-Invariant Forms on Irreducible Representations

We already have some formulas for c-invariant forms on some irreducible representations: (14.17)(h,k,n) on $J(x_9,3,1)$, $J(x_7,(\overline{1},2),(1,0))$ and $J(x_8,(\overline{1},2),(1,0))$, all at ρ.

18.1 The c-invariant form on $J(x_9, 1, 3)$

Let's use (14.17)(p) to get a formula for $J(x_9, 1, 3)_c$. Recall $J(x_9, 1, 3) = J(x_9, (1, 0), \frac{1}{2}(3, 3))$, irreducible representation `9` from the output of `block` (see Section 8). The formula for $J(x_9, 1, 3)_c$ is given by (11.2)(b) with $a = 2, b = 1$:

(18.1.1)

$$\begin{aligned} J(x_9, 1, 3)_c = {} & I(x_9, 1, 3)_c \\ & + I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_2, (2, 1))_c + I(x_3, (2, 1))_c \\ & - I(x_9, 3, 1)_c \\ & - I(x_7, (\overline{1}, 2), (1, 0))_c - I(x_8, (\overline{1}, 2), (1, 0))_c \end{aligned}$$

The terms on the right hand side are given by (14.17)(p), (14.13) (4 times), (14.14)(h) and (14.15)(d,e), respectively. The last three are also given in (14.16)(a,d,e). This gives:

(18.1.2)

$$\begin{aligned} J(x_9,1,3)_c = {} & \\ & \big\{I_K(x_9, 1)_c + (1-s)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c] \\ & + (1-s)\{2I_K(x_0, (2, 1))_c + 2I_K(x_1, (2, 1))_c\} \\ & + (1-s)\{I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c\} \\ & + (1-s)\{I_K(x_9, 3)_c + I_K(x_7, \overline{1}, 2)_c) + I_K(x_8, \overline{1}, 2)_c\}\big\} \\ & + I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c + I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c \\ & - \big\{I_K(x_9, 3)_c + (1-s)[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c]\big\} \\ & - \big\{I_K(x_7, (\overline{1}, 2))_c + (1-s)[I_K(x_0, (2, 1))_c + I_K(x_2, (2, 1))_c]\big\} \\ & - \big\{I_K(x_8, (\overline{1}, 2))_c + (1-s)[I_K(x_1, (2, 1))_c + I_K(x_3, (2, 1))_c]\big\} \end{aligned}$$

Here is a table of the terms:

k	λ	coefficients	total	(r,s)	δ
9	1	1	1	$(1,-1)$	1
0	$(1,0)$	$1-s$	$1-s$	$(2,0)$	s
1	$(1,0)$	$1-s$	$1-s$	$(0,-2)$	s
0	$(2,1)$	$2(1-s)+1-(1-s)-(1-s)$	1	$(3,-1)$	s
1	$(2,1)$	$2(1-s)+1-(1-s)-(1-s)$	1	$(1,-3)$	s
2	$(2,1)$	$(1-s)+1-(1-s)$	1	$(3,3)$	s
3	$(2,1)$	$(1-s)+1-(1-s)$	1	$(-3,-3)$	s
9	3	$(1-s)-1$	$-s$	$(2,-2)$	1
7	$(\overline{1},2)$	$(1-s)-1$	$-s$	$(3,1)$	1
8	$(\overline{1},2)$	$(1-s)-1$	$-s$	$(-1,-3)$	1

And the answer is:

(18.1.3)

$$\begin{aligned} J(x_9,1,3)_c = {} & I_K(x_9,1)_c + \\ & + (1-s)\big[I_K(x_0,(1,0))_c + I_K(x_1,(1,0))_c\big] \\ & + \big[I_K(x_0,(2,1))_c + I_K(x_1,(2,1))_c + I_K(x_2,(2,1))_c + I_K(x_3,(2,1))_c\big] \\ & - s\big[I_K(x_9,3)_c + I_K(x_7,(\overline{1},2))_c + I_K(x_8,(\overline{1},2))_c\big] \end{aligned}$$

Note that if $s=1$ this gives (11.2)(b) again, as it must.

Using Section 17, the invariant form is given as follows.

(18.1.4)

$$\begin{aligned} J(x_9,1,3)_0 = {} & I_K(x_9,1)_0 + \\ & + (s-1)\big[I_K(x_0,(1,0))_0 + I_K(x_1,(1,0))_0\big] \\ & + s\big[I_K(x_0,(2,1))_0 + I_K(x_1,(2,1))_0 + I_K(x_2,(2,1))_0 + I_K(x_3,(2,1))_0 \\ & - s\big[I_K(x_9,3)_0 + I_K(x_7,(\overline{1},2))_0 + I_K(x_8,\overline{1},2)_0\big] \end{aligned}$$

This is not unitary. Note that we can tell this from the previous expression because of the hyperbolic term $(1-s)$.

18.2 The c-invariant form on $J(x_{7,8},(\overline{2},1),(2,0))$

The character formula for $J(x_7,(\overline{2},1),(2,0))$ is (11.2)(e), and this holds as a formula for c-invariant forms:

$$J(x_7,(\overline{2},1),(2,0))_c = I(x_7,(\overline{2},1),(2,0))_c$$
(18.2.5)
$$+ I(x_0,(2,1))_c + I(x_1,(2,1))_c + I(x_2,(2,1))_c$$
$$- I(x_9,(3,1))_c - I(x_7,(\overline{1},2),(1,0))_c$$

The non-discrete series terms on the right hand side are given by (14.18)(d), (14.14)(h) and (14.16)(c). The result is:

(18.2.6)

$$\boxed{\begin{aligned} J(x_7,(\overline{2},1),(2,0))_c = & -s[I(x_9,3)_c + I(x_7,(\overline{1},2))_c] \\ & + [I(x_0,(1,0))_c + I(x_2,(1,0))_c] \\ & + [I(x_0,(2,1))_c + I(x_1,(2,1))_c + I(x_2,(2,1))_c] \end{aligned}}$$

The lowest K-types of these terms are: $(2,-2),(3,1),(2,0),(2,2),(3,-1),(1,-3),(3,3)$. We multiply by $1,1,s,1,s,s,s$ respectively, to give the invariant form:

(18.2.7)

$$\boxed{\begin{aligned} J(x_7,(\overline{2},1),(2,0))_0 = & -s[I(x_9,3)_0 + I(x_7,(\overline{1},2))_0] \\ & + [sI(x_0,(1,0))_0 + I(x_2,(1,0))_0] \\ & + s[I(x_0,(2,1))_0 + I(x_1,(2,1))_0 + I(x_2,(2,1))_0] \end{aligned}}$$

This representation is *not* unitary: the signs differ on the two lowest K-types $(2,2)$ and $(2,0)$.

Similarly:

(18.2.8)

$$\boxed{\begin{aligned} J(x_8,(\overline{2},1),(2,0))_c = & -s[I(x_9,3)_c + I(x_8,(\overline{1},2))_c] \\ & + [I(x_1,(1,0))_c + I(x_3,(1,0))_c] \\ & + [I(x_0,(2,1))_c + I(x_1,(2,1))_c + I(x_3,(2,1))_c] \end{aligned}}$$

19 The Trivial Representation

Let's prove the trivial representation $J(x_{10},(\overline{2},\overline{1}),(2,1))$ is unitary.

The deformation will go as follows. Let $\nu = (1+\epsilon)(2,1)$ for $\epsilon > 0$ small. Deforming ν to 0 we have the following reducibility points, up to small deformations.

(19.1)(a) $$\nu = (2,1), (\frac{4}{3}, \frac{2}{3}), (1, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3})$$

At the first point the integral root system is C_2; at all the others it is of type A_1.

Here is the reducibility at $\nu = (\frac{2}{3}, \frac{1}{3})$, obtained using `nblock`.

The input is $\lambda = \rho = (2,1)$, so $\lambda - \rho = (0,0)$.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 3
Choosing the unique KGB element for the Cartan class:
10:  3  [r,r]   10  10     *   *  (0,0)#3 1^2x1^e
rho = [1,1]/1
Give lambda-rho: 0 0
denominator for nu: 3
numerator for nu: 1 1
x=10, lambda=[1,1]/1, gamma=[1,1]/3.
Name an output file (return for stdout, ? to abandon):
Subsystem on dual side is of type A1, with roots 6.
Given parameters define element 1 of the following block:
0(0,2):  0  [i2]  0   (1,2)  *(x= 4, nu= [2,-1]/6,,lam=rho+ [-2,0])  2,1,2
1(1,0):  1  [r2]  2   (0,*)  *(x=10, nu=  [1,1]/3,,lam=rho+[-2,-2])  e
2(1,1):  1  [r2]  1   (0,*)  *(x=10, nu=  [1,1]/3,,lam=rho+[-2,-1])  e
KL polynomials (-1)^{l(1)-l(x)}*P_{x,1}:
0: -1
1: 1
```

This says:

(19.1)(b) $$I(x_{10}, (\overline{2}, \overline{1}), (\frac{2}{3}, \frac{1}{3})) = J(x_{10}, (\overline{2}, \overline{1}), (\frac{2}{3}, \frac{1}{3})) + J(x_9, 1, \frac{1}{3}).$$

Dangerous Bend: Here come some orientation numbers.

Applying (14.12) as usual, for the first time we have some nontrivial orientation numbers.

By Table 9.3.4, $(\overline{a} = 0, \overline{b} = 1, y = \frac{1}{3})$, $\ell_0(x_{10}, (\overline{2}, \overline{1}), (\frac{2}{3}, \frac{1}{3})) = 3$. On the other hand by Table (9.1).2, with $(c = 1, x = \frac{1}{3})$, $\ell_0(x_9, 1, \frac{1}{3}) = 1$. The orientation number contribution to (14.12) is $s^{\frac{1}{2}(3-1)} = s$, so:

(19.1)(c)
$$I(x_{10},(\overline{2},\overline{1}),(\frac{2}{3},\frac{1}{3}))_c = I_K(x_{10},(\overline{2},\overline{1}))_c + (1-s)sI_K(x_9,1)_c$$
$$= I_K(x_{10},(\overline{2},\overline{1}))_c + (s-1)I_K(x_9,1)_c.$$

The next reducibility point is $(1,\frac{1}{2})$. We're interested in $I(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2}))$. This is of type A^I_{sn} in the notation of [4, Section 6]. Here is `nblock`:

```
real: nblock
choose Cartan class (one of 0,1,2,3): 3
Choosing the unique KGB element for the Cartan class:
10:  3  [r,r]   10  10     *   *  (0,0)#3 1^2x1^e
rho = [1,1]/1
Give lambda-rho: 0 0
denominator for nu: 2
numerator for nu: 1 1
x=10, lambda=[1,1]/1, gamma=[1,1]/2.
Name an output file (return for stdout, ? to abandon):
Subsystem on dual side is of type A1, with roots 7.
Given parameters define element 2 of the following block:
0(0,1):  0  [i1]  1   (2,*)  *(x= 5, nu= [-1,1]/2,,lam=rho+ [0,-1])  1,2,1
1(1,1):  0  [i1]  0   (2,*)  *(x= 6, nu= [-1,1]/2,,lam=rho+ [0,-1])  1,2,1
2(2,0):  1  [r1]  2   (0,1)  *(x=10, nu=  [1,1]/2,,lam=rho+[-2,-2])  e
KL polynomials (-1)^{l(2)-l(x)}*P_{x,2}:
0: -1
1: -1
2: 1
```

This says

(19.1)(d)
$$I(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2})) = J(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2}))+J(x_5,(1,\overline{0}),(0,\frac{1}{2}))+J(x_6,(1,\overline{0}),(0,\frac{1}{2}))$$

or alternatively

(19.1)(e)
$$I(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2})) = J(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2}))+J(x_7,(\overline{0},1),(\frac{1}{2},0))+J(x_8,(\overline{0},1),(\frac{1}{2},0))$$

Now apply (14.12). There is an orientation number here: $\ell_0(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2})) = 2$ (Table 9.3.1) and $\ell_0(x_{7,8},(\overline{0},1),(\frac{1}{2},0)) = 0$ (Table 9.2.1). So:

(19.1)(f) $$I(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2}))_c = I(x_{10},(\overline{2},\overline{1})(\frac{2}{3},\frac{1}{3}))_c + (1-s)s\big[I_K(x_7,(\overline{0},1))_c + I_K(x_8,(\overline{0},1))_c\big]$$

Plugging in (c) gives

(19.1)(g) $$I(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2}))_c = I_K(x_{10},(\overline{2},\overline{1}))_c + (s-1)I_K(x_9,1)_c + (s-1)\big[I_K(x_7,(\overline{0},1))_c + I_K(x_8,(\overline{0},1))_c\big]$$

However there is one more step: $I(x_7,(\overline{0},1))$ and $I(x_8,(\overline{0},1))$ are not final. Use (5.4)(a) to give

(19.1)(h)

$$I(x_{10},(\overline{2},\overline{1}),(1,\frac{1}{2}))_c = I_K(x_{10},(\overline{2},\overline{1}))_c + (s-1)I_K(x_9,1)_c + (s-1)\big[I_K(x_0,(1,0))_c + I_K(x_1,(1,0))_c + I_K(x_2,(1,0))_c + I_K(x_3,(1,0))_c\big]$$

Let's move on to $(\frac{4}{3},\frac{2}{3})$, i.e. $I(x_{10},(\overline{2},\overline{1}),(\frac{4}{3},\frac{2}{3}))$. This standard module is irreducible. The integral real root is $(1,1)$. By the table in Section 6 this fails the parity condition, so doesn't give reducibility. This is also what `nblock` says:

```
real: nblock
choose Cartan class (one of 0,1,2,3): 3
Choosing the unique KGB element for the Cartan class:
10:  3  [r,r]   10  10     *   *  (0,0)#3 1^2x1^e
rho = [1,1]/1
Give lambda-rho: 0 0
denominator for nu: 3
numerator for nu: 2 2
x=10, lambda=[1,1]/1, gamma=[2,2]/3.
Name an output file (return for stdout, ? to abandon):
Subsystem on dual side is of type A1, with roots 6.
Given parameters define element 0 of the following block:
0(0,0):  0  [rn]  0   (*,*)  *(x=10, nu=  [2,2]/3,,lam=rho+[-2,-2])  e
KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1
```

Note that in [4, Section 6], case A^I_{sn}, this example doesn't appear (meaning it is irreducible); in that notation we'd be considering $J(x, \overline{2}e, (\frac{4}{3}, \frac{2}{3}))$, but what appears in the table is $J(x, \overline{2}o, (\frac{4}{3}, \frac{2}{3}))$.

Finally we come to $\nu = (2, 1)$. By (10.2)(j) with $a = 2, b = 1$:

(19.1)(i)
$$\begin{aligned}
&I(x_{10}, (\overline{2},\overline{1}), (2, 1))^3 = \\
&\qquad J(x_{10}, (\overline{2}, \overline{1}), (2, 1))^3 \\
&\qquad + J(x_0, (2, 1))^0 + J(x_1, (2, 1))^0 \\
&\qquad + 2 \times J(x_9, 3, 1)^1 \\
&\qquad + J(x_7, (\overline{1}, 2), (1, 0))^1 + J(x_8, (\overline{1}, 2), (1, 0))^1 \\
&\qquad + J(x_7, (\overline{2}, 1), (2, 0))^2 + J(x_8, (\overline{2}, 1), (2, 0))^2 \\
&\qquad + J(x_9, 1, 3)^2
\end{aligned}$$

Remember $J(x_9, c, x) = J(x_9, (c, 0), \frac{1}{2}(x, x))$, with infinitesimal character $\frac{1}{2}(x + c, x - c)$.

We're at integral infinitesimal character, so there are no orientation numbers, so (14.12) gives

(19.1)(j)
$$\begin{aligned}
I(x_{10}, (\overline{2},\overline{1}), (2, 1))_c &= I(x_{10}, (\overline{2}, \overline{1}), (1, \frac{1}{2}))_c \\
&+ (1 - s)\big[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c\big] \\
&+ (1 - s)\big[J(x_7, (\overline{2}, 1), (2, 0))_c + J(x_8, (\overline{2}, 1), (2, 0))_c + J(x_9, 1, 3)_c\big]
\end{aligned}$$

and plugging in (h) for $J(x_{10}, (\overline{2}, \overline{1}), (1, \frac{1}{2}))_c$ gives

(19.1)(k)
$$\begin{aligned}
I(x_{10}, (\overline{2},\overline{1}), (2, 1))_c &= I_K(x_{10}, (\overline{2}, \overline{1}))_c \\
&+ (s - 1)I_K(x_9, 1)_c \\
&+ (s - 1)\big[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c + I_K(x_2, (1, 0))_c + I_K(x_3, (1, 0))_c\big] \\
&+ (1 - s)\big[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c\big] \\
&+ (1 - s)\big[J(x_7, (\overline{2}, 1), (2, 0))_c + J(x_8, (\overline{2}, 1), (2, 0))_c + J(x_9, 1, 3)_c\big]
\end{aligned}$$

We still have to deal with the terms $J(x_7, (\overline{2}, 1), (2, 0))$ $J(x_8, (\overline{2}, 1), (2, 0))$ and $J(x_9, 1, 3)$.

These are available from (18.2.6), (18.2.8) and (18.1.3). So:

(19.1)(l)

$$I(x_{10},(\overline{2},\overline{1}),(2,1))_c = I_K(x_{10},(\overline{2},\overline{1}))_c$$
$$+ (s-1)I_K(x_9,1)_c$$
$$+ (s-1)\big[I_K(x_0,(1,0))_c + I_K(x_1,(1,0))_c + I_K(x_2,(1,0))_c + I_K(x_3,(1,0))_c\big]$$
$$+ (1-s)\big[I_K(x_0,(2,1))_c + I_K(x_1,(2,1))_c\big]$$

$$+ (1-s)\big\{$$
$$- s[I(x_9,3)_c + I(x_7,(\overline{1},2))_c]$$
$$+ I(x_0,(1,0))_c + I(x_2,(1,0))_c$$
$$+ I(x_0,(2,1))_c + I(x_1,(2,1))_c + I(x_2,(2,1))_c$$

$$- s[I(x_9,3)_c + I(x_8,(\overline{1},2))_c]$$
$$+ I(x_1,(1,0))_c + I(x_3,(1,0))_c$$
$$+ I(x_0,(2,1))_c + I(x_1,(2,1))_c + I(x_3,(2,1))_c$$

$$+ I_K(x_9,1)_c + (1-s)\big[I_K(x_0,(1,0))_c + I_K(x_1,(1,0))_c\big]$$
$$+ I_K(x_0,(2,1))_c + I_K(x_1,(2,1))_c + I_K(x_2,(2,1))_c + I_K(x_3,(2,1))_c$$
$$- s\big[I_K(x_9,3)_c + I_K(x_7,(\overline{1},2)_c + I_K(x_8,\overline{1},2)_c\big] \quad \big\}$$

Here is a table of the terms:

k	λ	coefficients	total
10	$(\overline{2},\overline{1})$	1	1
9	1	$(s-1)+(1-s)$	0
9	3	$(1-s)+(1-s)+(1-s)$	$3(1-s)$
8	$(\overline{1},2)$	$(1-s)+(1-s)$	$2(1-s)$
7	$(\overline{1},2)$	$(1-s)+(1-s)$	$2(1-s)$
3	$(2,1)$	$(1-s)+(1-s)$	$2(1-s)$
2	$(2,1)$	$(1-s)+(1-s)$	$2(1-s)$
1	$(2,1)$	$(1-s)+(1-s)+(1-s)+(1-s)$	$4(1-s)$
0	$(2,1)$	$(1-s)+(1-s)+(1-s)+(1-s)$	$4(1-s)$
3	$(1,0)$	$(s-1)+(1-s))$	0
2	$(1,0)$	$(s-1)+(1-s))$	0
1	$(1,0)$	$(s-1)+(1-s)+(1-s)^2$	$2(1-s)$
0	$(1,0)$	$(s-1)+(1-s)+(1-s)^2$	$2(1-s)$

The character formula for the trivial representation is (11.2)(j) with $a = 2, b = 1$, and this holds without change for c-invariant forms (see Section 13.1):

(19.1)(m)

$$\begin{aligned} J(x_{10},(\overline{2},\overline{1}),(2,1))_c = {} & I(x_{10},(\overline{2},\overline{1}),(2,1))_c \\ & - \left\{I(x_9,1,3)_c + I(x_7,(\overline{2},1),(2,0))_c + I(x_8,(\overline{2},1),(2,0))_c\right\} \\ & + \left\{I(x_7,(\overline{1},2),(1,0))_c + I(x_8,(\overline{1},2),(1,0))_c + I(x_9,3,1)_c\right\} \\ & - \left\{I(x_3,(2,1))_c + I(x_2,(2,1))_c + I(x_1,(2,1))_c + I(x_0,(2,1))_c\right\} \end{aligned}$$

We know the terms on the right hand side:

(1) $I(x_{10},(\overline{2},\overline{1}),(2,1))_c$: (19.1)(l)

(2) $I(x_9,1,3)_c$: (14.17)(p)

(3) $I(x_9,3,1)_c$: (14.14)(h)

(4) $I(x_k,(\overline{1},2),(1,0))_c$ ($k = 7,8$): (14.16)

(5) $I(x_k,(\overline{2},1),(2,0))_c$ ($k = 7,8$): (14.18)(d) and (e)

Of course the last four terms require no further comment.

So let's tabulate everything. The next table has a column for each standard module on the right hand side of (19.1)(m), each row is a limit K-representation $I_K(x, \lambda)$, and the columns give the multiplicities in the expression of the c-invariant form on the standard module.

	$I(x_{10}, (\overline{2}, \overline{1}), (2, 1))_c$	$I(x_9, 1, 3)_c$	$I(x_7, (\overline{2}, 1), (2, 0))_c$	$I(x_8, (\overline{2}, 1), (2, 0))_c$
$I_K(x_{10}, (\overline{2}, \overline{1}))_c$	1			
$I_K(x_9, 1)_c$	0	-1		
$I_K(x_9, 3)_c$	$3(1-s)$	$-(1-s)$	$-(1-s)$	$-(1-s)$
$I_K(x_8, (\overline{1}, 2))_c$	$2(1-s)$	$-(1-s)$		$-(1-s)$
$I_K(x_7, (\overline{1}, 2))_c$	$2(1-s)$	$-(1-s)$	$-(1-s)$	
$I_K(x_3, (2, 1))_c$	$2(1-s)$	$-(1-s)$		$-(1-s)$
$I_K(x_2, (2, 1))_c$	$2(1-s)$	$-(1-s)$	$-(1-s)$	
$I_K(x_1, (2, 1))_c$	$4(1-s)$	$-2(1-s)$	$-(1-s)$	$-2(1-s)$
$I_K(x_0, (2, 1))_c$	$4(1-s)$	$-2(1-s)$	$-2(1-s)$	$-(1-s)$
$I_K(x_3, (1, 0))_c$				-1
$I_K(x_2, (1, 0))_c$			-1	
$I_K(x_1, (1, 0))_c$	$2(1-s)$	$-(1-s)$		-1
$I_K(x_0, (1, 0))_c$	$2(1-s)$	$-(1-s)$	-1	

	$I(x_7, (\overline{1}, 2), (1, 0))_c$	$I(x_8, (\overline{1}, 2), (1, 0))_c$	$I(x_9, 3, 1)_c$	DS	Sum
$I_K(x_{10}, (\overline{2}, \overline{1}))_c$					1
$I_K(x_9, 1)_c$					-1
$I_K(x_9, 3)_c$			1		1
$I_K(x_8, (\overline{1}, 2))_c$		1			1
$I_K(x_7, (\overline{1}, 2))_c$	1				1
$I_K(x_3, (2, 1))_c$		$(1-s)$		-1	$-s$
$I_K(x_2, (2, 1))_c$	$(1-s)$			-1	$-s$
$I_K(x_1, (2, 1))_c$		$(1-s)$	$(1-s)$	-1	$-s$
$I_K(x_0, (2, 1))_c$	$(1-s)$		$(1-s)$	-1	$-s$
$I_K(x_3, (1, 0))_c$					-1
$I_K(x_2, (1, 0))_c$					-1
$I_K(x_1, (1, 0))_c$					$-s$
$I_K(x_0, (1, 0))_c$					$-s$

So:
(19.1)(n)

$$
\begin{aligned}
J(x_{10},(\overline{2},\overline{1}),(2,1))_c = {} & I_K(x_{10},(\overline{2},\overline{1}))_c \\
& + \big[I_K(x_9,3)_c - I_K(x_9,1)_c\big] \\
& + \big[I_K(x_7,(\overline{1},2))_c + I_K(x_8,(\overline{1},2))_c\big] \\
& - s\big[I_K(x_0,(2,1))_c + I_K(x_1,(2,1))_c + I_K(x_2,(2,1))_c + I_K(x_3,(2,1))_c\big] \\
& - \big[sI_K(x_0,(1,0))_c + sI_K(x_1,(1,0))_c + I_K(x_2,(1,0))_c + I_K(x_3,(1,0))_c\big]
\end{aligned}
$$

To get the invariant form, we give the lowest K-types and the corresponding correction factors:

$I_K(x_{10},(\overline{2},\overline{1}))$	$(0,0)$	1
$I_K(x_9,1)$	$(1,-1)$	1
$I_K(x_9,3)$	$(2,-2)$	1
$I_K(x_8,(\overline{1},2))$	$(-1,-3)$	1
$I_K(x_7,(\overline{1},2))$	$(3,1)$	1
$I_K(x_3,(2,1))$	$(-3,-3)$	s
$I_K(x_2,(2,1))$	$(3,3)$	s
$I_K(x_1,(2,1))$	$(1,-3)$	s
$I_K(x_0,(2,1))$	$(3,-1)$	s
$I_K(x_3,(1,0))$	$(-2,-2)$	1
$I_K(x_2,(1,0))$	$(2,2)$	1
$I_K(x_1,(1,0))$	$(0,-2)$	s
$I_K(x_0,(1,0))$	$(2,0)$	s

Taking this into account we compute the invariant form on the trivial representation $J(x_{10},(\overline{2},\overline{1}),(2,1))$:

(19.1)(o)

$$
\begin{aligned}
J(x_{10},(\overline{2},\overline{1}),(2,1))_0 = {} & I_K(x_{10},(\overline{2},\overline{1})) \\
& + \big[I_K(x_9,3) - I_K(x_9,1)\big] \\
& + \big[I_K(x_7,(\overline{1},2)) + I_K(x_8,(\overline{1},2))\big] \\
& - \big[I_K(x_0,(2,1)) + I_K(x_1,(2,1)) + I_K(x_2,(2,1)) + I_K(x_3,(2,1))\big] \\
& - \big[I_K(x_0,(1,0)) + I_K(x_1,(1,0)) + I_K(x_2,(1,0)) + I_K(x_3,(1,0))\big].
\end{aligned}
$$

Therefore the trivial representation of $Sp(4,\mathbb{R})$ is unitary. If anyone knows an easier proof please let me know.

References

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