ON THE HOWE CORRESPONDENCE FOR SYMPLECTIC-ORTHOGONAL DUAL PAIRS

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ABSTRACT. We reformulate some of Moeglin's results on the correspondence for the dual pairs $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$ with p and q even, and fill in the cases where p and q are both odd. We arrive at a complete and detailed description, in terms of Langlands parameters, of the dual pair correspondence for the cases $p+q=2n$ and $p+q=2n+2$. In addition, we point out and suggest a way to correct an error in Moeglin's paper.

1. INTRODUCTION

Let (G,G') be a reductive dual pair in $Sp(2n,\mathbb{R})$, let $\widetilde{Sp}(2n,\mathbb{R})$ be the connected double cover of $Sp(2n,\mathbb{R})$, and let $\widetilde{G}$ and $\widetilde{G}'$ be the inverse images of G and G' in $\widetilde{Sp}(2n,\mathbb{R})$ by the covering map. If π and π' are irreducible admissible representations of $\widetilde{G}$ and $\widetilde{G}'$, respectively, we say that π and π' *correspond* if $\pi\otimes\pi'$ is a quotient of the oscillator representation ω of $\widetilde{Sp}(2n,\mathbb{R})$, restricted to $\widetilde{G}\times\widetilde{G}'$. (To be precise, $\pi\leftrightarrow\pi'$ if the Harish-Chandra module of $\pi\otimes\pi'$ may be realized as a quotient of the Harish-Chandra module associated to ω.) Howe [7] showed that this defines a one-to-one correspondence between subsets of the admissible duals of $\widetilde{G}$ and $\widetilde{G}'$. It is of interest to compute this correspondence explicitly, e. g., in terms of Langlands parameters. One reason is given by applications to automorphic forms. Moreover, Li [12] showed that for a dual pair in the stable range (roughly, this means that the rank of G' is at least twice the rank of G), the correspondence preserves unitarity from G to G'. This provides a way to express part of the unitary dual of one group in terms of the unitary dual of a smaller group, so that knowing the stable range correspondence is especially important. One of the most powerful tools available at this point, the induction principle (due to Kudla [11]), is more suited to the equal rank case. However, knowing the equal rank correspondence can be a starting point for computing large parts the full correspondence (in terms of Langlands parameters). In [13], for example, the correspondence for all dual pairs of the form $\big(Sp(p,q),O^*(2n)\big)$ with $p+q\leq n$ followed fairly easily from the equal rank correspondence. The full correspondence for the type II dual pairs, as well as for the dual pairs of the form $\big(O(n,\mathbb{C}),Sp(2m,\mathbb{C})\big)$ ([2], [14], [13]) are additional examples of such cases. In this paper, we investigate the equal rank correspondence for dual pairs of the form $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$, and as a corollary (Theorem 6.2) we obtain a substantial part of the correspondence for $p+q\leq 2n$.

Knowing the correspondence for equal rank dual pairs is interesting in its own right. In [3], Adams and Barbasch show that the dual pair correspondence for the pairs $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$ with $p+q=2n+1$ gives rise to a bijection between the genuine representations of the metaplectic group and the representations of the odd special orthogonal groups of the same rank. This suggests a way to apply machinery that exists for linear groups only (e. g. the L-group) to the non-linear metaplectic group. As another application, we use in [4] both the same rank and the stable range correspondence to determine part (the 'pseudospherical' part) of the genuine unitary dual of $\widetilde{Sp}(2n,\mathbb{R})$ by expressing it in terms of the spherical unitary dual of $SO(n+1,n)$ (which is known due to Barbasch [6]).

Consider the dual pairs $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$ with $p+q$ even. Moeglin [14] has computed a significant part of the full correspondence for the case where p and q are both even; in particular, her results include the complete correspondence for the cases $p+q=2n$ and $2n+2$. In this paper, we reformulate her results and fill in the cases $p+q=2n$ and $2n+2$ for p and q odd, arriving at a complete (and even more explicit) description of the correspondence for symplectic-orthogonal dual pairs of these relative sizes. This completes the explicit description of the Howe correspondence for all dual pairs of equal and almost equal rank; see [2], [3], [16], [17], and [13] for the other dual pairs.

We use the following notation (see also §2.1): If π and π' are irreducible admissible representations of $Sp(2n,\mathbb{R})$ and $O(p,q)$ which correspond to each other, we write $\theta_{p,q}(\pi)=\pi'$ or $\theta_n(\pi')=\pi$ to take into account that $Sp(2n,\mathbb{R})$ is a member of many dual pairs, and similarly for $O(p,q)$. If π does not occur in the correspondence for the dual pair $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$ we write $\theta_{p,q}(\pi)=0$, and similarly for π'.

Since $p+q$ is even, we can interpret the Howe correspondence as a correspondence between representations of $Sp(2n,\mathbb{R})$ and representations of $O(p,q)$ (as in [14]). The picture that emerges when looking at all dual pairs with $p+q=2n$ or $2n+2$ at the same time is much cleaner than the one that was apparent before. In particular, we have the following result.

Theorem 1.1 (Corollary 4.16). *Let π be an irreducible admissible representation of $Sp(2n,\mathbb{R})$. There are precisely four pairs of integers (p,q) with $p+q=2n$ or $2n+2$ such that $\theta_{p,q}(\pi)\neq 0$.*

If we start with a fixed representation of the orthogonal group, we get the following result which Moeglin already noticed for the case p and q even.

Theorem 1.2. *Let p and q be non-negative integers such that $p+q=2n$ is even, and let π be an irreducible admissible representation of $O(p,q)$. Then either π or $\pi\otimes det$ (possibly both) occur in the correspondence for the dual pair $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$.*

For comparison recall from [3] the analogous results for the case $p+q=2n+1$.

Theorem 1.3 (Adams and Barbasch).
1. *Let π be a genuine irreducible admissible representation of $\widetilde{Sp}(2n,\mathbb{R})$. Then there are precisely two pairs of integers (p,q) with $p+q=2n+1$ such that $\theta_{p,q}(\pi)\neq 0$.*
2. *Let π' be an irreducible admissible representation of $O(p,q)$ with $p+q=2n+1$. Then precisely one of π' and $\pi'\otimes det$ occurs in the correspondence for the dual pair $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$.*

Notice that in contrast to the case of odd orthogonal groups, in the even case we need to look at groups of two different sizes simultaneously in order to obtain a uniform statement. The explanation lies probably on the dual side; the groups considered by Adams and Barbasch are essentially duals of each other; however, if $p+q=2n$ and $r+s=2n+2$ then although $O(p,q)$ and and $Sp(2n,\mathbb{R})$ have the same rank, the dual group $SO(2n,\mathbb{C})$ of $SO(p,q)$ is properly contained in the dual group $SO(2n+1,\mathbb{C})$ of $Sp(2n,\mathbb{R})$; in fact, we have a chain $SO(2n,\mathbb{C})\subset SO(2n+1,\mathbb{C})\subset SO(2n+2,\mathbb{C})$ of dual groups for $SO(p,q)$, $Sp(2n,\mathbb{R})$, and $SO(r,s)$. From this point of view, it is reasonable to expect a more symmetric picture when considering both $p+q=2n$ and $2n+2$. There are similar pictures (of dual groups and correspondences) for the dual pairs $\big(Sp(p,q),O^*(2n)\big)$ with $p+q=2n$ or $2n+2$, and for the dual pairs $\big(U(p,q),U(r,s)\big)$ with $p+q=r+s\pm 1$.

Using Adams' definition [1] of the dual and L-groups for the disconnected orthogonal groups, we get a corresponding (although in general not completely canonical) containment of L-groups. One can check that in terms of L-parameters (i. e., admissible homomorphisms from the Weil group of $\mathbb{R}$ into the L-group as described in [5]) the Howe correspondence for $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$ with $p+q=2n,2n+2$ is essentially the composition of the Langlands map with inclusion of L-groups. (See [15] for a similar result in the non-archimedean case.)

The paper is organized as follows. After setting up notation and reviewing some facts about the correspondence, in particular those concerning the space of joint harmonics, we give in §3 a careful and explicit description of the Langlands parametrization for the admissible duals of $Sp(2n,\mathbb{R})$ and $O(p,q)$, using the version set up in [20], and explain how to compute the lowest K-types. In §4, we display the full correspondence for $p+q=2n$ and $2n+2$ in terms of this parametrization, starting with limits of discrete series representations. After discussing and proposing how to remove an error in [14], we set up the induction principle for these dual pairs, and perform the calculations needed for the proof of the correspondence. Our proof relies heavily on Moeglin's results, combined with the techniques of [2].

2. Preliminaries and Notation

2.1. **Notation and Root Systems.** Let n, p, and q be non-negative integers such that $p+q$ is even, and let $G = Sp(2n,\mathbb{R})$ or $O(p,q)$, the group of isometries of the bilinear form on $\mathbb{R}^n$ or $\mathbb{R}^{p+q}$ given by

$$\begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_p & O_{p\times q} \\ O_{q\times p} & -I_q \end{pmatrix}, \tag{2.1}$$

where I_m and O_m are the $m\times m$ identity and zero matrices respectively, and $O_{r\times s}$ is the $r\times s$ zero matrix. We let $\mathfrak{g}_0$ be the Lie algebra of G, and $\mathfrak{g}$ its complexification. Let $K \cong U(n)$ or $O(p)\times O(q)$ be the maximal compact subgroup of G corresponding to the Cartan involution $X \mapsto -{}^tX$, with Lie algebra $\mathfrak{k}_0$ and complexification $\mathfrak{k}$. We choose a Cartan subgroup T of K with Lie algebra $\mathfrak{t}_0$ and complexification $\mathfrak{t}$ as follows: if $G = Sp(2n,\mathbb{R})$ then

$$\mathfrak{t}_0 = \left\{ \begin{pmatrix} O_n & diag(t_1,\dots,t_n) \\ diag(-t_1,\dots,-t_n) & O_n \end{pmatrix} : t_i \in \mathbb{R}, 1\le i\le n \right\}. \tag{2.2}$$

If $G = O(p,q)$ then

$$\begin{aligned} \mathfrak{t}_0 &= \{diag(g(t_1),\dots,g(t_{p_0}),g(s_1),\dots,g(s_{q_0})) : t_i, s_i \in \mathbb{R}\} \quad \text{or} \\ \mathfrak{t}_0 &= \{diag(g(t_1),\dots,g(t_{p_0}),1,1,g(s_1),\dots,g(s_{q_0})) : t_i, s_i \in \mathbb{R}\}, \end{aligned} \tag{2.3}$$

depending on whether p and q are even or odd, and where $p_0 = [\frac{p}{2}]$, $q_0 = [\frac{q}{2}]$, and $g(t) = \left(\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix}\right)$ for all $t \in \mathbb{R}$.

The roots of $\mathfrak{t}$ in $\mathfrak{g}$ are

$$\Delta(\mathfrak{g},\mathfrak{t}) = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \cup \{\pm 2e_i : 1\le i\le n\} \tag{2.4}$$

if $G = Sp(2n,\mathbb{R})$,

$$\begin{aligned} \Delta(\mathfrak{g},\mathfrak{t}) = \{\pm e_i \pm e_j : 1 \le i < j \le p_0\} &\cup \{\pm f_i \pm f_j : 1\le i<j\le q_0\} \\ &\cup \{\pm e_i \pm f_j : 1 \le i \le p_0, 1\le j\le q_0\} \end{aligned} \tag{2.5}$$

if $G = O(p,q)$ and p, q are even, and

$$\begin{aligned} \Delta(\mathfrak{g},\mathfrak{t}) = \{\pm e_i \pm e_j : 1 \le i < j \le p_0\} &\cup \{\pm f_i \pm f_j : 1\le i<j\le q_0\} \\ &\cup \{\pm e_i \pm f_j, \pm e_i, \pm f_j : 1 \le i \le p_0, 1\le j\le q_0\} \end{aligned} \tag{2.6}$$

with the roots of the form $\pm e_i$ and $\pm f_j$ each occurring twice if $G = O(p,q)$ with p, q odd. We denote the sets of compact and noncompact roots Δ_c and Δ_n respectively, and fix a set of positive compact roots

$$\Delta_c^+ = \{e_i - e_j : 1 \le i < j \le n\} \tag{2.7}$$

if $G = Sp(2n,\mathbb{R})$,

$$\Delta_c^+ = \{e_i \pm e_j : 1 \le i < j \le p_0\} \cup \{f_i \pm f_j : 1 \le i < j \le q_0\} \tag{2.8}$$

if $G = O(p,q)$ with p, q even, and Δ_c^+ as in (2.8) with $\{e_i, f_j : 1 \le i \le p_0, 1 \le j \le q_0\}$ added if $G = O(p,q)$ with p, q odd.

We write $<,>$ for the trace form on $\mathfrak{g}$, and we use the same notation for its restrictions and dualization.

If H is a Lie group with maximal compact subgroup K_H, we will refer to K_H-types (i. e., irreducible representations of K_H) as *K-types for H*, or, if the group is clearly understood from the context, as simply as *K-types*. We identify K-types for connected groups with their highest weights, and for a representation π of H, we will use the abbreviation *LKT* to refer to a lowest K-type of π (in the sense of Vogan [19]).

We identify infinitesimal characters of representations of G with elements of the dual of a Cartan subalgebra of $\mathfrak{g}$ (modulo the Weyl group action), via the Harish-Chandra map. For $Sp(2n,\mathbb{R})$ and $O(p,q)$ with p and q even, we can choose $\mathfrak{t}$ for our Cartan subalgebra, for $O(p,q)$ with p and q odd we choose a maximally compact CSA $\mathfrak{t} \oplus \mathfrak{a}_c$.

If $pq \neq 0$ then $O(p,q)$ has four one-dimensional representations: the trivial representation $\mathbb{1}$, the sign or determinant representation *det*, and two characters whose restriction to $SO(p,q)$ is nontrivial, which we denote $\chi_{+,-}$ and $\chi_{-,+}$ depending on whether the restriction to $O(p)$ is trivial or not.

On a number of occasions, we will construct new parameters from pairs of parameters by "tacking" them together, so we set up some notation for this process. If $\mu = (a_1, a_2, \ldots, a_k) \in \mathbb{C}^k$ and $\nu = (b_1, b_2, \ldots, b_m) \in \mathbb{C}^m$, then $(\mu|\nu)$ will be the element of $\mathbb{C}^{k+m}$ given by

$$(\mu|\nu) = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_m). \tag{2.9}$$

Given a dual pair of the form $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$, let π and π' be irreducible admissible representations of $Sp(2n,\mathbb{R})$ and $O(p,q)$, respectively. Let $\omega_{n,p,q}$ be the oscillator representation of $\widetilde{Sp}(2n(p+q),\mathbb{R})$. (There are two oscillator representations; we make the same choice as Moeglin does in [14].) We say that π *corresponds to* π' if the Harish-Chandra module associated to $\pi \otimes \pi'$ may be realized as a quotient of the Harish-Chandra module associated to $\omega_{n,p,q}$; i. e., if there is a nonzero $\big((\mathfrak{g},K) \times (\mathfrak{g}',K')\big)$-map from the Harish-Chandra module of $\omega_{n,p,q}$ to the Harish-Chandra module of $\pi \otimes \pi'$. Here $\mathfrak{g}$ and $\mathfrak{g}'$ are the complexified Lie algebras of $Sp(2n,\mathbb{R})$ and $O(p,q)$, respectively, and K and K' are maximal compact subgroups. We denote the Howe correspondence by θ; if π corresponds to π' we write $\theta_{p,q}(\pi) = \pi'$ and $\theta_n(\pi') = \pi$. If π does not occur in the correspondence, we write $\theta_{p,q}(\pi) = 0$, and similarly $\theta_n(\pi') = 0$ if π' does not occur.

2.2. **K-Types and the Space of Joint Harmonics.** Let p and q be non-negative integers, and recall that $p_0 = [\frac{p}{2}]$, and $q_0 = [\frac{q}{2}]$. As in [3], we list irreducible representations of $O(p)$ by parameters $\lambda = (\lambda_0; \epsilon)$, where $\lambda_0 = (a_1, \ldots, a_{p_0}) \in i\mathfrak{t}_0^*$ and $\epsilon = \pm 1$, with the a_i integers such that $a_1 \ge a_2 \ge \ldots a_{p_0} \ge 0$. The parameters $(\lambda_0;\epsilon)$ and $(\lambda_0;-\epsilon)$ correspond to the same representation of $O(p)$ if and only if p is even and $a_{p_0} > 0$. The weight λ_0 is the highest weight of (one of the representations in) the restriction of μ to $SO(p)$. If p is odd then $-Id$ in $O(p)$ acts by $(-1)^{\sum_{i=1}^{p_0} a_i}\epsilon$; if p is even then we use the convention of [10], §6. For example, the trivial representation corresponds to $(0,\ldots,0;1)$, the sign representation of $O(p)$ corresponds to $(0,\ldots,0;-1)$, and $(a_1,\ldots,a_{p_0};\epsilon) \otimes det = (a_1,\ldots,a_{p_0};-\epsilon)$. We parametrize the representations of $O(q)$ in a the same way, and we write K-types for $O(p,q)$ in the form $(a_1,\ldots,a_{p_0};\epsilon) \otimes (b_1,\ldots,b_{q_0};\eta)$. We will refer to $(a_1,\ldots,a_{p_0};b_1,\ldots,b_{q_0})$ as the *highest weight*, and to $(\epsilon;\eta)$ as the *signs* of the K-type.

We parametrize K-types for $Sp(2n,\mathbb{R})$, i. e., irreducible representations of $U(n)$, by non-increasing n-tuples of integers $(a_1, a_2, \ldots, a_n)$.

We now describe the correspondence of K-types in the space of joint harmonics $\mathcal{H}$, a subspace of the Fock space $\mathcal{F}$ associated to the oscillator representation for the dual pair (G, G') (see [7]). Recall that each K-type μ which occurs in $\mathcal{F}$ has associated to it a degree (the minimum degree of polymomials in the μ-isotypic subspace), and that if π and π' are representations of G and G', respectively, which correspond to each other, then each K-type for G which is of minimal degree in π will occur in $\mathcal{H}$ and correspond to a K-type for G' of minimal degree in π'.

Proposition 2.10. *Let p, q, and n be non-negative integers such that $p+q$ is even. The correspondence of K-types in the space of joint harmonics $\mathcal{H}$ for the dual pair $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$ is given as follows.*

(1) *Let*

$$\mu = (a_1, a_2, \dots, a_x, 0, \dots, 0; \epsilon) \otimes (b_1, b_2, \dots, b_y, 0, \dots, 0; \eta) \tag{2.11}$$

be a K-type for $O(p,q)$, with $a_x > 0$ and $b_y > 0$. Then μ occurs in $\mathcal{H}$ if and only if $n \geq x + \frac{1-\epsilon}{2}(p-2x) + \frac{1-\eta}{2}(q-2y) + y$. In that case, μ corresponds to

$$\big(\tfrac{p-q}{2}, \tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}\big) + (a_1, \dots, a_x, \underbrace{1, \dots, 1}_{\frac{1-\epsilon}{2}(p-2x)}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\frac{1-\eta}{2}(q-2y)}, -b_y, \dots, -b_1). \tag{2.12}$$

(2) *If a K-type μ for $O(p,q)$ as in (2.11) occurs in the Fock space, then the degree of μ is*

$$\sum_{i=1}^{x} a_i + \sum_{i=1}^{y} b_i + \frac{1-\epsilon}{2}(p-2x) + \frac{1-\eta}{2}(q-2y). \tag{2.13}$$

For a K-type for $Sp(2n,\mathbb{R})$ ξ which occurs in $\mathcal{F}$, write

$$\xi = \big(\tfrac{p-q}{2}, \tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}\big) + (a_1, a_2, \dots, a_n). \tag{2.14}$$

Then the degree of ξ is $\sum_{i=1}^{n} |a_i|$.

Proof. This is well known, and may be easily obtained from [10] using the theory of [7] (see also Cor. I.4 of [14]). ☐

Remark 2.15. Notice that if $p+q \leq 2n$ then every K-type for $O(p,q)$ with signs $(1;1)$ occurs in $\mathcal{H}$. Moreover, it follows from Proposition 2.10 (1) that if ξ is a K-type for $Sp(2n,\mathbb{R})$, and

$$\xi = \big(\tfrac{p-q}{2}, \tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}\big) + (a_1, \dots, a_x, \underbrace{1, \dots, 1}_{k}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{l}, -b_y, \dots, -b_1) \tag{2.16}$$

with $a_x > 1$ and $b_y > 1$, then ξ occurs in $\mathcal{H}$ if and only if $x \leq p_0$, $k \leq p-2x$, $y \leq q_0$, and $l \leq q-2y$.

3. Langlands Parameters and Lowest K-Types

We describe the Langlands classification (using Vogan's version [20]) for $Sp(2n,\mathbb{R})$ and $O(p,q)$, and explain how to compute the lowest K-types.

3.1. **The Representations of $Sp(2n,\mathbb{R})$.** Let $G = Sp(2n,\mathbb{R})$, $\mathfrak{g}$ the complexified Lie algebra of G with $\mathfrak{k}$ and $\mathfrak{t}$ the complexified Lie algebras of a maximal compact subgroup K of G and Cartan subgroup T of K respectively. Limits of discrete series ρ of G may be parametrized by pairs (λ_d, Ψ) where $\lambda_d \in i\mathfrak{t}_0^*$ is the Harish-Chandra parameter of ρ and $\Psi \subset \Delta(\mathfrak{g},\mathfrak{t})$ the corresponding set of positive roots. The parameter λ_d is of the form

$$\lambda_d = (\underbrace{a_1,\ldots,a_1}_{k_1},\underbrace{a_2,\ldots,a_2}_{k_2},\ldots,\underbrace{a_b,\ldots,a_b}_{k_b},\underbrace{0,\ldots,0}_{z},\underbrace{-a_b,\ldots,-a_b}_{l_b},\ldots,\underbrace{-a_1,\ldots,-a_1}_{l_1}), \tag{3.1}$$

where $a_i \in \mathbb{Z}$, $a_1 > a_2 > \cdots > a_b > 0$, and $|k_i - l_i| \leq 1$ for all i. The root system Ψ satisfies that $\Delta_c^+ \subset \Psi$, λ_d is dominant with respect to Ψ, and for all simple roots $\alpha \in \Psi$ we have that if $< \lambda_d, \alpha >= 0$ then α is noncompact (this is condition F-1 of [20]). Consequently, there are 2^r nonequivalent limit of discrete series representations of $Sp(2n,\mathbb{R})$ with Harish-Chandra parameter λ_d as in (3.1), where r is the number of indices i such that $0 < k_i = l_i$, plus 1 if $z > 0$. These representations may be distinguished by their (unique) LKT's, given by

$$\Lambda = \lambda_d + \rho_n - \rho_c, \tag{3.2}$$

where ρ_n and ρ_c are one half the sums of the noncompact and compact roots in Ψ respectively. The representation $\rho = \rho(\lambda_d, \Psi)$ is a discrete series representation if $z = 0$ and $k_i + l_i = 1$ for all i.

Cuspidal parabolic subgroups (i. e., those of the form $P = MAN$ such that the Lie algebra $\mathfrak{m}_0$ of M has a theta stable Cartan subalgebra in $\mathfrak{k}_0$) of $Sp(2n,\mathbb{R})$ are of the form $P = MAN$ with

$$MA \cong Sp(2v,\mathbb{R}) \times GL(2,\mathbb{R})^s \times GL(1,\mathbb{R})^t \tag{3.3}$$

and $n = v + 2s + t$.

Relative limits of discrete series of $GL(2,\mathbb{R})$ are parametrized by pairs (μ,ν), where μ is a non-negative integer and ν a complex number. We denote the equivalence class of this representation $\tau(\mu,\nu)$. The representation $\tau(\mu,\nu)$ has infinitesimal character $(\frac{1}{2}(\mu+\nu), \frac{1}{2}(-\mu+\nu))$ (as an element of the dual of the diagonal, split Cartan), and LKT $(\mu+1;1)$. The character $x \mapsto sgn(x)^{\frac{\epsilon-1}{2}}|x|^\kappa$ of $GL(1,\mathbb{R})$ will be denoted $\chi_{\epsilon,\kappa}$.

Every irreducible admissible representation π of $Sp(2n,\mathbb{R})$ is equivalent to the unique irreducible quotient of a standard module

$$Ind_P^{Sp(2n,\mathbb{R})}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1}), \tag{3.4}$$

where $P = MAN$ is a cuspidal parabolic subgroup of $Sp(2n,\mathbb{R})$ with MA as in (3.3), $\rho = \rho(\lambda_d,\Psi)$ a limit of discrete series of $Sp(2m,\mathbb{R})$, $\tau = \bigotimes_{i=1}^s \tau(\mu_i,\nu_i)$ a relative limit of discrete series representation of $GL(2,\mathbb{R})^s$, $\chi = \bigotimes_{i=1}^t \chi_{\epsilon_i,\kappa_i}$ a character of $GL(1,\mathbb{R})^t$, and $\mathbb{1}$ the trivial representation of N. We use normalized induction so that infinitesimal characters are preserved. Write $\mu = (\mu_1,\ldots,\mu_s) \in \mathbb{Z}^s$, and similarly for $\nu \in \mathbb{C}^s$, $\epsilon \in \{\pm 1\}^t$, and $\kappa \in \mathbb{C}^t$. We can regard ν and κ as elements of $\mathfrak{a}^*$, where $\mathfrak{a}$ is the Lie algebra of the vector group A. Then we assume that $P = MAN$ is chosen so that we have $Re < \alpha, \nu > \geq 0$ and $Re < \alpha, \kappa > \geq 0$ for all roots α in $\Delta(\mathfrak{g},\mathfrak{a})$. The non-parity condition (F-2 of [20]) amounts to the following requirements:

$$\text{for} \quad 0 \leq i \leq s, \quad \text{if } \nu_i = 0 \text{ then } \mu_i \text{ is odd;} \tag{3.5}$$

$$\text{for} \quad 0 \leq i \leq t, \quad \text{if } \kappa_i = 0 \text{ then } \epsilon_i = (-1)^m; \tag{3.6}$$

$$\text{for} \quad 0 \leq i,j \leq t, \quad \text{if } \kappa_i = \pm\kappa_j \text{ then } \epsilon_i = \epsilon_j. \tag{3.7}$$

We write $\pi = \pi(\lambda_d,\Psi,\mu,\nu,\epsilon,\kappa)$, and refer to the data $(\lambda_d,\Psi,\mu,\nu,\epsilon,\kappa)$ as the Langlands parameters of π. Two representations $\pi(\lambda_d,\Psi,\mu,\nu,\epsilon,\kappa)$ and $\pi(\lambda_d',\Psi',\mu',\nu',\epsilon',\kappa')$ are equivalent if and only if

$\lambda_d = \lambda'_d$, $\Psi = \Psi'$, (μ', ν') is obtained from (μ, ν) by a simultaneous permutation of the coordinates of μ and ν, and by possibly multiplying some of the entries of ν by -1, and similarly (ϵ', κ') is obtained from (ϵ, κ) by permutations and multiplying coordinates of κ by -1. Parameters that do not occur will be written 0, or $\emptyset$ for Ψ; for example, a limit of discrete series of $Sp(2n, \mathbb{R})$ has Langlands parameters of the form $(\lambda_d, \Psi, 0, 0, 0, 0)$, a principal series looks like $\pi(0, \emptyset, 0, 0, \epsilon, \kappa)$.

The infinitesimal character γ of $\pi(\lambda_d, \Psi, \mu, \nu, \epsilon, \kappa)$ is the element of $\mathfrak{t}^*$ given by $\gamma = (\lambda_d|\beta)$ (see (2.9) for notation), where

(3.8)
$$\beta = (\tfrac{1}{2}(\mu_1+\nu_1), \tfrac{1}{2}(\mu_2+\nu_2), \dots, \tfrac{1}{2}(\mu_s+\nu_s), \kappa_1, \kappa_2, \dots, \kappa_t, \tfrac{1}{2}(-\mu_s+\nu_s), \tfrac{1}{2}(-\mu_{s-1}+\nu_{s-1}) \dots, \tfrac{1}{2}(-\mu_1+\nu_1)).$$

To each irreducible admissible representation (or set of Langlands parameters) we assign a parameter $\lambda_a \in i\mathfrak{t}_0^*$ (this is Vogan's λ of [19], §5.3) as follows: Let

$$\alpha = \left(\tfrac{\mu_1}{2}, \tfrac{\mu_2}{2}, \dots, \tfrac{\mu_s}{2}, \underbrace{0, 0, \dots, 0}_{t}, -\tfrac{\mu_s}{2}, -\tfrac{\mu_{s-1}}{2}, \dots, -\tfrac{\mu_1}{2}\right). \tag{3.9}$$

Then λ_a is obtained from $(\lambda_d|\alpha)$ by reordering of the coordinates so the resulting parameter is Δ_c^+-dominant (i. e., nonincreasing entries). Write

$$\lambda_a = (\underbrace{\alpha_1, \dots, \alpha_1}_{u_1}, \dots, \underbrace{\alpha_m, \dots, \alpha_m}_{u_m}, \underbrace{0, \dots, 0}_{w}, \underbrace{-\alpha_m, \dots, -\alpha_m}_{r_m}, \dots, \underbrace{-\alpha_1, \dots, -\alpha_1}_{r_1}) \tag{3.10}$$

with $\alpha_1 > \alpha_2 > \dots > \alpha_m > 0$. Then we have for all i that $|u_i - r_i| \le 1$, $\alpha_i \in \frac{1}{2}\mathbb{Z}$, and if $u_i \ne r_i$ then α_i is an integer. Let $u = \sum_{i=1}^m u_i$ and $r = \sum_{i=1}^m r_i$.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the theta stable parabolic subalgebra of $\mathfrak{g}$ associated to λ_a, and

$$L \cong \prod_{i=1}^m U(u_i, r_i) \times Sp(2w, \mathbb{R}) \tag{3.11}$$

the Levi subgroup of G corresponding to $\mathfrak{l}$. By the standard theory of [19] and [8], the LKT's of $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \epsilon, \kappa)$ (and of the standard module (3.4)) are those of the form

$$\Lambda = \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L. \tag{3.12}$$

Here $\rho(\mathfrak{u}\cap\mathfrak{p})$ and $\rho(\mathfrak{u}\cap\mathfrak{k})$ are one half the sums of the noncompact and compact roots in $\Delta(\mathfrak{g}, \mathfrak{t})$ with respect to which λ_a is strictly dominant, respectively, and δ_L is a fine K-type for L (see Definition 4.3.9 of [19]), given explicitly below.

Proposition 3.13. *Retain the notation of this section. Let $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \epsilon, \kappa)$ be an irreducible admissible representation of $G = Sp(2n, \mathbb{R})$, write λ_d as in (3.1) with k_i, l_i, b, and z as defined there, and let $\tilde{k}_j = \sum_{c=1}^j k_c$ and $\tilde{l}_j = \sum_{c=j}^b l_c$ for $1 \le j \le b$. Write $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ from (3.12) as*

$$(\underbrace{\beta_1, \dots, \beta_1}_{u_1}, \underbrace{\beta_2, \dots, \beta_2}_{u_2}, \dots, \underbrace{\beta_m, \dots, \beta_m}_{u_m}, \underbrace{u-r, \dots, u-r}_{w}, \underbrace{\gamma_m, \dots, \gamma_m}_{r_m}, \dots, \underbrace{\gamma_1, \dots, \gamma_1}_{r_1}). \tag{3.14}$$

Then the LKT's of π are precisely those of the form (3.12) with

$$\delta_L = (\underbrace{\delta_1, \dots, \delta_1}_{u_1}, \dots, \underbrace{\delta_m, \dots, \delta_m}_{u_m}, \eta_1, \eta_2, \dots, \eta_w, \underbrace{\delta_m, \dots, \delta_m}_{r_m}, \dots, \underbrace{\delta_1, \dots, \delta_1}_{r_1}) \tag{3.15}$$

satisfying the following conditions:

(1) *If* β_i *is an integer then* $\delta_i = 0$.

(2) *Suppose* $\beta_i \in \mathbb{Z} + \frac{1}{2}$. *Then* $\delta_i = \frac{1}{2}$ *or* $-\frac{1}{2}$; *if* α_i *does not occur as an entry in* λ_d *then both choices occur. If* $\alpha_i = a_j$, *then* $\delta_i = \frac{1}{2}$ *if* $e_{\tilde{k}_{j-1}+1} + e_{\tilde{l}_j} \in \Psi$, *and* $\delta_i = -\frac{1}{2}$ *otherwise.*

(3) *We have* $\eta_i \in \{-1, 0, 1\}$. *Let* h *be the number of indices* j *such that* $\epsilon_j = (-1)^{\tilde{k}_b + \tilde{l}_1 + 1} = (-1)^{u-r+1}$, *plus the number of indices* j *such that* $\mu_j = 0$, *plus* $[\frac{z+1}{2}]$. *Then* $(\eta_1, \eta_2, \dots, \eta_w) = (\underbrace{1, \dots, 1}_{h}, 0, \dots, 0)$ *or* $(0, \dots, 0, \underbrace{-1, \dots, -1}_{h})$. *If* $z = 0$ *then both choices occur. If* $z > 0$ *then* $(\eta_1, \dots, \eta_w)$ *is of the first form whenever* $e_{\tilde{k}_b+1} + e_{\tilde{k}_b+z} \in \Psi$ *(this includes the case* $z = 1$ *where the condition becomes* $2e_{\tilde{k}_b+1} \in \Psi$*), and of the second form otherwise.*

Proof. Using the definition, one can check that the fine K-types for $L \cong \prod_{i=1}^m U(u_i, r_i) \times Sp(2w, \mathbb{R})$ are those of the form (3.15) with $\delta_i \in \{0, \pm\frac{1}{2}\}$ and $\delta_i = 0$ if $u_i \neq r_i$, and

$$(\eta_1, \eta_2, \dots, \eta_w) = (\underbrace{1, \dots, 1}_{\xi}, 0, \dots, 0) \quad \text{or} \quad (0, \dots, 0, \underbrace{-1, \dots, -1}_{\xi}) \tag{3.16}$$

for some $0 \leq \xi \leq w$. Part (1) follows from integrality considerations. Also, if π is a limit of discrete series representation then this is a straightforward calculation using (3.2).

The general case uses Frobenius' Reciprocity and Proposition 8.1 of [21] ("the lowest K-types of the induced are contained in the induced from the lowest"). Consequently, if Λ is a LKT of π then Λ must be of the form (3.12) and contained in the induced representation of the LKT of $\rho \otimes \tau \otimes \chi$ (see (3.4)) to $U(n)$. This means that the entries of Λ consist of those of the LKT of the limit of discrete series $\rho = \rho(\lambda_d, \Psi)$, plus a pair of entries for each factor $GL(2, \mathbb{R})$, plus an entry for each factor of $GL(1, \mathbb{R})$, subject to the following conditions: for $GL(2, \mathbb{R})$, if the corresponding μ_i is an even integer, we get a pair of entries in Λ with opposite parity, if μ_i is odd then the pair of entries has the same parity; for $GL(1, \mathbb{R})$, the entry is even or odd depending on whether the corresponding $\epsilon_i = 1$ or -1. For example, the number h of nonzero entries in $(\eta_1, \eta_2, \dots, \eta_w)$ is $[\frac{z+1}{2}]$ from the LKT of ρ, plus one for each $GL(2, \mathbb{R})$ factor with $\mu_i = 0$ (since these yield an entry each of parity the same and opposite to that of $u - r$), plus one for each $GL(1, \mathbb{R})$ factor with $\epsilon_i = (-1)^{u-r+1}$. The form of (3.15) implies then that $(\eta_1, \eta_2, \dots, \eta_w)$ is as in (3.16) with $\xi = h$, and if $[\frac{z+1}{2}] \neq 0$, i. e., the limit of discrete series parameter λ_d contains a zero, then Ψ determines the choice. □

Example 3.17. Let $G = Sp(22, \mathbb{R})$, $m = 4$, $s = 2$, and $t = 3$. Suppose $\lambda_d = (2, 2, 0, -2)$, Ψ such that $2e_3 \in \Psi$, $\mu = (4, 2)$, and $\epsilon = (1, 1, -1)$. Then $h = 3$ (notice that $\tilde{k}_b = 2$ and $\tilde{l}_1 = 1$),

$$\begin{aligned} \lambda_a &= (2, 2, 2, 1, 0, 0, 0, 0, -1, -2, -2), \\ \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{p}) &= (3, 3, 3, \tfrac{5}{2}, 1, 1, 1, 1, -\tfrac{1}{2}, -2, -2), \end{aligned}$$

and the possible fine K-types are

$$\delta_L = (0, 0, 0, \tfrac{1}{2}, 1, 1, 1, 0, \tfrac{1}{2}, 0, 0) \quad \text{and} \quad \delta_L = (0, 0, 0, -\tfrac{1}{2}, 1, 1, 1, 0, -\tfrac{1}{2}, 0, 0), \tag{3.18}$$

so that π has LKT's

$$\Lambda_1 = (3, 3, 3, 3, 2, 2, 2, 1, 0, -2, -2) \quad \text{and} \quad \Lambda_2 = (3, 3, 3, 2, 2, 2, 2, 1, -1, -2, -2). \tag{3.19}$$

Example 3.20. Let $n = 5$, $s = 1$, $t = 3$, $\mu = (0)$ so that $\lambda_a = \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{p}) = (0, 0, 0, 0, 0)$. If $\epsilon = (1, 1, 1)$ then $h = 1$ and π has LKT's $(1, 0, 0, 0, 0)$ and $(0, 0, 0, 0, -1)$. If $\epsilon = (-1, -1, -1)$ then $h = 4$ and the LKT's are $(1, 1, 1, 1, 0)$ and $(0, -1, -1, -1, -1)$.

3.2. The Representations of $O(p, q)$.

For this section, let p and q be non-negative integers such that $p + q$ is even, and let $G = O(p, q)$. Let $n = \frac{p+q}{2}$, $p_0 = [\frac{p}{2}]$, and $q_0 = [\frac{q}{2}]$. As in the last section, we let $\mathfrak{g}$ be the complexified Lie algebra of G with $\mathfrak{k}$ and $\mathfrak{t}$ the complexified Lie algebras of a maximal compact subgroup K of G and a Cartan subgroup T of K respectively. In describing the irreducible admissible representations of G, we must account for the fact that G does not belong to Harish-Chandra's class. If G is realized as the set of n by n real invertible matrices preserving the symmetric form on $\mathbb{R}^n$ given by the matrix $I_{p,q} = diag(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$, let $J = diag(1, \ldots, 1, -1) \in O(p, q) - SO(p, q)$, and let σ be the automorphism of $O(p, q)$ or $SO(p, q)$ given by conjugation by J. Note that σ also acts on representations, Cartan and parabolic subgroups, and on Langlands parameters for $SO(p, q)$. Recall that $O(p, q) \cong SO(p, q) \rtimes \{Id, J\}$. We can parametrize the irreducible admissible representations of $SO(p, q)$ using the theory of [20]. Then using Frobenius' Reciprocity, the representations of $O(p, q)$ are obtained as follows: for each representation π of $SO(p, q)$ such that $\sigma(\pi)$ is equivalent to π we get two representations ρ and $\rho \otimes det$ of $O(p, q)$, both of which restrict to π on $SO(p, q)$; if $\sigma(\pi) \cong \pi'$ with π and π' nonequivalent, then we get one representation of $O(p, q)$ whose restriction to $SO(p, q)$ is $\pi \oplus \pi'$.

If p and q are even then σ acts on the Harish-Chandra parameter of a limit of discrete series representation of $SO(p, q)$ by changing the sign of one of the entries. Since the Weyl group can act by changing two signs at a time, we may parametrize the limit of discrete series representations ρ of $O(p, q)$ by triples (λ_d, ξ, Ψ) as follows. The parameter $\lambda_d \in i\mathfrak{t}_0^*$ is of the form

(3.21)
$$\lambda_d = (\underbrace{a_1, \ldots, a_1}_{k_1}, \underbrace{a_2, \ldots, a_2}_{k_2}, \ldots, \underbrace{a_b, \ldots, a_b}_{k_b}, \underbrace{0, \ldots, 0}_{z}; \underbrace{a_1, \ldots, a_1}_{l_1}, \underbrace{a_2, \ldots, a_2}_{l_2}, \ldots, \underbrace{a_b, \ldots, a_b}_{l_b}, \underbrace{0, \ldots, 0}_{z'}),$$

where $a_i \in \mathbb{Z}$, $a_1 > a_2 > \cdots > a_b > 0$, $|k_i - l_i| \le 1$, and $|z - z'| \le 1$. As for $Sp(2n, \mathbb{R})$ we have that $\Psi \subset \Delta(\mathfrak{g}, \mathfrak{t})$ is a positive root system containing Δ_c^+, such that λ_d is dominant with respect to Ψ, and satisfying condition F-1 of [20]. The parameter ξ takes values $+1$ or -1, and we sometimes absorb it into λ_d by writing $\lambda_d = (a_1, \ldots, a_1, \ldots, 0, \ldots, 0; b_1, \ldots, 0)_\xi$. The representation $\rho = \rho(\lambda_d, \xi, \Psi)$ is one of the irreducible representations of $O(p, q)$ whose restriction to $SO(p, q)$ contains the limit of discrete series with Harish-Chandra parameter λ_d and positive root system Ψ. The highest weight of the LKT Λ is given by $\Lambda_0 = (\lambda_0; \mu_0) = \lambda_d + \rho_n - \rho_c$ with ρ_n and ρ_c as for $Sp(2n, \mathbb{R})$. If $z + z' = 0$ then the entries of Λ_0 are all nonzero, so the signs of Λ are arbitrary; in this case we choose $\xi = 1$, and there is only one limit of discrete series of $O(p, q)$ corresponding to (λ_d, Ψ). If $z + z' > 0$ then there are two; in this case, precisely one of λ_0 and μ_0 has a zero entry (λ_0 if $-e_{p_0} + f_{q_0} \in \Psi$, and μ_0 otherwise), so there are two possible LKT's, corresponding to two representations of $O(p, q)$. We choose $\xi = 1$ for the representation whose LKT has signs $(1; 1)$, and $\xi = -1$ for the other one. For a given parameter λ_d as in (3.21), there are 2^r limit of discrete representations of $O(p, q)$, where r is the number of indices i such that $0 < k_i = l_i$, plus 1 if $0 < z = z'$, plus 1 if $z + z' > 0$. The representation $\rho = \rho(\lambda_d, \xi, \Psi)$ is a discrete series if $k_i + l_i = 1$ for all i, and $z + z' \le 1$.

Example 3.22. Let $G = O(6, 8)$, and $\lambda_d = (1, 0, 0; 2, 1, 0, 0)$. Then $r = 3$ since $k_2 = l_2 = 1$, $z = z' = 2$, and $z + z' = 4 > 0$, so there are 8 limit of discrete series representations $O(p, q)$ with parameter λ_d. We list them by giving the four sets of simple roots determining distinct positive root systems Ψ_i, along with the highest weights Λ_i they determine, and the two possible pairs of signs each.

$$
\begin{gathered}
\{f_1 - e_1, e_1 - f_2, f_2 - e_2, e_2 - f_3, f_3 - e_3, e_3 - f_4, e_3 + f_4\} \subset \Psi_1, \\
\Lambda_1 = (2,1,1;2,1,0,0); \quad \text{signs } (1;1) \text{ or } (1;-1); \\
\{f_1 - e_1, e_1 - f_2, f_2 - f_3, f_3 - e_2, e_2 - f_4, f_4 - e_3, e_3 + f_4\} \subset \Psi_2, \\
\Lambda_2 = (2,0,0;2,1,1,1); \quad \text{signs } (1;1) \text{ or } (-1;1); \\
\{f_1 - f_2, f_2 - e_1, e_1 - e_2, e_2 - f_3, f_3 - e_3, e_3 - f_4, e_3 + f_4\} \subset \Psi_3, \\
\Lambda_3 = (1,1,1;2,2,0,0); \quad \text{signs } (1;1) \text{ or } (1;-1); \\
\{f_1 - f_2, f_2 - e_1, e_1 - f_3, f_3 - e_2, e_2 - f_4, f_4 - e_3, e_3 + f_4\} \subset \Psi_4, \\
\Lambda_4 = (1,0,0;2,2,1,1); \quad \text{signs } (1;1) \text{ or } (-1;1).
\end{gathered}
$$

Up to conjugation, the theta-stable Cartan subalgebras of $SO(p,q)$ are

$$
\left\{ \mathfrak{h}_\zeta^{r,s,t} : 0 \le r = 2s + t \le min\{p,q\},\ p - r \text{ is even},\ \zeta = \pm 1 \right\}, \tag{3.23}
$$

where an element of $\mathfrak{h}_\zeta^{r,s,t}$ is of the form

$$
\begin{pmatrix}
A &&&&&&& \\
&&&&&& Z & \\
&&& Y && X' && \\
&& -Y && X &&& \\
&&& X && -Y' && \\
&& X' && Y' &&& \\
& Z &&&&&& \\
&&&&&&& B
\end{pmatrix}. \tag{3.24}
$$

Here $A = diag\left(g(t_1), \dots, g(t_{\frac{p-r}{2}})\right)$, $B = diag\left(g(s_1), \dots, g(s_{\frac{q-r}{2}})\right)$ (see (2.3) for notation), $Z = diag\,(z_1, \dots, z_t)$, $X = diag\,(x_1, \dots, x_s)$, $Y = diag\,(y_1, \dots, y_s)$, $X' = diag\,(x_1, \dots, x_{s-1}, \zeta x_s)$, and $Y' = diag\,(y_1, \dots, y_{s-1}, \zeta y_s)$ with all entries real numbers. We have that $\mathfrak{h}_\zeta^{r,s,t}$ is conjugate to $\mathfrak{h}_{-\zeta}^{r,s,t}$ by an element of $O(p,q) - SO(p,q)$. The cuspidal parabolic subgroups of $SO(p,q)$ have Levi factors of the form

$$
MA \cong SO(p-r, q-r) \times GL(2,\mathbb{R})^s_\zeta \times GL(1,\mathbb{R})^t. \tag{3.25}
$$

Here the subscript ζ refers to the embedding of the last $GL(2,\mathbb{R})$-factor. If we choose a limit of discrete series ρ of $SO(p-r, q-r)$, a relative limit of discrete series τ of $GL(2,\mathbb{R})^s_\zeta$, and a character χ of $GL(1,\mathbb{R})^t$, the effect of σ on these data will be (up to the Weyl group action of $SO(p,q)$), to change the sign of one entry in the Harish-Chandra parameter of ρ, to change the embedding of the last $GL(2,\mathbb{R})$, or to change the sign of one entry of the continuous part of χ. Two sets of data which differ by an even number of such changes are conjugate by the Weyl group. Consequently, the parametrization of irreducible admissible representations of $O(p,q)$ is as follows.

Cuspidal parabolic subgroups of $O(p,q)$ are of the form $P = MAN$ with

$$
MA \cong O(2a, 2d) \times GL(2,\mathbb{R})^s \times GL(1,\mathbb{R})^t, \tag{3.26}
$$

$p = 2a + 2s + t$, and $q = 2d + 2s + t$.

Every irreducible admissible representation π of $O(p,q)$ is equivalent to an irreducible quotient of a standard module

$$Ind_P^{O(p,q)}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1}), \tag{3.27}$$

where $P = MAN$ is a cuspidal parabolic subgroup of $O(p,q)$ with MA as in (3.26), $\rho = \rho(\lambda_d, \xi, \Psi)$ a limit of discrete series representation of $O(2a, 2d)$, $\tau = \bigotimes_{i=1}^s \tau(\mu_i, \nu_i)$ a relative limit of discrete series representation of $GL(2,\mathbb{R})^s$, $\chi = \bigotimes_{i=1}^t \chi_{\epsilon_i,\kappa_i}$ a character of $GL(1,\mathbb{R})^t$, and $\mathbb{1}$ the trivial representation of N. Let μ, ν, ϵ, and κ be as in §3.1, and we assume that we have chosen $P = MAN$ as we did there (according to the real parts of the parameters ν and κ). For $O(p,q)$, the non-parity condition F-2 becomes:

$$\text{for} \quad 0 \le i \le s, \quad \text{if } \nu_i = 0 \text{ then } \mu_i \text{ is odd;} \tag{3.28}$$

$$\text{for} \quad 0 \le i, j \le t, \quad \text{if } \kappa_i = \pm\kappa_j \text{ then } \epsilon_i = \epsilon_j. \tag{3.29}$$

Under the above conditions, the induced representation (3.27) has a unique irreducible quotient unless $\kappa_i = 0$ for some $0 \le i \le t$, and either $a = d = 0$ or the parameter λ_d satisfies $z + z' = 0$ (i. e., contains no zero entry). In this case, we have two irreducible quotients which may be distinguished by the signs of their LKT's (as described in Proposition 3.43 below), and which we denote $\pi_{+1}(\lambda_d, 1, \Psi, \mu, \nu, \epsilon, \kappa)$ and $\pi_{-1}(\lambda_d, 1, \Psi, \mu, \nu, \epsilon, \kappa)$ respectively. In the first case, we denote the unique irreducible quotient $\pi_{+1}(\lambda_d, \xi, \Psi, \mu, \nu, \epsilon, \kappa)$. In either case, we refer to the corresponding data as the Langlands parameters of the representation. We have

$$\pi_1(\lambda_d, -\xi, \Psi, \mu, \nu, \epsilon, \kappa) = \pi_1(\lambda_d, \xi, \Psi, \mu, \nu, \epsilon, \kappa) \otimes det, \quad \text{and} \tag{3.30}$$

$$\pi_{-1}(\lambda_d, 1, \Psi, \mu, \nu, \epsilon, \kappa) = \pi_1(\lambda_d, 1, \Psi, \mu, \nu, \epsilon, \kappa) \otimes det. \tag{3.31}$$

As for representations of $Sp(2n,\mathbb{R})$, we have that two representations $\pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \epsilon, \kappa)$ and $\pi_{\zeta'}(\lambda'_d, \xi', \Psi', \mu', \nu', \epsilon', \kappa')$ are equivalent if and only if $\lambda_d = \lambda'_d$, $\xi = \xi'$, $\Psi = \Psi'$, $\zeta = \zeta'$, (μ', ν') is obtained from (μ, ν) by a simultaneous permutation of the coordinates of μ and ν, and by possibly multiplying some of the entries of ν by -1, and similarly (ϵ', κ') is obtained from (ϵ, κ) by permutations and multiplying coordinates of κ by -1. As for $Sp(2n,\mathbb{R})$, parameters that do not occur (e. g., μ for a discrete series) are written 0 or $\emptyset$.

The infinitesimal character of $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \epsilon, \kappa)$ is the element of $\mathfrak{t}^*$ (or $(\mathfrak{t} \oplus \mathfrak{a}_c)^*$ if p and q are odd) given by $\gamma = (\lambda_d|\beta)$, where

$$\beta = (\tfrac{1}{2}(\mu_1+\nu_1), \tfrac{1}{2}(\mu_2+\nu_2), \ldots, \tfrac{1}{2}(\mu_s+\nu_s), \kappa_1, \kappa_2, \ldots, \kappa_t, \tfrac{1}{2}(\mu_1-\nu_1), \tfrac{1}{2}(\mu_2-\nu_2) \ldots, \tfrac{1}{2}(\mu_s-\nu_s)). \tag{3.32}$$

The Vogan parameter $\lambda_a \in i\mathfrak{t}_0^*$ which we assign to π is again obtained by reordering according to Δ_c^+, of $(\lambda_d|\alpha)$, where

$$\alpha = \big(\tfrac{\mu_1}{2}, \tfrac{\mu_2}{2}, \ldots, \tfrac{\mu_s}{2}, \underbrace{0, 0, \ldots, 0}_{[\frac{t}{2}]}; \tfrac{\mu_1}{2}, \tfrac{\mu_2}{2}, \ldots, \tfrac{\mu_s}{2}, \underbrace{0, 0, \ldots, 0}_{[\frac{t}{2}]}\big). \tag{3.33}$$

Write

$$\lambda_a = (\underbrace{\alpha_1, \ldots, \alpha_1}_{u_1}, \ldots, \underbrace{\alpha_m, \ldots, \alpha_m}_{u_m}, \underbrace{0, \ldots, 0}_{x}; \underbrace{\alpha_1, \ldots, \alpha_1}_{r_1}, \ldots, \underbrace{\alpha_m, \ldots, \alpha_m}_{r_m}, \underbrace{0, \ldots, 0}_{y}) \tag{3.34}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$. Then we have for all i that $|u_i - r_i| \le 1$, $|x - y| \le 1$, $\alpha_i \in \frac{1}{2}\mathbb{Z}$, and if $u_i \ne r_i$ then α_i is an integer. Let $u = \sum_{i=1}^m u_i$ and $r = \sum_{i=1}^m r_i$.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the theta stable parabolic subalgebra of $\mathfrak{g}$ associated to λ_a, and

$$L \cong \prod_{i=1}^{m} U(u_i, r_i) \times O(p-2u, q-2r) \tag{3.35}$$

the Levi subgroup of $O(p,q)$ corresponding to $\mathfrak{l}$. Notice that $(p-2u, q-2r) = (2x, 2y)$ or $(2x+1, 2y+1)$ depending on whether p and q are even or odd. Up to signs, the LKT's of π are again given by (3.12), with $\rho(\mathfrak{u}\cap\mathfrak{p})$, $\rho(\mathfrak{u}\cap\mathfrak{k})$, and δ_L defined as they are there. We describe the fine K-types below.

Proposition 3.36. *Retain the notation of this section. Let $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \epsilon, \kappa)$ be an irreducible admissible representation of $G = O(p,q)$. If λ_d is as in (3.21) let $\tilde{k}_j = \sum_{c=1}^{j} k_c$ and $\tilde{l}_j = \sum_{c=1}^{j} l_c$ for $1 \le j \le b$. Write $\lambda_a + \rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{k})$ from (3.12) as*

$$(\underbrace{\beta_1,\dots,\beta_1}_{u_1}, \underbrace{\beta_2,\dots,\beta_2}_{u_2}, \dots, \underbrace{\beta_m,\dots,\beta_m}_{u_m}, \underbrace{0,\dots,0}_{x}; \underbrace{\gamma_1,\dots,\gamma_1}_{r_1}, \dots, \underbrace{\gamma_m,\dots,\gamma_m}_{r_m}, \underbrace{0,\dots,0}_{y}). \tag{3.37}$$

Then the highest weights of the LKT's of π are precisely those of the form (3.12) with

$$\delta_L = (\underbrace{\delta_1,\dots,\delta_1}_{u_1}, \dots, \underbrace{\delta_m,\dots,\delta_m}_{u_m}, \eta_1, \eta_2, \dots, \eta_x; \underbrace{-\delta_1,\dots,-\delta_1}_{r_1}, \dots, \underbrace{-\delta_m,\dots,-\delta_m}_{r_m}, \xi_1, \dots, \xi_y) \tag{3.38}$$

satisfying the following conditions:

1. *If β_i is an integer then $\delta_i = 0$.*
2. *Suppose $\beta_i \in \mathbb{Z} + \frac{1}{2}$. Then $\delta_i = \frac{1}{2}$ or $-\frac{1}{2}$; if α_i does not occur as an entry in λ_d then both choices occur. If $\alpha_i = a_j$ then $\delta_i = \frac{1}{2}$ if $e_{\tilde{k}_j} - f_{\tilde{l}_j} \in \Psi$, and $\delta_i = -\frac{1}{2}$ otherwise.*
3. *We have $\eta_i, \xi_i \in \{0,1\}$. Let $h = min\{z, z'\}$, plus the number of indices $j \le s$ such that $\mu_j = 0$, plus $min\{\beta, \gamma\}$, where β and γ are the numbers of indices $j \le t$ such that $\epsilon_j = 1$ and $\epsilon_j = -1$ respectively. Then $(\eta_1, \eta_2, \dots, \eta_x; \xi_1, \xi_2, \dots, \xi_y) = (\underbrace{1,\dots,1}_{h}, 0, \dots, 0; 0, \dots, 0)$ or $(0,\dots,0; \underbrace{1,\dots,1}_{h}, 0, \dots, 0)$. If $z = z' = 0$ (i. e., no zero entry in λ_d), then both choices occur. If $z + z' > 0$ then only the first possibility occurs whenever $e_{\tilde{k}_b+z} - f_{\tilde{l}_b+z'} \in \Psi$, the second possibility otherwise.*

Proof. The proof is similar to that of Proposition 3.13. The fine K-types for $L \cong \prod_{i=1}^{m} U(u_i, r_i) \times O(p-2u, q-2r)$ are those of the form (3.38) with $\delta_i \in \{0, \pm\frac{1}{2}\}$ and $\delta_i = 0$ if $u_i \ne r_i$, and

$$(\eta_1, \eta_2, \dots, \eta_x; \xi_1, \xi_2, \dots, \xi_y) = (\underbrace{1,\dots,1}_{\xi}, 0, \dots, 0; 0, \dots, 0) \quad \text{or} \quad (0,\dots,0; \underbrace{1,\dots,1}_{\xi}, 0, \dots, 0) \tag{3.39}$$

for some $\xi \ge 0$, with only the first form allowed if $x < y$, and only the second if $y < x$. This time each $GL(2,\mathbb{R})$ factor with $\mu_i = 0$ and each pair of $GL(1,\mathbb{R})$ factors with corresponding $(\epsilon_i, \epsilon_j) = (1, -1)$ contribute a pair of entries with opposite parity in the LKT Λ of π. □

Example 3.40. Let $G = O(17,13)$, $a = 4$, $d = 2$, $s = 3 = t$, $\lambda_d = (3,2,0,0;2,0)$ with $f_1 - e_2 \in \Psi$, $\mu = (6,4,0)$, and $\epsilon = (1,1,-1)$. Then $h = 3$,

$$\lambda_a = (3,3,2,2,0,0,0,0;3,2,2,0,0,0),$$
$$\lambda_a + \rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{p}) = (2,2,\tfrac{3}{2},\tfrac{3}{2},0,0,0,0;5,\tfrac{7}{2},\tfrac{7}{2},0,0,0),$$
$$\delta_L = (0,0,-\tfrac{1}{2},-\tfrac{1}{2},0,0,0,0;0,\tfrac{1}{2},\tfrac{1}{2},1,1,1),$$

so that π has a LKT with highest weight

$$\Lambda = (2,2,1,1,0,0,0,0;5,4,4,1,1,1). \tag{3.41}$$

Example 3.42. Let $G = O(8,8)$, $a = d = 0$, $s = 2$, $t = 4$, $\mu = (2,0)$, and $\epsilon = (-1,-1,-1,-1)$. Then $h = 1$, $\lambda_a = (1,0,0,0;1,0,0,0)$, $\lambda_a + \rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{p}) = (\tfrac{3}{2},0,0,0;\tfrac{3}{2},0,0,0)$, and we have 4 choices for δ_L, given by $(\pm\tfrac{1}{2},1,0,0;\mp\tfrac{1}{2},0,0,0)$ and $(\pm\tfrac{1}{2},0,0,0;\mp\tfrac{1}{2},1,0,0)$, resulting in LKT's with highest weights $(2,1,0,0;1,0,0,0)$, $(1,1,0,0;2,0,0,0)$, $(2,0,0,0;1,1,0,0)$ and $(1,0,0,0;2,1,0,0)$.

If we change ϵ to $(1,1,1,1)$, we get the same four highest weights.

Proposition 3.43. *Let $\pi = \pi_\zeta(\lambda_d,\xi,\Psi,\mu,\nu,\epsilon,\kappa)$ be an irreducible admissible representation of $O(p,q)$, and $\Lambda_0 = (\Lambda_1;\Lambda_2)$ the highest weight of a LKT of π, as in Proposition 3.36. Let z and z' be as in (3.21), α the number of indices $i \le s$ such that $\mu_i = 0$, β be the number of indices $i \le t$ such that $\epsilon_i = 1$, γ the number of indices $i \le t$ such that $\epsilon_i = -1$. Then π has a LKT Λ with highest weight Λ_0 and signs given as follows:*

1. *Suppose $z + z' = 0$ and $\kappa_i \neq 0$ for all i. If $\beta \ge \gamma$ then both $(1;1)$ and $(-1;-1)$ occur as signs with Λ_0. (The resulting two K-types may coincide.) If $\beta < \gamma$ then both $(1;-1)$ and $(-1;1)$ occur as signs with Λ_0.*
2. *Suppose $z + z' = 0$ and $(\epsilon_i,\kappa_i) = (1,0)$ for some i. If $\beta \ge \gamma$ then the signs of Λ are $(\zeta;\zeta)$. If $\beta < \gamma$ then the signs are $(\zeta;-\zeta)$ if Λ_1 has more zeros then Λ_2, and $(-\zeta;\zeta)$ otherwise.*
3. *Suppose $z + z' = 0$ and $(\epsilon_i,\kappa_i) = (-1,0)$ for some i. If $\beta \ge \gamma$ then the signs are $(\zeta;\zeta)$ if Λ_1 has more zeros than Λ_2, and $(-\zeta,-\zeta)$ otherwise. If $\beta < \gamma$ then the signs are $(\zeta;-\zeta)$.*
4. *Suppose $z + z' > 0$ and $\beta \ge \gamma$. Then Λ has signs $(\xi;\xi)$.*
5. *Suppose $z + z' > 0$ and $\beta < \gamma$. The signs of Λ are $(\xi;-\xi)$ if Λ_1 has more zeros than Λ_2, and $(-\xi;\xi)$ otherwise.*

Proof. We can take parts (2) and (3) as the definition of $\pi_\zeta(\lambda_d,\dots)$, but we need to show that this definition makes sense. We defer this proof and the proof of parts (1),(4), and (5) to Section 5 since most of these assertions may be deduced from the full correspondence, along with the correspondence of K-types in the space of joint harmonics. ☐

Example 3.40 (continued). The representation $\pi = \pi_1(\lambda_d,1,\Psi,\mu,\nu,\epsilon,\kappa)$ of $O(17,13)$ has as a LKT

$$(2,2,1,1,0,0,0,0;1)\otimes(5,4,4,1,1,1;1), \tag{3.44}$$

and $\pi = \pi_1(\lambda_d,-1,\Psi,\mu,\nu,\epsilon,\kappa)$ has as a LKT

$$(2,2,1,1,0,0,0,0;-1)\otimes(5,4,4,1,1,1;-1). \tag{3.45}$$

Example 3.46. The one-dimensional representations of $O(t,t)$ (see §2.1) have Langlands parameters as follows: Let $\kappa = (0,1,2,\dots,t-1)$. Then $\mathbb{1} = \pi_{+1}(0,1,\emptyset,0,0,\epsilon,\kappa)$ and $det = \pi_{-1}(0,1,\emptyset,0,0,\epsilon,\kappa)$ with $\epsilon = (1,\dots,1)$; the characters whose restriction to $SO(t,t)$ is nontrivial are given by $\chi_{+,-} = \pi_{+1}(0,1,\emptyset,0,0,\epsilon,\kappa)$ and $\chi_{-,+} = \pi_{-1}(0,1,\emptyset,0,0,\epsilon,\kappa)$, with $\epsilon = (-1,\dots,-1)$.

4. The Correspondence

We now describe the correspondence for the dual pairs $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$ with $p+q=2n$ and $p+q=2n+2$ explicitly, in terms of Langlands parameters as in §3. We first fix a limit of discrete series ρ of $Sp(2n,\mathbb{R})$ and give four theta lifts at ranks n and $n+1$. Only one of these four lifts will occur at rank n if the parameter does not contain any zeros, and two if it does. If p and q are both even then $\theta_{p,q}(\rho)$ is again a limit of discrete series of $O(p,q)$.

Theorem 4.1. *Let $\rho=\rho(\lambda_d,\Psi)$ be a limit of discrete series representation of $Sp(2n,\mathbb{R})$, with λ_d as in (3.1). Let $k=\sum_{i=1}^b k_i$, $l=\sum_{i=1}^b l_i$, and $w=[\frac{z}{2}]$. Let $p_1=k+w$ and $q_1=l+w$.*

(1) *If $z=2w$ then $\theta_{2p_1,2q_1}(\rho)=\rho(\lambda_{0,0},1,\Psi_{0,0})$, where*

$$\lambda_{0,0}=(\underbrace{a_1,\dots,a_1}_{k_1},\dots,\underbrace{a_b,\dots,a_b}_{k_b},\underbrace{0,\dots,0}_{w};\underbrace{a_1,\dots,a_1}_{l_1},\dots,\underbrace{a_b,\dots,a_b}_{l_b},\underbrace{0,\dots,0}_{w}), \tag{4.2}$$

and $\Psi_{0,0}$ is obtained from Ψ as follows: for $1\le i\le p_1$ and $1\le j\le q_1$, the root $e_i-f_j\in\Psi_{0,0}$ if and only if $e_i+e_{n-j+1}\in\Psi$. (This determines $\Psi_{0,0}$ completely.) We also have $\theta_{2p_1+1,2q_1+1}(\rho)=\pi_1(\lambda_{0,0},1,\Psi_{0,0},0,0,(1),(0))$, i. e., it is obtained from $\theta_{2p_1,2q_1}(\rho)=\rho(\lambda_{0,0},1,\Psi_{0,0})$ by adding a factor of $GL(1,\mathbb{R})$ with the trivial character to the data.

(2) *If $z=2w>0$ then we have a second occurrence of ρ at rank n, depending on the root system Ψ. If $e_{k+1}+e_{k+z}\in\Psi$ then $\theta_{2p_1+1,2q_1-1}(\rho)\neq 0$, and if $-e_{k+1}-e_{k+z}\in\Psi$ then $\theta_{2p_1-1,2q_1+1}(\rho)\neq 0$. In the first case, the lift is $\pi_1(\lambda_{1,-1},1,\Psi_{1,-1},0,0,(-1),(0))$, where $\lambda_{1,-1}$ is obtained from $\lambda_{0,0}$ (4.2) by removing the last zero, and $\Psi_{1,-1}\subset\Psi_{0,0}$. Then $\theta_{2p_1+2,2q_1}(\rho)=\rho(\lambda_{2,0},1,\Psi_{2,0})$ is the limit of discrete series of $O(2p_1+2,2q_1)$ with Harish-Chandra parameter $\lambda_{2,0}$ obtained from $\lambda_{0,0}$ by adding a zero on the left, and $\Psi_{0,0}\subset\Psi_{2,0}$. Similarly for the second case, where a zero is removed from $\lambda_{0,0}$ on the left to get $\lambda_{-1,1}$, a zero is added on the right to get $\lambda_{0,2}$, and the root systems are obtained by restriction or (unique) extension.*

(3) *If $z=w=0$ then $\theta_{2p_1+2,2q_1}(\rho)=\rho(\lambda_{2,0},1,\Psi_{2,0})$ and $\theta_{2p_1,2q_1+2}(\rho)=\rho(\lambda_{0,2},1,\Psi_{0,2})$, where $\lambda_{2,0}$ and $\lambda_{0,2}$ are obtained from $\lambda_{0,0}$ by adding a zero on the left and right respectively, and $\Psi_{0,0}\subset\Psi_{2,0},\Psi_{0,2}$.*

(4) *If $z=2w+1$ is odd then $\theta_{2p_1+1,2q_1+1}(\rho)=\pi_\zeta(\lambda_{0,0},1,\Psi_{0,0},0,0,(-1),(0))$ with $\lambda_{0,0}$ and $\Psi_{0,0}$ as in (1) above. Here $\zeta=-1$ if $w=0$ and $2e_{k+1}\in\Psi$, and $\zeta=1$ otherwise. If $e_{k+1}+e_{k+z}\in\Psi$ then $\theta_{2p_1+2,2q_1+2}(\rho)=\rho(\lambda_{1,1},1,\Psi_{1,1})$, the limit of discrete series of $O(2p_1+2,2q_1+2)$ with $\lambda_{1,1}$ obtained from $\lambda_{0,0}$ by adding a zero on each side of the semicolon, and $\Psi_{0,0}\cup\{e_{k+w+1}-f_{l+w+1}\}\subset\Psi_{1,1}$. Moreover, $\theta_{2p_1+2,2q_1}(\rho)=\rho(\lambda_{1,0},1,\Psi_{1,0})$ with $\lambda_{1,0}$ obtained from $\lambda_{0,0}$ by adding a zero on the left, and $\Psi_{0,0}\subset\Psi_{1,0}$; and $\theta_{2p_1+3,2q_1+1}(\rho)=\pi_1(\lambda_{1,0},1,\Psi_{1,0},0,0,(1),(0))$. Similarly, if $-e_{k+1}-e_{k+z}\in\Psi$, we get $\theta_{2p_1+2,2q_1+2}(\rho)=\rho(\lambda_{1,1},1,\Psi_{1,1})$ (with $-e_{k+w+1}+f_{l+w+1}\in\Psi_{1,1}$), $\theta_{2p_1,2q_1+2}(\rho)=\rho(\lambda_{0,1},1,\Psi_{0,1})$, and $\theta_{2p_1+1,2q_1+3}(\rho)=\pi_1(\lambda_{0,1},1,\Psi_{0,1},0,0,(1),(0))$ in an analogous way.*

Moreover, in all the above cases we have that the LKT of ρ is of minimal degree and corresponds in the space of joint harmonics to a LKT of $\theta_{p,q}(\rho)$.

Example 4.3. The limit of discrete series ρ_1 of $SL(2,\mathbb{R})$ with LKT (1) occurs at ranks 1 and 2 as given below:

$$\begin{array}{cc} O(3,1) & O(2,2) \\ O(2,0) & O(1,1) \end{array} \tag{4.4}$$

We have that $\theta_{1,1}(\rho_1)=\chi_{-,+}$, $\theta_{2,0}(\rho_1)=\mathbb{1}$, $\theta_{2,2}(\rho_1)$ is a limit of discrete series with Harish-Chandra parameter $(0;0)$ and LKT $(1;1)\otimes(0;1)$, and $\theta_{3,1}(\rho_1)$ is the spherical representation with infinitesimal character zero.

Example 4.5. The occurrence of the discrete series ρ_2 of $SL(2,\mathbb{R})$ with LKT (2) is given by:

$$\begin{array}{ccc} O(4,0) & O(3,1) & O(2,2) \\ & O(2,0) & \end{array} \tag{4.6}$$

Now $\theta_{2,0}(\rho_2)$ is the representation with highest weight (1), $\theta_{4,0}(\rho_2) = \mathbb{1}$, $\theta_{2,2}(\rho_2)$ is the discrete series of $O(2,2)$ with parameter $(1;0)_1$ (and LKT $(2;1) \otimes (0;1)$), and $\theta_{3,1}(\rho_2)$ is the constituent π_+ of the principal series $Ind_{O(2)\times GL(1,\mathbb{R})}^{O(3,1)}((1) \otimes \mathbb{1})$ with LKT $(1;1) \otimes (\_;1)$. (This induced has a second constituent π_- with LKT $(1;-1) \otimes (\_;-1)$.)

The general case is determined by the correspondence for limits of discrete series so that the picture for occurrence at ranks $2n$ and $2n+2$ will look like

$$\begin{array}{cc} O(p+1,q+1) & O(p,q+2) \\ O(p,q) & O(p-1,q+1) \end{array} \quad \text{or} \quad \begin{array}{ccc} O(p+2,q) & O(p+1,q+1) & O(p,q+2) \\ & O(p,q) & \end{array} \tag{4.7}$$

as in (4.4) and (4.6), depending on whether the limit of discrete series parameter contains a zero or not.

Theorem 4.8. *Let $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \epsilon, \kappa)$ be an irreducible admissible representation of $Sp(2n,\mathbb{R})$, let v, s, and t be as in (3.3), and let $\rho = \rho(\lambda_d, \Psi)$ be the limit of discrete series representation of $Sp(2v,\mathbb{R})$ determined by λ_d and Ψ. Let p_1 and q_1 be integers such that $p_1 + q_1 = 2v$ or $2v+2$ and ρ occurs in the correspondence for the dual pair $\big(Sp(2v,\mathbb{R}), O(p_1,q_1)\big)$, as in Theorem 4.1. Let $p = p_1 + 2s + t$, $q = q_1 + 2s + t$, and write $\theta_{p_1,q_1}(\rho) = \pi_\zeta(\lambda'_d, 1, \Psi', 0, 0, \epsilon_0, \kappa_0)$. Let $\epsilon_{p,q} = \big(\epsilon_1(-1)^{\frac{p-q}{2}}, \ldots, \epsilon_t(-1)^{\frac{p-q}{2}}\big)$, $\epsilon' = (\epsilon_0|\epsilon_{p,q})$, and $\kappa' = (\kappa_0|\kappa)$. Then*

$$\theta_{p,q}(\pi) = \pi_\zeta(\lambda'_d, 1, \Psi', \mu, \nu, \epsilon', \kappa'). \tag{4.9}$$

Moreover, some LKT of π is of minimal degree and corresponds in the space of joint harmonics to a LKT of $\theta_{p,q}(\pi)$. If $p+q=2n$ then this last statement applies to every *LKT of π.*

Remark 4.10. It is straightforward to check that the correspondence as given in Theorem 4.8 preserves the non-parity condition; i. e., if the parameters for π are such that (3.5) through (3.7) hold, then (3.28) and (3.29) hold for $\pi_\zeta(\lambda'_d, 1, \Psi', \mu, \nu, \epsilon', \kappa')$.

The following results about the correspondence are standard (see Lemmas 1.5, Lemma 1.7 and 1.8 of [3]; the proofs outlined there will go through with no or little adjustment).

Lemma 4.11. *Let n, p, and q be non-negative integers with $p+q$ even. Let π be an irreducible admissible representation of $Sp(2n,\mathbb{R})$, and π^* the contragredient of π. Then $\theta_{q,p}(\pi) = \theta_{p,q}(\pi^*)$.*

Lemma 4.12. *Let n, p, q, p', and q' be non-negative integers such that $p+q$ and $p'+q'$ are even. Let $\omega_{n,p,q}$ be the oscillator representation of $\widetilde{Sp}(2n(p+q),\mathbb{R})$, restricted to the dual pair $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$, and similarly for $\omega_{n,p',q'}$ and $\omega_{n,p+p',q+q'}$. Then*

$$\omega_{n,p,q} \otimes \omega_{n,p',q'} \cong \omega_{n,p+p',q+q'} \tag{4.13}$$

as representations of $Sp(2n,\mathbb{R}) \times O(p,q) \times O(p',q')$, with $Sp(2n,\mathbb{R})$ acting diagonally on the left-hand side.

Proposition 4.14. *Let n, p, and q be non-negative integers such that $p+q$ is even, and π an irreducible admissible representation of $Sp(2n,\mathbb{R})$ such that $\theta_{p,q} \neq 0$. If r and s are integers such that $r+s$ is even, $r \geq -p$, $s \geq -q$, $|r-s| \geq 4$, and $min\{r,s\} < 2n-p-q$, then $\theta_{p+r,q+s}(\pi) = 0$.*

Proof. The proof is similar to that of Lemma 1.8 of [3]; we just outline the main steps. If $\theta_{p+r,q+s}(\pi) \neq 0$, then by Lemma 4.11, $\theta_{q+s,p+r}(\pi^*) \neq 0$. Then it follows using Lemma 4.12 that $\theta_{p+q+s,p+q+r}(\mathbb{1}) \neq 0$. Using the correspondence of K-types in the space of joint harmonics (Proposition 2.10), for example, one sees easily that with $|r-s| \geq 4$, the trivial representation of $Sp(2n,\mathbb{R})$ does not occur below stable range, i. e., we must have $min\{p+q+s, p+q+r\} \geq 2n$. The proposition follows. □

Let π be an irreducible admissible representation of $Sp(2n,\mathbb{R})$, and suppose p and q are such that $p+q=2n$ and π occurs with the following groups:

$$\begin{array}{cc} O(p+1,q+1) & O(p,q+2) \\ O(p,q) & O(p-1,q+1) \end{array} \tag{4.15}$$

as given by Theorem 4.8. Then $2n-p-q=0$, so Proposition 4.14 with $r=-2$ and $s=2$ gives that $\theta_{p-2,q+2}(\pi)=0$, and similarly $\theta_{p-j,q+j}(\pi)=0$ for $j \geq 3$. Replacing (p,q) by $(p-1,q+1)$ in the argument yields that $\theta_{p+j,q-j}(\pi)=0$ for $j \geq 1$ as well, so we have no additional occurrence at rank n. Moreover, $\theta_{p-1,q+3}(\pi)=0$ (with $r=-1$, $s=3$), and so are all other lifts at rank $n+1$. We leave it as an exercise for the diligent reader to check that if the occurrence of π looks like the diagram on the right of (4.7), no other occurrence are possible at ranks n and $n+1$ in that case either. So we have the following result.

Corollary 4.16. *Every irreducible admissible representation of $Sp(2n,\mathbb{R})$ occurs precisely four times at ranks n and $n+1$; i. e., Theorems 4.1 and 4.8 give the complete correspondence for the dual pairs $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$ with $p+q=2n$ and $p+q=2n+2$.*

Now that we know the full correspondence at equal rank, we can turn things around and look at which representations of $O(p,q)$ occur.

Corollary 4.17. *Let $\pi = \pi_\zeta(\lambda_d,\xi,\Psi,\mu,\nu,\epsilon,\kappa)$ be an irreducible admissible representation of $G = O(p,q)$, with λ_d, z, and z' as in (3.21), and let $n = \frac{p+q}{2}$. We have*

1. *$\theta_n(\pi) \neq 0$ if and only if $\zeta = \xi = 1$, or $z+z'=0$ and $(\epsilon_i,\kappa_i) = (-1,0)$ for some $i \leq t$.*
2. *$\theta_{n-1}(\pi) \neq 0$ if and only if $\theta_n(\pi) \neq 0$ and the parameters $(\lambda_d,\xi,\Psi,\mu,\nu,\epsilon,\kappa)$ satisfy $z+z'>0$ or $(\epsilon_i,\kappa_i)=(1,0)$ for some $i \leq t$ (see §3.2).*

5. The Proof of Theorems 4.1 and 4.8

5.1. **Some Comments about Moeglin's Paper.** Theorems 4.1 and 4.8 are restatements of results of Moeglin [14] plus extensions of these results to the groups $O(p,q)$ with p and q odd. Much of the proof amounts to checking that Moeglin's proof can be adapted to cover these cases. Before proceeding, we take this opportunity to point out an error in [14], and to suggest a way to fix it.

In [14], Moeglin defines conditions (†) and (∗) (depending on p_0 and q_0) for representations of $Sp(2n,\mathbb{R})$, and (†)′ and (∗)′ (the latter depending on n) for representations of $O(2p_0,2q_0)$. We state conditions (†), (†)′ and (∗) here.

Definition 5.1. Let n, $p=2p_0$, and $q=2q_0$ be given. Let π be an irreducible admissible representation of $Sp(2n,\mathbb{R})$ with associated Vogan-parameter λ_a as in (3.10). Write $u=\sum_{i=1}^m u_i$ and $r=\sum_{i=1}^m r_i$, and let $\sigma=0$ if $p_0-q_0+r-u=0$, $\sigma=1$ otherwise. Let π' be an irreducible admissible representation of $O(p,q)$.

(1) We say that π satisfies (†) if π has a LKT $\Lambda = (c_1, c_2, \dots, c_n)$ such that the following three conditions are satisfied:
  (α): $p_0 - q_0 + r - u = 1$, 0, or -1.
  (β): If $n > u + r$ (i. e., if $w > 0$) then $|c_j + q_0 - p_0| \leq 1$ for $u + 1 \leq j \leq u + w$.
  (γ): $p_0 \geq u$, $q_0 \geq r$, $\sharp\{u+1 \leq j \leq u+w \,:\, c_j + q_0 - p_0 > 0\} \leq 2(p_0 - u)$, and $\sharp\{u+1 \leq j \leq u+w \,:\, c_j + q_0 - p_0 < 0\} \leq 2(q_0 - r)$.
  We will sometimes say that the pair (π, Λ) satisfies (†) since it may happen that some but not all LKT's of π satisfy conditions (β) and (γ).
(2) We say that π' satisfies (†)$'$ if π' has a LKT $\Lambda' = (a_1, \dots, a_\xi, 0, \dots, 0; \epsilon) \otimes (b_1, \dots, b_h, 0, \dots, 0; \eta)$ with $a_\xi > 0$, $b_h > 0$, and $\epsilon(p_0 - \xi) + \eta(q_0 - h) \geq 0$.
(3) We say that π satisfies ($*$) if $n \geq p_0 + q_0$, and if π is a LKT constituent of an induced representation of the form $Ind_P^{Sp(2n,\mathbb{R})}(\pi_0 \otimes \chi)$, where $P = LN$ is a parabolic subgroup of $Sp(2n, \mathbb{R})$ with Levi factor $L \cong Sp(2(p_0 + q_0 - \sigma), \mathbb{R}) \times GL(n - p_0 - q_0 + \sigma, \mathbb{R})$, π_0 an irreducible admissible representation of $Sp(2(p_0 + q_0 - \sigma), \mathbb{R})$, and χ is the character of $GL(n - p_0 - q_0 + \sigma, \mathbb{R})$ given by $\chi = |det|^{\frac{n - p_0 - q_0 - \sigma + 1}{2}} sgn(det)^{p_0 - q_0}$.

Notice that (†) and (†)$'$ are conditions which can be checked by looking at the LKT's only (since λ_a may be recovered using the Vogan algorithm), while ($*$) is a condition on the inducing data, and requires knowledge of the continuous parameter.

The strategy is then to define maps Ψ_n and Φ_{p_0,q_0}; for $p_0 + q_0 \leq n$, Ψ_n assigns to each representation of $O(2p_0, 2q_0)$ satisfying (†)$'$ a representation of $Sp(2n, \mathbb{R})$ satisfying (†) and ($*$), and for $n \leq p_0 + q_0$, Φ_{p_0,q_0} assigns to each representation of $Sp(2n, \mathbb{R})$ satisfying (†) a representation $O(2p_0, 2q_0)$ which then satisfies (†)$'$ and ($*$)$'$. Moreover, every representation of $Sp(2n, \mathbb{R})$ satisfying (†) and ($*$) is in the image of Ψ_n, and similarly for $O(2p_0, 2q_0)$. Theorem III.13 then says that the maps Ψ_n and Φ_{p_0,q_0} give the Howe correspondence; in particular, that if $n \leq p_0 + q_0$ then every representation π of $Sp(2n, \mathbb{R})$ satisfying (†) occurs in the correspondence for the dual pair $\big(Sp(2n, \mathbb{R}), O(2p_0, 2q_0)\big)$, and $\theta_{2p_0,2q_0}(\pi) = \Phi_{p_0,q_0}(\pi)$, and a similar statement for $p_0 + q_0 \leq n$ and representations of $O(2p_0, 2q_0)$. In particular, occurrence can be determined by looking at the LKT's only. Implicit in these statements is the assertion that for $n = p_0 + q_0$, the condition (†) implies the condition ($*$) (and similarly for (†)$'$ and ($*$)$'$). A simple example shows that this is not true in general.

Example 5.2. Let $n = 1$, $p_0 = 1$, and $q_0 = 0$, and let π be a non-spherical principal series of $SL(2, \mathbb{R})$ with generic continuous parameter ν. Then $u = r = 0$, $\sigma = 1$, and π has LKT's (1) and (-1). It is easy to check that using the K-type (1), the condition (†) is satisfied. However, the condition ($*$) is not (condition ($*$) requires the representation to be a principal series with $\nu = 0$), and it is easy to see (by using the infinitesimal character correspondence as given in [18], for example) that π does not occur in the correspondence for the dual pair $\big(SL(2, \mathbb{R}), O(2, 0)\big)$.

More generally, if $n = p_0 + q_0$ and $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \epsilon, \kappa)$ is a representation of $Sp(2n, \mathbb{R})$ satisfying (†) and with the associated λ_a as in (3.10), then whenever w is odd then the parameter σ defined in 5.1 equals 1. In this case, the definition of Φ_{p_0,q_0} in §III.3 does not make sense since we would need $p_0 + q_0 - n - \sigma = -1$ $GL(1)$-factors in the Levi factor of $P_0 \subset O(2p_0, 2q_0)$. The condition ($*$) then amounts to there being an index i such that $(\epsilon_i, \kappa_i) = ((-1)^{p_0 - q_0}, 0)$, which certainly does not follow from (†), a condition on a LKT that is independent of the continuous parameter.

However, if $n < p_0 + q_0$ then the condition (†) does indeed imply ($*$), so one might correct the error by changing the condition $n \leq p_0 + q_0$ in Theorem III.13(i) to $n < p_0 + q_0$, and by making the analogous changes throughout the paper; most notably in the introduction, in conditions ($*$)$'$ and ($**$)$'$ in §III.1, in the definition of Φ_{p_0,q_0} in §III.3, in Lemma III.12(i), and in Theorem IV.3 (since for given n, p_0, and q_0, only one of Ψ_n and Φ_{p_0,q_0} is defined). Unfortunately, though not completely surprisingly in light of our discussion of dual groups in the introduction, this diminishes the beautiful symmetry of Moeglin's statements. However, because of the redundancy contained in the present statements, we will still get the complete correspondence for the equal rank case.

For the purposes of this paper, we reformulate some of the results of Lemmas II.3, II.6, II.7, and II.12, and Theorem III.13 of [14] below. Moeglin defines the map Φ_{p_0,q_0} as follows: If π is an irreducible admissible representation of $Sp(2n,\mathbb{R})$ with LKT Λ such that (π,Λ) satisfies $(\dagger)$ (and $(*)$), she assigns to π a standard module (induced from discrete series) of $O(2p_0,2q_0)$, and specifies that $\Phi_{p_0,q_0}(\pi)$ is the unique subquotient containing as LKT the K-type corresponding in $\mathcal{H}$ to Λ. Analougously for the map Ψ_n. Using the LKT calculations of §3, one can check that for the cases $p+q=2n$ and $2n+2$ with p and q even, the map described in Theorems 4.1 and 4.8 coincides with Ψ_n^{-1} and Φ_{p_0,q_0} respectively.

Theorem 5.3 (Moeglin). *Let n, p, and q be non-negative integers such that $p=2p_0$ and $q=2q_0$ are even.*

(1) *Let π be an irreducible admissible representation of $Sp(2n,\mathbb{R})$, and Λ a LKT of π such that (π,Λ) satisfies $(\dagger)$. Then Λ is of minimal degree in π. Suppose that in addition $2n\le p+q$, and if $2n=p+q$ then we also have that π satisfies $(*)$. Then $\theta_{p,q}(\pi)=\Phi_{p_0,q_0}(\pi)$, and Λ corresponds in the space of joint harmonics to a LKT of $\Phi_{p_0,q_0}(\pi)$.*
(2) *Let π' be an irreducible admissible representation of $O(p,q)$, and Λ' a LKT of π' such that (π',Λ') satisfies $(\dagger)'$. The Λ' is of minimal degree in π'. If in addition $p+q\le 2n$ then $\theta_n(\pi')=\Psi_n(\pi')$, and Λ' corresponds in the space of joint harmonics to a LKT of $\Psi_n(\pi')$.*

5.2. **The Induction Principle.** Let $(W,<,>)$ be a symplectic space over $\mathbb{R}$ of dimension $2n$ with isometry group $Sp(W)\cong Sp(2n,\mathbb{R})$, and $(V,(,))$ a real vector space with nondegenerate symmetric bilinear form $(,)$ of signature (p,q), with isometry group $O(V)\cong O(p,q)$ $(p+q$ even$)$. Let

$$\{0\}\subset W_1\subset W_2\subset\cdots\subset W_r \tag{5.4}$$

be an isotropic flag in W, and let

$$\{0\}\subset V_1\subset V_2\subset\cdots\subset V_r \tag{5.5}$$

be an isotropic flag in V. For $1\le i\le r$, let d_i be the dimension of W_i, and d_i' the dimension of V_i. Set $d_0=d_0'=0$, and for $1\le i\le r$, $n_i=d_i-d_{i-1}$ and $n_i'=d_i'-d_{i-1}'$. Let $P=MAN$ be the stabilizer of the flag (5.4) in $Sp(W)$, and let $P'=M'A'N'$ be the stabilizer of the flag (5.5) in $O(V)$. Then P has Levi factor

$$MA\cong Sp(2v,\mathbb{R})\times\prod_{i=1}^{r}GL(n_i,\mathbb{R}) \tag{5.6}$$

with $v=n-d_r$, and P' has Levi factor

$$M'A'\cong O(a,b)\times\prod_{i=1}^{r}GL(n_i',\mathbb{R}) \tag{5.7}$$

with $a=p-d_r'$ and $b=q-d_r'$. For $1\le i\le r$, let ξ_i and ξ_i' be the characters of $GL(n_i,\mathbb{R})$ and $GL(n_i',\mathbb{R})$, respectively, given by

$$\xi_i(g) = sgn(det(g))^{\frac{p-q}{2}} |det(g)|^{\frac{p+q}{2} - n - d'_{i-1} - \frac{n'_i}{2} + d_{i-1} + \frac{n_i}{2} - \frac{1}{2}} \quad \text{for } g \in GL(n_i, \mathbb{R}), \tag{5.8}$$

$$\xi'_i(g') = |det(g')|^{n - \frac{p+q}{2} - d_{i-1} - \frac{n_i}{2} + d'_{i-1} + \frac{n'_i}{2} + \frac{1}{2}} \quad \text{for } g' \in GL(n'_i, \mathbb{R}). \tag{5.9}$$

Theorem 5.10 (Induction Principle, First Version). *Let π and π' be irreducible admissible representations of $Sp(2v,\mathbb{R})$ and $O(a,b)$, respectively, such that π corresponds to π' in the correspondence for the dual pair $\big(Sp(2v,\mathbb{R}), O(a,b)\big)$. For $1 \le i \le r$, let σ_i and σ'_i be irreducible admissible representations of $GL(n_i,\mathbb{R})$ and $GL(n'_i,\mathbb{R})$, respectively, such that σ_i corresponds to σ'_i in the correspondence for the dual pair $\big(GL(n_i,\mathbb{R}), GL(n'_i,\mathbb{R})\big)$. Write $\sigma\xi = \bigotimes_{i=1}^r \sigma_i\xi_i$ and $\sigma'\xi' = \bigotimes_{i=1}^r \sigma'_i\xi'_i$. Let ω be the oscillator representation for the dual pair $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$. Then there exists a nonzero $(Sp(2n,\mathbb{R}) \times O(p,q))$-equivariant map (on the level of $(\mathfrak{g},K)$-modules)*

$$\phi : \omega \longrightarrow Ind_P^{Sp(2n,\mathbb{R})}(\pi \otimes \sigma\xi \otimes \mathbb{1}) \otimes Ind_{P'}^{O(p,q)}(\pi' \otimes \sigma'\xi' \otimes \mathbb{1}). \tag{5.11}$$

Proof. The proof is very much like that of Corollary 3.21 of [2], Theorem 4.5.5 of [16], and Theorem 4.20 of [13], using ideas of [11] (see also Corollary III.8 of [14]).

□

The correspondence for the dual pairs $\big(GL(k,\mathbb{R}), GL(l,\mathbb{R})\big)$ is described in Proposition III.9 of [14]. For convenience, we record it for the case $k = l$ here.

Proposition 5.12. *The correspondence for the dual pairs $\big(GL(k,\mathbb{R}), GL(k,\mathbb{R})\big)$ is given as follows. Let σ be an irreducible admissible representation of $GL(k,\mathbb{R})$. Then σ occurs in the correspondence and corresponds to σ^*, the contragredient representation. The unique LKT of σ is of minimal degree and corresponds in the space of joint harmonics to the unique LKT of σ^*.*

Taking the oscillator representation of $\widetilde{Sp}(0,\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ to be the nontrivial character, we get the dual pair correspondence for $\big(GL(k,\mathbb{R}), GL(0,\mathbb{R})\big)$ as $\mathbb{1} \leftrightarrow \mathbb{1}$; this allows us to choose n_i or n'_i to be zero for some i in Theorem 5.10. Keeping in mind that $\xi'_i = \xi^*_i sgn(det(g))^{\frac{p-q}{2}}$, and that $\tau \otimes sgn(det) \cong \tau$ for a relative limit of discrete series of $GL(2,\mathbb{R})$, we can deduce the following result.

Theorem 5.13 (Induction Principle, Second Version). *Let n, p, and q be non-negative integers such that $p+q$ is even, π an irreducible admissible representation of $Sp(2n,\mathbb{R})$, and π' an irreducible admissible representation of $O(p,q)$ such that $\theta_{p,q}(\pi) = \pi'$. Let s, t, and m be non-negative integers, τ a relative limit of discrete series representation of $GL(2,\mathbb{R})^s$, and χ a character of $GL(1,\mathbb{R})^t$. Write $\chi_{p,q} = \otimes_{i=1}^t sgn^{\frac{p-q}{2}}$, a character of $GL(1,\mathbb{R})^t$.*

(1) *Let $G = Sp(2(n+2s+t+m),\mathbb{R})$, $G' = O(p+2s+t, q+2s+t)$, ω the oscillator representation for the dual pair $\big(G, G'\big)$, and let ξ be the character of $GL(m,\mathbb{R})$ given by*

$$\xi(g) = |det(g)|^{\frac{p+q-2n-m-1}{2}} sgn(det(g))^{\frac{p-q}{2}}. \tag{5.14}$$

Then there are parabolic subgroups $P = MAN$ and $P' = M'A'N'$ of G and G' with Levi factors

$$MA \cong Sp(2n,\mathbb{R}) \times GL(2,\mathbb{R})^s \times GL(1,\mathbb{R})^t \times GL(m,\mathbb{R}) \quad \textit{and} \tag{5.15}$$

$$M'A' \cong O(p,q) \times GL(2,\mathbb{R})^s \times GL(1,\mathbb{R})^t \tag{5.16}$$

such that there exists a nonzero $G \times G'$-equivariant map (on the level of $(\mathfrak{g},K)$-modules)

$$\phi : \omega \longrightarrow Ind_P^G(\pi \otimes \tau^* \otimes \chi^* \otimes \chi_{p,q} \otimes \xi \otimes 1\!\!1) \otimes Ind_{P'}^{G'}(\pi' \otimes \tau \otimes \chi \otimes 1\!\!1). \tag{5.17}$$

(2) *Let* $G = Sp(2(n+2s+t), \mathbb{R})$, $G' = O(p+2s+t+m, q+2s+t+m)$, ω *the oscillator representation for the dual pair* (G, G'), *and let* ξ' *be the character of* $GL(m, \mathbb{R})$ *given by*

$$\xi(g) = |det(g)|^{\frac{2n-p-q-m+1}{2}}. \tag{5.18}$$

Then there are parabolic subgroups $P = MAN$ *and* $P' = M'A'N'$ *of* G *and* G' *with Levi factors*

$$MA \cong Sp(2n, \mathbb{R}) \times GL(2, \mathbb{R})^s \times GL(1, \mathbb{R})^t \quad and \tag{5.19}$$

$$M'A' \cong O(p,q) \times GL(2, \mathbb{R})^s \times GL(1, \mathbb{R})^t \times GL(m, \mathbb{R}) \tag{5.20}$$

such that there exists a nonzero $G \times G'$*-equivariant map (on the level of* $(\mathfrak{g}, K)$*-modules)*

$$\phi : \omega \longrightarrow Ind_P^G(\pi \otimes \tau \otimes \chi \otimes 1\!\!1) \otimes Ind_{P'}^{G'}(\pi' \otimes \tau^* \otimes \chi^* \otimes \chi_{p,q} \otimes \xi' \otimes 1\!\!1). \tag{5.21}$$

Proposition 5.22. *In the setting of Theorem 5.13 (2) with* $m = 0$, *let* I *and* I' *be the induced representations of* G *and* G' *given in (5.21). Suppose* $\pi \otimes \tau \otimes \chi$ *has a LKT* Λ *that is of minimal degree for the dual pair* $(MA, M'A')$. *Suppose also that* π_I *is a subquotient of* I *containing a LKT* Λ_I *of the induced representation such that the highest weight of* Λ_I *restricts to the highest weight of* Λ. *If* Λ_I *is of minimal degree for the dual pair* (G, G'), *occurs in the space of joint harmonics for the dual pair* (G, G') *and corresponds to the* K*-type* $\Lambda_{I'}$ *then* $\Lambda_I \otimes \Lambda_{I'}$ *is in the image of* ϕ. *Consequently, if* π_I *is a quotient of* I *then* π_I *occurs in the correspondence, and corresponds to a constituent of* I' *containing the* K*-type* $\Lambda_{I'}$.

Proof. This follows from the analogue of Proposition 3.25 of [2] (the extended induction principle of Adams/Barbasch); the proof given there goes through in the case of real symplectic-orthogonal dual pairs. □

Lemma 5.23. *In the setting of Proposition 5.22, the restriction of the highest weight of* Λ_I *is always the highest weight of a LKT of* $\pi \otimes \tau \otimes \chi$. *In particular, if* $\pi \otimes \tau \otimes \chi$ *has a unique LKT* Λ *then this condition is satisfied for all LKT's of* I.

Proof. This is a straightforward calculation using the explicit description of LKT's for $Sp(2n, \mathbb{R})$ given in Proposition 3.13. □

Remark 5.24. There is a result analogous to Proposition 5.22 for LKT's of π' and I'.

5.3. **The Proofs.** We are now ready to prove Theorems 4.1 and 4.8. We start with the case where π is a limit of discrete series representation of $Sp(2n, \mathbb{R})$, using Moeglin's results for the case p, q even, and applying the induction principle to the even case to obtain the cases p, q odd. Moeglin parametrizes irreducible admissible representations by inducing data for representations induced from discrete series, rather than from limits of discrete series. The resulting standard module may have several irreducible quotients which may be distinguished by their LKT's. Recall (see, e. g., [9]) that a limit of discrete series may be obtained by induction from a discrete series by violating the non-parity condition F-2 of [20] (see (3.5) through (3.7), and (3.28) and 3.29)); one such induced module will then be the direct sum of all limit of discrete series representations with the same Harish-Chandra parameter λ_d, but different root systems Ψ. We illustrate using an example.

Example 5.25. Let $\rho = \rho(\lambda_d, \Psi)$ be the limit of discrete series representation of $Sp(12,\mathbb{R})$ with parameter $\lambda_d = (2,1,0,0,-1,-3)$ and positive root system Ψ containing as simple roots $\{-e_1 - e_6, e_1 - e_2, e_2 + e_5, e_4 - e_5, -e_3 - e_4, 2e_3\}$. (So ρ has LKT $\Lambda = (2,2,0,-1,-1,-4)$.) Then ρ is a summand of the induced representation

$$I = Ind_{P_1}^{Sp(12,\mathbb{R})}(\rho_1 \otimes \tau_1 \otimes \mathbb{1}) \cong Ind_{P_2}^{Sp(12,\mathbb{R})}(\rho_1 \otimes \tau_2 \otimes \chi \otimes \mathbb{1}), \tag{5.26}$$

where $P_1 = M_1A_1N_1$ and $P_2 = M_2A_2N_2$ are parabolic subgroups of $Sp(12,\mathbb{R})$ with Levi factors $M_1A_1 \cong Sp(4,\mathbb{R}) \times GL(2,\mathbb{R})^2$ and $M_2A_2 \cong Sp(4,\mathbb{R}) \times GL(2,\mathbb{R}) \times GL(1,\mathbb{R})^2$, ρ_1 is the discrete series of $Sp(4,\mathbb{R})$ with Harish-Chandra parameter $\lambda_1 = (2,-3)$, $\tau_1 = \tau(2,0) \otimes \tau(0,0)$, $\tau_2 = \tau(2,0)$, and $\chi = \chi_{1,0} \otimes \chi_{-1,0}$. These induced representations have three more summands, corresponding to the other three positive root systems (see the comments after (3.1)).

Notice also that if $\rho_3 = \rho(\lambda_3, \Psi_3)$ is the limit of discrete series representation of $Sp(10,\mathbb{R})$ with $\lambda_3 = (2,1,0,-1,-3)$ and Ψ_3 containing $\{-e_1-e_5, e_1-e_2, e_2+e_4, -e_3-e_4, -2e_3\}$ (this representation has LKT $(2,2,-1,-1,-4)$), and $P_3 = M_3A_3N_3$ is a parabolic subgroup of $Sp(12,\mathbb{R})$ with $M_3A_3 \cong Sp(10,\mathbb{R}) \times GL(1,\mathbb{R})$, then we have

$$\rho \cong Ind_{P_3}^{Sp(12,\mathbb{R})}(\rho_3 \otimes \chi_{1,0} \otimes \mathbb{1}). \tag{5.27}$$

More generally, we can add coordinates to the Harish-Chandra parameter by adding $GL(1,\mathbb{R})$ and $GL(2,\mathbb{R})$ factors and using parabolic induction, and if the new coefficients already occur in the parameter for the smaller group (here we added a 0 which already occurred in the parameter λ_3), then the induced representation is irreducible.

Lemma 5.28. *Let $\rho = \rho(\lambda_d, \Psi)$ be a limit of discrete series representation of $Sp(2n,\mathbb{R})$. Theorem 4.1 gives a list of four pairs (p,q) of integers, depending on λ_d and Ψ, with $p+q = 2n$ or $2n+2$ and such that $\theta_{p,q}(\rho) \neq 0$. (E. g., if λ_d does not contain any zeros then this list consists of $(2p_1, 2q_1)$, $(2p_1+1, 2q_1+1)$, $(2p_1+2, 2q_1)$, and $(2p_1, 2q_1+2)$.) Choose such a pair (p,q) and let π' be the asserted theta lift $\theta_{p,q}(\rho)$.*

1. *The LKT Λ of ρ corresponds in the space of joint harmonics $\mathcal{H}$ to a LKT of π'.*
2. *If p and q are even then ρ satisfies the condition (†) of [14]; if $p+q = 2n$ then ρ also satisfies $(*)$.*
3. *Let $\Phi_{p,q}$ be as in §III.3 of [14] and Theorem 5.3. If p and q are even then $\pi' = \Phi_{p_0,q_0}(\rho)$, so that Theorem 4.1 for this case follows from Theorem 5.3.*

Proof. We use unprimed letters for parameters associated to ρ and $Sp(2n,\mathbb{R})$, and primed ones (λ'_d, $M'A'$, etc.) for those associated to π' and $O(p,q)$.

Since ρ is a limit of discrete series representation of $Sp(2n,\mathbb{R})$, we have $\lambda_a = \lambda_d$, and if λ_d is given by (3.1) then

$$\begin{aligned}\lambda_a + \rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{k}) = {} & (k-l.k-l.\dots,k-l) \\ & + (\underbrace{a_1+\alpha_1+\tfrac{1}{2},\dots,a_1+\alpha_1+\tfrac{1}{2}}_{k_1},\dots,\underbrace{a_b+\alpha_b+\tfrac{1}{2},\dots,a_b+\alpha_b+\tfrac{1}{2}}_{k_b},\underbrace{0,\dots,0}_{z}, \\ & \underbrace{-a_b+\alpha_b-\tfrac{1}{2},\dots,-a_b+\alpha_b-\tfrac{1}{2}}_{l_b},\dots,\underbrace{-a_1+\alpha_1-\tfrac{1}{2},\dots,-a_1+\alpha_1-\tfrac{1}{2}}_{l_1}),\end{aligned} \tag{5.29}$$

where for $1 \le i \le b$,

$$\alpha_i = \frac{l_i - k_i}{2} + \sum_{j=i+1}^{b} (l_j - k_j). \tag{5.30}$$

Now suppose first that $\frac{p-q}{2} = k - l$, so that the Vogan-parameter associated to π' is of the form

$$\lambda'_a = (\underbrace{a_1, \ldots, a_1}_{k_1}, \ldots, \underbrace{a_b, \ldots, a_b}_{k_b}, \underbrace{0, \ldots, 0}_{d}; \underbrace{a_1, \ldots, a_1}_{l_1}, \ldots, \underbrace{a_b, \ldots, a_b}_{l_b}, \underbrace{0, \ldots, 0}_{e}) \tag{5.31}$$

with $d = e$. (Notice that even if p and q are both odd so that π' is not a limit of discrete series, we have $\lambda'_a = \lambda'_d$.) Then

$$\begin{aligned} (5.32)\quad & \lambda'_a + \rho(\mathfrak{u}' \cap \mathfrak{p}') - \rho(\mathfrak{u}' \cap \mathfrak{k}') \\ &= (\underbrace{a_1 + \alpha_1 + \tfrac{1}{2}, \ldots, a_1 + \alpha_1 + \tfrac{1}{2}}_{k_1}, \ldots, \underbrace{a_b + \alpha_b + \tfrac{1}{2}, \ldots, a_b + \alpha_b + \tfrac{1}{2}}_{k_b}, \underbrace{0, \ldots, 0}_{d}; \\ & \qquad \underbrace{a_1 - \alpha_1 + \tfrac{1}{2}, \ldots, a_1 - \alpha_1 + \tfrac{1}{2}}_{l_1}, \ldots, \underbrace{a_b - \alpha_b + \tfrac{1}{2}, \ldots, a_b - \alpha_b + \tfrac{1}{2}}_{l_b}, \underbrace{0, \ldots, 0}_{d}) \end{aligned}$$

with α_i as in (5.30). Write the fine K-types δ_L and $\delta_{L'}$ that we have to add to (5.29) and (5.32) to obtain the highest weights of LKT's of ρ and π', respectively, as in (3.15) and (3.38) (the latter with primed entries). It is now easy to check that if the root systems Ψ and Ψ' are related as described in the theorem, then $\delta_i = \delta'_i$ for all $1 \leq i \leq b$. So to see that the K-types correspond, it remains to look at the parts $(\eta_1, \eta_2, \ldots, \eta_z)$ of δ_L and $(\eta'_1, \eta'_2, \ldots, \eta'_d; \xi'_1, \xi'_2, \ldots, \xi_d)$ of $\delta_{L'}$. It can be checked case by case that the relationship between Ψ and Ψ' is always such that the resulting K-types correspond; maybe the most interesting case is when $z = 2w + 1$, $p = 2k + 2w + 1$, and $q = 2l + 2w + 1$, and the signs of the LKT('s) of π' are not necessarily trivial. Assume that $e_{k+1} + e_{k+z} \in \Psi$ (the other case being analogous). Then (with $w = d$) $(\eta_1, \eta_2, \ldots, \eta_z) = (\underbrace{1, \ldots, 1}_{d+1}, \underbrace{0, \ldots, 0}_{d})$, and $(\eta'_1, \eta'_2, \ldots, \eta'_d; \xi'_1, \xi'_2, \ldots, \xi_d) = (1, \ldots, 1; 0, \ldots, 0)$. We have that the two K-types correspond provided that the signs of the LKT of π' are $(-1; 1)$. According to Proposition 3.43, this is indeed one of the LKT's.

If p and q are both even then (†) is easy to check (note that Moeglin's p'_{r+1} and q'_{r+1} are k and l in our notation, and that her condition (γ) is equivalent with Λ occurring in $\mathcal{H}$), and if in addition $p + q = 2n$ then $(*)$ holds because Moeglin's $\sigma = 0$.

For the third part, recall that ρ is a LKT constituent (in fact, a direct summand) of the following induced representation I: Let $\bar{s} = \sum_{i=1}^{b} |k_i - l_i|$, $a = \sum_{i=1}^{b} min\{k_i, l_i\}$, $P = MAN$ with

$$MA \cong Sp(2\bar{s}, \mathbb{R}) \times GL(2, \mathbb{R})^a \times GL(1, \mathbb{R})^z, \tag{5.33}$$

ρ_0 the discrete series representation of $Sp(2\bar{s}, \mathbb{R})$ whose Harish-Chandra parameter contains, for each i with $k_i \neq l_i$, the entry a_i if $k_i > l_i$ and $-a_i$ otherwise, τ the representation of $GL(2, \mathbb{R})^a$ with $min\{k_i, l_i\}$ factors of $\tau(2a_i, 0)$ for each $1 \leq i \leq b$, and χ the character of $GL(1, \mathbb{R})^z$ given by $\chi = \bigotimes_{i=1}^{z} \chi_{(-1)^{\bar{s}+i}, 0}$. Then

$$I = Ind_P^{Sp(2n, \mathbb{R})}(\rho_0 \otimes \tau \otimes \chi \otimes \mathbb{1}). \tag{5.34}$$

Let $\bar{p}$ be the number of indices $i \leq b$ such that $k_i > l_i$, $\bar{q}$ the number of indices i such that $k_i < l_i$, $c = p - 2\bar{p} - 2a = q - 2\bar{q} - 2a$, $P' = M'A'N'$ a parabolic subgroup of $O(p, q)$ with

$$M'A' \cong O(2\bar{p}, 2\bar{q}) \times GL(2,\mathbb{R})^a \times GL(1,\mathbb{R})^c, \tag{5.35}$$

ρ_0' the discrete series representation of $O(2\bar{p}, 2\bar{q})$ whose Harish-Chandra parameter contains, for each i such that $k_i \neq l_i$, the entry a_i, on the left if $k_i > l_i$ and on the right otherwise, and χ' the character of $GL(1,\mathbb{R})^c$ given by $\chi' = \bigotimes_{i=1}^c \chi_{(-1)^i,0}$. (Notice that c is even.) Then Moeglin's map Φ_{p_0,q_0} assigns to ρ the constituent (summand) of

$$I' = Ind_{P'}^{O(p,q)}(\rho_0' \otimes \tau \otimes \chi' \otimes \mathbb{1}) \tag{5.36}$$

that contains the K-type corresponding to Λ in $\mathcal{H}$. This is easily seen to be the limit of discrete series representation π'.

This proves the lemma for the case $\frac{p-q}{2} = k - l$.

Now suppose $\left|\frac{p-q}{2} + l - k\right| = 1$. We analyze the case $\frac{p-q}{2} = k-l-1$ only, the case $\frac{p-q}{2} = k-l+1$ being completely analogous. Rewrite

$$\begin{aligned}\lambda_a + \rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{k}) &= (k-l-1.k-l-1.\dots,k-l-1)\\ &+ (\underbrace{a_1+\alpha_1+\tfrac{3}{2},\dots,a_1+\alpha_1+\tfrac{3}{2}}_{k_1},\dots,\underbrace{a_b+\alpha_b+\tfrac{3}{2},\dots,a_b+\alpha_b+\tfrac{3}{2}}_{k_b},\underbrace{1,\dots,1}_{z},\\ &\underbrace{-a_b+\alpha_b+\tfrac{1}{2},\dots,-a_b+\alpha_b+\tfrac{1}{2}}_{l_b},\dots,\underbrace{-a_1+\alpha_1+\tfrac{1}{2},\dots,-a_1+\alpha_1+\tfrac{1}{2}}_{l_1}),\end{aligned} \tag{5.37}$$

with α_i as in (5.30). The parameter λ_a' will be as in (5.31) with $e = d+1$, and

$$\begin{aligned}&\lambda_a' + \rho(\mathfrak{u}'\cap\mathfrak{p}') - \rho(\mathfrak{u}'\cap\mathfrak{k}')\\ &= (\underbrace{a_1+\alpha_1+\tfrac{3}{2},\dots,a_1+\alpha_1+\tfrac{3}{2}}_{k_1},\dots,\underbrace{a_b+\alpha_b+\tfrac{3}{2},\dots,a_b+\alpha_b+\tfrac{3}{2}}_{k_b},\underbrace{0,\dots,0}_{d};\\ &\underbrace{a_1-\alpha_1-\tfrac{1}{2},\dots,a_1-\alpha_1-\tfrac{1}{2}}_{l_1},\dots,\underbrace{a_b-\alpha_b-\tfrac{1}{2},\dots,a_b-\alpha_b-\tfrac{1}{2}}_{l_b},\underbrace{0,\dots,0}_{e}).\end{aligned} \tag{5.38}$$

It is easy to see that, just as in the previous case, the root systems for ρ and π' are related in such a way that if we write the fine K-type δ_L for $Sp(2n,\mathbb{R})$ and $\delta_{L'}$ for $O(p,q)$ as above, then we have $\delta_i = \delta_i'$ for $1 \leq i \leq b$. In order to check that the LKT's of ρ and π' correspond, we must therefore only look at the entries $\eta = (\eta_1,\dots,\eta_z)$ of δ_L and $\eta' = (\eta_1',\dots,\eta_d';\xi_1',\dots,\xi_e')$ of $\delta_{L'}$.

First we look at the case where z is odd (this is case (4) of the theorem) and $-e_{k+1}-e_{k+z} \in \Psi$. If $p+q = 2n$ then $d = \frac{z-1}{2}$, $e = \frac{z+1}{2}$, $p = 2k+z-1$, and $q = 2l+z+1$, both even. According to Proposition 3.13 we have $\eta = (\underbrace{0,\dots,0}_{d},\underbrace{-1,\dots,-1}_{e})$, so that the corresponding z entries of Λ are $(k-l-1,\dots,k-l-1) + (\underbrace{1,\dots,1}_{d},\underbrace{0,\dots,0}_{e})$. By Proposition 3.36, the LKT of the limit of discrete series π' has $\eta' = (\underbrace{1,\dots,1}_{d};\underbrace{0,\dots,0}_{e})$ and signs $(1;1)$, so the two correspond. The condition (†) is again straightforward. To check $(*)$, notice that Moeglin's $\sigma = 1$ so that we need to make sure that ρ is a constituent of an induced representation $Ind_{M_1A_1N_1}^{Sp(2n,\mathbb{R})}(\pi_1 \otimes \chi_{(-1)^{\frac{p-q}{2}}} \otimes \mathbb{1})$ with $M_1A_1 \cong Sp(2n-2,\mathbb{R}) \times GL(1,\mathbb{R})$, and π_1 a representation of $Sp(2n-2,\mathbb{R})$. But ρ is a constituent of I in (5.34), and since $z > 0$ and $(-1)^{\frac{p-q}{2}} = (-1)^{k+l+1} = (-1)^{\bar{s}+1}$, $(*)$ follows by induction in stages. This time Φ_{p_0,q_0} maps ρ to the constituent containing the K-type corresponding to Λ in $\mathcal{H}$ of an induced representation I' as in (5.36) with

$$M'A' \cong O(2\bar{p}, 2\bar{q}+2) \times GL(2,\mathbb{R})^a \times GL(1,\mathbb{R})^{z-1} \tag{5.39}$$

with a as defined above, ρ_0' the discrete series whose Harish-Chandra is obtained from that above by adding a zero on the right, and $\chi = \bigotimes_{i=1}^{z-1} \chi_{(-1)^i,0}$. This representation is indeed π'.

Next assume that $p+q = 2n+2$ (still with z odd and $-e_{k+1} - e_{k+z} \in \Psi$). Then $p = 2k+z$ and $q = 2l+z+2$ are both odd. The LKT of π has the same highest weight as in the previous case, with signs $(1;1)$, so it corresponds to Λ.

Now assume $z = 2w > 0$, $-e_{k+1} - e_{k+z} \in \Psi$, and $p+q = 2n$. Then $p = 2k+z-1$ and $q = 2l+z+1$ are odd so that we only have to check (1) of the lemma. In this case, $\eta = (\underbrace{0,\ldots,0}_{w}, \underbrace{-1,\ldots,-1}_{w})$, so that the corresponding z entries of Λ are $(k-l-1,\ldots,k-l-1) + (\underbrace{1,\ldots,1}_{w}, \underbrace{0,\ldots,0}_{w})$. We have $d = w-1$, $e = w$, and $\eta' = (\underbrace{1,\ldots,1}_{w-1}; \underbrace{0,\ldots,0}_{w})$, and the signs of the LKT of π (according to Proposition 3.43) are $(-1;1)$. It follows (using Proposition 2.10) that the LKT's correspond.

The case $z = 2w$, $p+q = 2n+2$, and $-e_{k+1} - e_{k+z} \in \Psi$ if $w > 0$, is similar to the case z odd and $p+q = 2n$. We omit the details. □

Proof of Theorem 4.1. In light of Lemma 5.28 and Theorem 5.3, it only remains to prove the cases where p and q are odd. We first consider the cases $p+q = 2n+2$, i. e., the cases $(p,q) = (2p_1+1, 2q_1+1)$ in part (1) and $(p,q) = (2p_1+3, 2q_1+1)$ or $(2p_1+1, 2q_1+3)$ in case (4) of the theorem. Notice that in this situation we have by Lemma 5.28 that $\theta_{p-1,q-1}(\rho) \neq 0$ and is a limit of discrete series ρ' of $O(p-1,q-1)$ (with $p-1$ and $q-1$ even). Consequently, we can use Theorem 5.13 (2) with $s = t = 0$ and $m = 1$ to conclude that $\theta_{p,q}(\rho)$ is a constituent of

$$Ind_P^{O(p,q)}(\rho' \otimes \chi_{1,0} \otimes \mathbb{1}) \tag{5.40}$$

for some parabolic subgroup $P = MAN$ of $O(p,q)$ with Levi factor $MA \cong O(p-1,q-1) \times GL(1,\mathbb{R})$. If we write $\rho' = \rho(\lambda_d', 1, \Psi')$ then this (tempered) induced representation is either irreducible or consists of two summands $\pi_\zeta(\lambda_d', \Psi', 1, 0, 0, (1), (0))$ with $\zeta = \pm 1$ (if z=0). Since the LKT Λ of ρ satisfies (†) for $(p-1, q-1)$ and the degree of a K-type depends on the difference between p and q only, we know by Lemma 5.28 that Λ is of minimal degree in ρ. Consequently, ρ corresponds to the constituent of the induced (5.40) containing the K-type corresponding to Λ in $\mathcal{H}$, which (by Lemma 5.28) is a LKT of $\pi_1(\lambda_d', \Psi', 1, 0, 0, (1), (0))$ and has multiplicity one in the induced representation.

Now we look at the cases where $p+q = 2n$. First assume that $z = 2w > 0$ is even, that $e_{k+1} + e_{k+z} \in \Psi$ (the case $-e_{k+1} - e_{k+z} \in \Psi$ is analogous), and we want to show that $\theta_{2p_1+1, 2q_1-1}(\rho) = \pi_1(\lambda_{1,-1}, 1, \Psi_{1,-1}, 0, 0, (-1), (0))$ as in part (2) of the theorem. Let $\rho_0 = \rho((\lambda_d)_0, \Psi_0)$ be the limit of discrete series representation of $Sp(2n-2,\mathbb{R})$ with Harish-Chandra parameter $(\lambda_d)_0$ obtained from λ_d by removing the last (z^{th}) zero, and Ψ_0 obtained from Ψ in the "natural" way, i. e., by removing all roots of the form $\pm e_{k+z} \pm e_j$ or $\pm 2e_{k+z}$, and then subtracting 1 from each subscript greater than $k+z$. Let $\rho' = \rho(\lambda_{1,-1}, 1, \Psi_{1,-1})$, a limit of discrete series of $O(2p_1, 2q_1-2)$. By Lemma 5.28, we know that $\rho' = \theta_{2p_1, 2q_1-2}(\rho_0)$. Since $z \geq 2$ so that $(\lambda_d)_0$ contains at least one zero (see the comments in Example 5.25), we have $\rho \cong Ind_P^{Sp(2n,\mathbb{R})}(\rho_0 \otimes \chi_{(-1)^{k+l}} \otimes \mathbb{1})$, where $P = MAN$ is a parabolic subgroup of $Sp(2n,\mathbb{R})$ with Levi factor $MA \cong Sp(2n-2,\mathbb{R}) \times GL(1,\mathbb{R})$ (see [9]). Notice that $\frac{(2p_1+1)-(2q_1-1)}{2} = p_1 - q_1 + 1$, and $(-1)^{p_1-q_1+1} = (-1)^{k+l+1}$. So by Theorem 5.13, $\theta_{2p_1+1,2q_1-1}(\rho)$ is a constituent (summand) of $Ind_{P'}^{O(2p_1+1,2q_1-1)}(\rho' \otimes \chi_{-1,0} \otimes \mathbb{1})$ for some parabolic subgroup $P' = M'A'N'$ of $O(2p_1+1, 2q_1-1)$ with Levi factor $M'A' \cong O(2p_1, 2q_1-2) \times GL(1,\mathbb{R})$.

As above we know that Λ is of minimal degree in ρ, so that the theta lift of ρ must contain the K-type corresponding to Λ in $\mathcal{H}$, which by Lemma 5.28 is a LKT of $\pi_1(\lambda_{1,-1}, 1, \Psi_{1,-1}, 0, 0, (-1), (0))$ (a constituent of the induced). Since this K-type has multiplicity one in the induced, we must have $\theta_{2p_1+1,2q_1-1}(\rho) = \pi_1(\lambda_{1,-1}, 1, \Psi_{1,-1}, 0, 0, (-1), (0))$.

Now suppose $z = 2w + 1$ is odd. We need to prove the first statement of (4), i. e., that $\theta_{2p_1+1,2q_1+1}(\rho) = \pi_\zeta(\lambda_{0,0}, 1, \Psi_{0,0}, 0, 0, (-1), (0))$, with ζ, $\lambda_{0,0}$, and $\Psi_{0,0}$ as described in the theorem. Let π' be this representation of $O(2p_1 + 1, 2q_1 + 1)$. By Lemma 5.28, we know that ρ satisfies (†) for $p = 2p_1 + 2$ and $q = 2q_1 + 2$, so that we know the LKT is of minimal degree for the dual pair $\big(Sp(2n, \mathbb{R}), O(2p_1 + 1, 2q_1 + 1)\big)$ as well. Let ρ_0 be the limit of discrete series representation of $Sp(2n - 2, \mathbb{R})$ that corresponds to $\rho' = \rho(\lambda_{0,0}, 1, \Psi_{0,0})$ in the correspondence for the dual pair $\big(Sp(2n - 2, \mathbb{R}), O(2p_1, 2q_1)\big)$; its Harish-Chandra parameter is obtained from λ_d by removing the middle zero, and the positive root system is obtained from Ψ in an obvious way. Then ρ is a summand of an induced representation $I = Ind_P^{Sp(2n,\mathbb{R})}(\rho_0 \otimes \chi_{(-1)^n,0} \otimes \mathbb{1})$ for some parabolic subgroup $P = MAN$ of $Sp(2n, \mathbb{R})$ with Levi factor $MA \cong Sp(2n - 2, \mathbb{R}) \times GL(1, \mathbb{R})$, and π' is a summand of $I' = Ind_{P'}^{O(2p_1+1,2q_1+1)}(\rho' \otimes \chi_{-1,0} \otimes \mathbb{1})$ with $P' = M'A'N'$ a parabolic subgroup of $O(2p_1 + 1, 2q_1 + 1)$ with Levi factor $M'A' \cong O(2p_1, 2q_1) \times GL(1, \mathbb{R})$. Since by Lemma 5.28 the LKT of ρ corresponds to a LKT of π', ρ corresponds to π' by Theorem 5.13, Proposition 5.22, and Lemma 5.23. □

Lemma 5.41. *In the setting of Theorem 4.8, let $\pi' = \pi_\zeta(\lambda'_d, 1, \Psi', \mu, \nu, \epsilon', \kappa')$, the proposed theta lift of π.*

1. *The representation π has a LKT of minimal degree in π.*
2. *If $p + q = 2n$ then every LKT of π is of minimal degree in π.*
3. *Each LKT of π that is of minimal degree in π corresponds to a LKT of π' in the space of joint harmonics.*
4. *If $p + q = 2n + 2$ then for every highest weight Λ_0 of a LKT of π' (as in Proposition 3.36) there is LKT Λ' of π' with highest weight Λ_0 such that Λ' corresponds in $\mathcal{H}$ to a LKT (of minimal degree) of π.*

Proof. Let π be as above, with λ_d (and a_i, k_i, l_i, k, l, z, etc.) as in (3.1), and the associated λ_a (and α_i, u_i, r_i, w, u, etc.) as in (3.10). Let Λ be a LKT of π, $\Lambda = \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L$ as in (3.12). Write

$$\begin{aligned}(5.42)\quad \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) &= (u - r, u - r. \dots, u - r) \\ &+ (\underbrace{\alpha_1 + \omega_1 + \tfrac{1}{2}, \dots, \alpha_1 + \omega_1 + \tfrac{1}{2}}_{u_1}, \dots, \underbrace{\alpha_m + \omega_m + \tfrac{1}{2}, \dots, \alpha_m + \omega_m + \tfrac{1}{2}}_{u_m}, \underbrace{0, \dots, 0}_{w}, \\ &\quad \underbrace{-\alpha_m + \omega_m - \tfrac{1}{2}, \dots, -\alpha_m + \omega_m - \tfrac{1}{2}}_{r_m}, \dots, \underbrace{-\alpha_1 + \omega_1 - \tfrac{1}{2}, \dots, -\alpha_1 + \omega_1 - \tfrac{1}{2}}_{r_1}),\end{aligned}$$

where for $1 \le i \le m$,

$$(5.43)\qquad \omega_i = \frac{r_i - u_i}{2} + \sum_{j=i+1}^{m} (r_j - u_j).$$

The K-type Λ is then obtained by adding a fine K-type δ_L as in (3.15), so that $\Lambda - (u - r, u - r. \dots, u - r)$ is a weight with the first u entries integers ≥ 1, the last r entries integers ≤ -1, and with w entries $\eta = (\eta_1, \eta_2, \dots, \eta_w)$ in between, of the form

$$\eta = (\underbrace{1, \ldots, 1}_{\xi}, 0, \ldots, 0) \quad \text{or} \tag{5.44}$$

$$\eta = (0, \ldots, 0, \underbrace{-1, \ldots, -1}_{\xi}) \tag{5.45}$$

for some ξ.

To check that Λ is of minimal degree in π for (p, q) we use Theorem 5.3; it is sufficient to check that (π, Λ) satisfies (†) for some (p', q') with $p' - q' = p - q$ since the degree only depends on this difference.

Assume first that $z = 0$, i. e., that λ_d does not have any zeros. Then both (5.44) and (5.45) occur (see Proposition 3.13), and we need to consider $(p, q) = (2u + w, 2r + w)$, $(2u + w + 1, 2r + w + 1)$, $(2u + w + 2, 2r + w)$, and $(2u + w, 2r + w + 2)$. It is easy to check that (π, Λ) satisfies (†) for either $(p, q) = (2u+w, 2r+w)$ or $(2u+w+1, 2r+w+1)$ (whichever is a pair of even integers). If Λ has (5.44) then (π, Λ) satisfies (†) for $(p, q) = (2u+w+2, 2r+w)$ if w is even, and for $(p, q) = (2u+w+3, 2r+w+1)$ otherwise, since then $p_0 - q_0 = u - r + 1$ so that we can write Λ of the form

$$\Lambda = (p_0 - q_0, p_0 - q_0, \ldots, p_0 - q_0) + (d_1, \ldots, d_u, \underbrace{0, \ldots, 0}_{\xi}, \underbrace{-1, \ldots, -1}_{w-\xi}, d_{u+w+1}, \ldots, d_{u+w+r}) \tag{5.46}$$

with $d_u \geq 0$ and $d_{u+w+1} \leq -2$ and check conditions (β) and (γ) of Definition 5.1 easily. Similarly, we have that if Λ has (5.45) then (π, Λ) satisfies (†) for $(p, q) = (2u + w, 2r + w + 2)$ if w is even, and for $(p, q) = (2u + w + 1, 2r + w + 3)$ otherwise. So we have (1) and (2) of the lemma for this case.

Now assume that $z > 0$. Then by Proposition 3.13, either all LKT's of π have (5.44), or all LKT's have (5.45), depending on the root system Ψ. In the first case, we need to consider $(p, q) = (2u+w, 2r+w)$, $(2u+w+1, 2r+w-1)$, $(2u+w+1, 2r+w+1)$, and $(2u+w+2, 2r+w)$, in the second $(p, q) = (2u+w, 2r+w)$, $(2u+w-1, 2r+w+1)$, $(2u+w+1, 2r+w+1)$, and $(2u+w, 2r+w+2)$. It is easy th see that in either case, (π, Λ) satisfies (†) for $(p, q) = (2u+w, 2r+w)$ or $(2u+w+1, 2r+w+1)$ (depending on the parity of w), and with (5.44), (π, Λ) satisfies (†) for $(p, q) = (2u+w+1, 2r+w-1)$ or $(2u + w + 2, 2r + w)$. Similarly if Λ has (5.45). So in fact, if $z > 0$ then all LKT's of π are of minimal degree in π, for $p + q = 2n$ and $2n + 2$.

It remains to show that the LKT's of π that are of minimal degree in π correspond in $\mathcal{H}$ to LKT's of π', and that whenever $p + q = 2n + 2$ then, up to signs, all LKT's of π' occur this way. We display some details of this calculation for one case only; the remaining cases will be very similar.

Assume that z is odd, and that $-e_{k+1} - e_{k+z} \in \Psi$. Then $\zeta = 1$, and we must consider $(p, q) = (2u + w, 2r + w)$, $(2u + w + 1, 2r + w + 1)$, $(2u + w - 1, 2r + w + 1)$, and $(2u + w, 2r + w + 2)$. As in the proof of Lemma 5.28, we use unprimed letters for parameters associated to π and $Sp(2n, \mathbb{R})$, and primed ones (λ'_a, etc.) for those associated to π' and $O(p, q)$.

We start with $(p, q) = (2u + w, 2r + w)$. Using the theory described in §3, we have that if λ_a is of the form (3.10), then λ'_a is of the form (3.34) with $x = y = [\frac{w}{2}]$. Here $w = z + 2b + t$, where b is the number of indices $i \leq s$ such that $\mu_i = 0$. We have $u - r = \frac{p-q}{2}$, and we write $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ as in (5.42). Similarly, we have

$$(5.47)\quad \lambda'_a + \rho(\mathfrak{u}'\cap\mathfrak{p}') - \rho(\mathfrak{u}'\cap\mathfrak{k}') = (\underbrace{\alpha_1+\omega_1+\tfrac{1}{2},\dots,\alpha_1+\omega_1+\tfrac{1}{2}}_{u_1},\dots,\underbrace{\alpha_m+\omega_m+\tfrac{1}{2},\dots,\alpha_m+\omega_m+\tfrac{1}{2}}_{u_m},\underbrace{0,\dots,0}_{[\frac{w}{2}]}; \underbrace{\alpha_1-\omega_1+\tfrac{1}{2},\dots,\alpha_1-\omega_1+\tfrac{1}{2}}_{r_1},\dots,\underbrace{\alpha_m-\omega_m+\tfrac{1}{2},\dots,\alpha_m-\omega_m+\tfrac{1}{2}}_{r_m},\underbrace{0,\dots,0}_{[\frac{w}{2}]})$$

with ω_i as in (5.43). Write the fine K-types δ_L and $\delta_{L'}$ that we have to add to (5.42) and (5.47) to obtain the highest weights of LKT's of π and π', respectively, as in (3.15) and (3.38) (the latter with primed entries). The root systems Ψ and Ψ' are related in such a way that by Propositions 3.13 and 3.36, the choices for the δ_i's in δ_L correspond exactly to the choices for the δ_i''s in $\delta_{L'}$. Since $z>0$, $\eta=(\eta_1,\dots,\eta_w)$ is uniquely determined, and when $z\geq 3$, so is $\eta'=(\eta_1',\dots,\eta'_{[\frac{w}{2}]};\xi_1',\dots,\xi'_{[\frac{w}{2}]})$, so that the number of LKT's of π equals the number of distinct highest weights of LKT's of π'. (When $z=1$ the parameter λ_d' does not contain any zeros, so there may be two choices for η'.) We need to check that η and η' are such that they are consistent with the correspondence in the space of joint harmonics. Since $-e_{k+1}-e_{k+z}\in\Psi$,

$$(5.48)\qquad \eta=(0,\dots,0,\underbrace{-1,\dots,-1}_{h}).$$

Here $h=\frac{z+1}{2}+b+a$, where a is the number of indices $i\geq t$ such that $\epsilon_i=(-1)^{u+r+1}=(-1)^{\frac{p-q}{2}+1}$. Write

$$(5.49)\qquad \epsilon=(\underbrace{(-1)^{\frac{p-q}{2}+1},\dots,(-1)^{\frac{p-q}{2}+1}}_{a},\underbrace{(-1)^{\frac{p-q}{2}},\dots,(-1)^{\frac{p-q}{2}}}_{t-a}).$$

Then

$$(5.50)\qquad \epsilon'=(\underbrace{-1,\dots,-1}_{a+1},\underbrace{1,\dots,1}_{t-a}).$$

(Recall that by Theorem 4.1 (4), $t'=t+1$ since z is odd.) We have

$$(5.51)\qquad \eta'=(0,\dots,0;\underbrace{1,\dots,1}_{\xi},0,\dots,0)$$

as one of the choices (the only one if $z\geq 3$). If $a+1\leq t-a$ then $\xi=\frac{z-1}{2}+b+a+1=h$ and the signs may be taken to be $(1;1)$ (see Prop. 3.43). In this case, (5.48) and (5.51) indeed match up so that the LKT's correspond in $\mathcal{H}$.

If $t-a>a+1$ then $\xi=\frac{z-1}{2}+b+t-a$ and the signs may be chosen to be $(1;-1)$. The K-types match up in $\mathcal{H}$ provided that $h=\xi+q-2(r+\xi)$ (see Proposition 2.10). But

$$(5.52)\qquad \xi+q-2(r+\xi)=q-2r-\xi=z+2b+t-\tfrac{z-1}{2}-b-t+a=\tfrac{z+1}{2}+b+a=h,$$

so we are done for the case $(p,q)=(2u+w,2r+w)$.

For $(p,q)=(2u+w+1,2r+w+1)$, the calculation is very similar; η' now has $[\frac{w+1}{2}]$ entries on each side, ϵ' has only a entries of -1, and $\xi=\frac{z+1}{2}+b+a=h$ or $\frac{z+1}{2}+b+t-a$ (with signs as before), and the K-types match up as before. In addition, λ_d' now always contains zeros, so that only LKT's with η' satisfying (5.51) occur. Therefore, up to signs, all LKT's of π' actually correspond to LKT's of π.

Now assume $(p,q)=(2u+w-1,2r+w+1)$. Then if λ_a is of the form (3.10), then λ'_a will be of the form (3.34) with $x=[\frac{w-1}{2}]$ and $y=[\frac{w+1}{2}]=[\frac{w-1}{2}]+1$. Since $\frac{p-q}{2}=u-r-1$, we write $\lambda_a+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})$ in the form

$$\begin{aligned}(5.53)\quad \lambda_a+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k}) &= (u-r-1,u-r-1.\dots,u-r-1)\\ &+(\underbrace{\alpha_1+\omega_1+\tfrac{3}{2},\dots,\alpha_1+\omega_1+\tfrac{3}{2}}_{u_1},\dots,\underbrace{\alpha_m+\omega_m+\tfrac{3}{2},\dots,\alpha_m+\omega_m+\tfrac{3}{2}}_{u_m},\underbrace{1,\dots,1}_{w},\\ &\underbrace{-\alpha_m+\omega_m+\tfrac{1}{2},\dots,-\alpha_m+\omega_m+\tfrac{1}{2}}_{r_m},\dots,\underbrace{-\alpha_1+\omega_1+\tfrac{1}{2},\dots,-\alpha_1+\omega_1+\tfrac{1}{2}}_{r_1}),\end{aligned}$$

where the ω_i are given by (5.43). We then have

$$\begin{aligned}(5.54)\quad &\lambda'_a+\rho(\mathfrak{u}'\cap\mathfrak{p}')-\rho(\mathfrak{u}'\cap\mathfrak{k}')\\ &=(\underbrace{\alpha_1+\omega_1+\tfrac{3}{2},\dots,\alpha_1+\omega_1+\tfrac{3}{2}}_{u_1},\dots,\underbrace{\alpha_m+\omega_m+\tfrac{3}{2},\dots,\alpha_m+\omega_m+\tfrac{3}{2}}_{u_m},\underbrace{0,\dots,0}_{[\frac{w-1}{2}]};\\ &\underbrace{\alpha_1-\omega_1-\tfrac{1}{2},\dots,\alpha_1-\omega_1-\tfrac{1}{2}}_{r_1},\dots,\underbrace{\alpha_m-\omega_m-\tfrac{1}{2},\dots,\alpha_m-\omega_m-\tfrac{1}{2}}_{r_m},\underbrace{0,\dots,0}_{[\frac{w+1}{2}]}).\end{aligned}$$

As in the previous cases, Ψ and Ψ' are related in such a way that for the entries in δ_L and $\delta_{L'}$, for each i, the choices for δ_i are the same as the choices for δ'_i. So we need to check that $\eta+1=(\eta_1+1,\eta_2+1,\dots,\eta_w+1)$ and $\eta'=(\eta'_1,\dots,\eta'_{[\frac{w-1}{2}]};\xi'_1,\dots,\xi'_{[\frac{w+1}{2}]})$ match up correctly. Since this time λ'_d contains a zero, there is only choice each for η and η'. If ϵ is as in (5.49), then (now with $(-1)^{\frac{p-q}{2}+1}=(-1)^{u+r}$)

$$(5.55)\qquad \epsilon'=(\underbrace{-1,\dots,-1}_{a},\underbrace{1,\dots,1}_{t-a}).$$

Let $h=\frac{z+1}{2}+b+t-a$. Then

$$(5.56)\qquad \eta+1=(\underbrace{1,\dots,1}_{w-h},\underbrace{0,\dots,0}_{h}).$$

Write

$$(5.57)\qquad \eta'=(\underbrace{1,\dots,1}_{\xi},0,\dots,0;,0,\dots,0).$$

If $a\le t-a$ then $\xi=\frac{z-1}{2}+b+a=w-h$ and the signs may be chosen to be $(1;1)$, so $\eta+1$ and η' do indeed match up. If $t-a<a$ then $\xi=\frac{z-1}{2}+b+t-a$, and the signs may be chosen to be $(1;-1)$. So $\eta+1$ and η' match up if $w-h=\xi+p-2(u+\xi)$. But

$$(5.58)\quad \xi+p-2(u+\xi)=p-2u-\xi=w-1-\tfrac{z-1}{2}-b-t+a=w-(\tfrac{z+1}{2}+b+t-a)=w-h,$$

so we are done for this case. The case $(p,q)=(2u+w,2r+w+2)$ is easily obtained from this last case very much like the case $(p,q)=(2u+w+1,2r+w+1)$ was obtained from $(p,q)=(2u+w,2r+w)$.

As mentioned above, the calculations for z even are very similar. In this case, it turns out that (4) of the lemma holds even for $p+q=2n$. As before, note that whether π and π' have (up to signs) the same number of LKT's depends on whether there are one or two choices for η and η'. If $z=0$ then there are two choices for η (provided that $h>0$). In that case, there are three possibilities: there are two choices for η', there is only one choice for η' but only half the LKT's of π are of minimal degree, or $\eta'=0$. In this last case pairs of LKT's of π with different η will correspond to pairs of LKT's of π' with the same highest weight but different signs. If $z>0$ and even, then there is always only one choice for η and η', so that the numbers of LKT's (up to signs) match, and one can check that they correspond in $\mathcal{H}$.

□

Proof of Theorem 4.8. Let π' denote the proposed theta lift of π, and $\rho'=\theta_{p_1,q_1}(\rho)$. Let $P=MAN$ be a parabolic subgroup of $Sp(2n,\mathbb{R})$ with Levi factor $MA\cong Sp(2v,\mathbb{R})\times GL(2,\mathbb{R})^s\times GL(1,\mathbb{R})^t$, imbedded as in Theorem 5.13(2), $\tau=\tau(\mu,\nu)$, a representation of $GL(2,\mathbb{R})^s$, and $\chi=\chi(\epsilon,\kappa)$, a character of $GL(1,\mathbb{R})^t$. Let $I=Ind_P^{Sp(2n,\mathbb{R})}(\rho\otimes\tau\otimes\chi\otimes\mathbb{1})$. Then π is the unique LKT constituent of I. By replacing some of the ν_i and some of the κ_i, if necessary, by their negatives, we can arrange that π is actually a quotient of I. Let $P'=M'A'N'$ be a parabolic subgroup of $O(p,q)$ with Levi factor $M'A'\cong O(2p_1,2q_1)\times GL(2,\mathbb{R})^s\times GL(1,\mathbb{R})^t$, imbedded as in Theorem 5.13(2). Then π' is a LKT constituent of $I'=Ind_{P'}^{O(p,q)}(\rho'\otimes\tau^*\otimes\chi^*\otimes\chi_{p,q}\otimes\mathbb{1})$; note that replacing τ by τ^* and χ by χ^* in the induced does not change its composition series. By Theorem 5.13, there is a nonzero $(G\times G')$-map $\phi:\omega\longrightarrow I\otimes I'$. By Lemma 5.41, π has a LKT Λ that is of minimal degree in π. In fact, Λ is of minimal degree in I since the K-structure of I does not depend on the continuous parameter, and Λ is of minimal degree in I if ν and κ are generic so that I is irreducible. By Proposition 5.12, the unique LKT σ_0 of τ is of minimal degree in τ, and of course, χ has only one K-type δ. Moreover, by Lemma 5.28, the unique LKT Λ_0 of ρ is of minimal degree in ρ. So $\Lambda_0\otimes\sigma_0\otimes\delta$ is the unique LKT of $\rho\otimes\tau\otimes\chi$ and of minimal degree in that representation. Consequently, the hypotheses of Proposition 5.22 and Lemma 5.23 are satisfied for Λ and $\Lambda_0\otimes\sigma_0\otimes\delta$, so that π occurs in the correspondence and lifts to the constituent of I' containing the K-type which corresponds in $\mathcal{H}$ to Λ. By Lemma 5.41, this is a LKT of π', and hence a LKT of I'. Since the LKT's of I' have multiplicity one, $\theta_{p,q}(\pi)=\pi'$.

□

Proof of Proposition 3.43. For limits of discrete series, the signs are clear, determined by ξ (as described before Example 3.22). Recall that the parameter ζ only has meaning if $z+z'=0$ and $\kappa_i=0$ for some i, and ξ only has meaning if $z+z'>0$. For (1), note that we have $\pi\otimes det\cong\pi$, so if Λ_0 occurs with signs $(\eta_1;\eta_2)$ then it will occur with signs $(-\eta_1;-\eta_2)$ as well. In (2) through (5), however, we have that if Λ_0 occurs with signs $(\eta_1;\eta_2)$ in π then it will not occur with signs $(-\eta_1;-\eta_2)$; this will be a LKT of $\pi\otimes det$ which is the representation obtained from π by replacing ζ by $-\zeta$ if we are in case (2) or (3), and by replacing ξ by $-\xi$ for (4) and (5). Consequently, we only have to prove these statements for one choice of ζ or ξ each. We choose $\xi=1$ and ζ such that $\pi_0=\theta_{\frac{p+q}{2}}(\pi)\neq 0$ according to Theorem 4.8 or Theorem 4.1. (Since we are assuming $2n=p+q$ and π is not a limit of discrete series, the only case from Theorem 4.1 we need to consider is the first case in (4), where p and q are odd and $\pi=\pi_\zeta(\lambda_{0,0},1,\Psi_{0,0},0,0,(-1),(0))$.) The proof of the theorems is set up so that by the induction principle, π_0 must correspond to a constituent of some induced representation I' which has π either as the unique LKT constituent, or as one of only two LKT constituents which have the same parameters except for ζ. Lemmas 5.28 and 5.41 then show that the LKT's of π_0 that are of minimal degree indeed correspond to LKT's of I', so the lift must be one of the LKT constituents. Moreover, the correspondence in $\mathcal{H}$ gives the signs of the LKT's of π which occur that way. When for some highest weight as in Proposition 3.36, no K-type with this highest weight occurs in $\mathcal{H}$, then π must have a LKT Λ with that highest weight, but with signs such that the degree of Λ is not minimal. For instance, in case (3) with $\beta>\gamma$, the K-type with signs $(-1;-1)$ will typically not be of minimal degree. If the representation obtained from π by replacing ζ by $-\zeta$

lifts to $Sp(p+q,\mathbb{R})$, then this representation will have a LKT with the same highest weight but signs $(1;1)$, and it will be of minimal degree. Using such arguments and the calculations in Lemmas 5.28 and 5.41, we get the signs as described in the Proposition.

□

6. Some Consequences

In this section we list two results for symplectic-orthogonal dual pairs of unequal sizes that follow easily from the work in the previous sections.

Corollary 6.1. *Let n, p, and q be non-negative integers such that $p+q$ is even, and let π and π' be irreducible admissible representations of $Sp(2n,\mathbb{R})$ and $O(p,q)$, respectively, which correspond in the Howe correspondence for the dual pair $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$.*

(1) *If $2n \geq p+q$ then every LKT of π is of minimal degree in π.*
(2) *If $p+q \geq 2n+2$ then for every highest weight Λ_0 of a LKT of π' (as in Proposition 3.36) there is a LKT Λ of π' with highest weight Λ_0 such that Λ is of minimal degree in π'.*

Proof. For $p+q=2n$, part (1) this follows from Lemma 5.41. If $2n>p+q$ then let $k=\frac{1}{2}(2n-p-q)$. By Theorem 5.13 we know that $\theta_{p+k,q+k}(\pi)\neq 0$. (This is Kudla's Persistence Principle.) Thus the LKT's of π are of minimal degree for the dual pair $\big(Sp(2n,\mathbb{R}),O(p+k,q+k)\big)$. Since the degree of a K-type for $Sp(2n,\mathbb{R})$ depends on the difference $(p+k)-(q+k)=p-q$ only, the LKT's of π are of minimal degree for the original dual pair as well. The argument for (2) is analogous. □

Theorem 6.2. (1) *Let p, q, and n be non-negative integers such that $p+q=2n$. Let π' be an irreducible admissible representation of $O(p,q)$, and suppose that $\theta_n(\pi')=\pi(\lambda_d,\Psi,\mu,\nu,\epsilon,\kappa)$. If k is a positive integer, write $\epsilon^{(k)}=(\underbrace{(-1)^{\frac{p-q}{2}},\dots,(-1)^{\frac{p-q}{2}}}_{k})$ and $\kappa^{(k)}=(1,2,\dots,k)$. Then*

$$\theta_{n+k}(\pi')=\pi_k=\pi(\lambda_d,\Psi,\mu,\nu,(\epsilon|\epsilon^{(k)}),(\kappa|\kappa^{(k)})). \tag{6.3}$$

(2) *Let p, q, and n be non-negative integers such that $p+q=2n+2$. Let π be an irreducible admissible representation of $Sp(2n,\mathbb{R})$, and suppose that $\theta_{p,q}(\pi)=\pi_1(\lambda_d,\xi,\Psi,\mu,\nu,\epsilon,\kappa)$. (Notice that $\zeta=1$ whenever $p+q=2n+2$.) If k is a positive integer, let $\epsilon^{(k)}=(\underbrace{1,1,\dots,1}_{k})$ and $\kappa^{(k)}=(1,2,\dots,k)$. Then*

$$\theta_{p+k,q+k}(\pi)=\pi'_k=\pi_1(\lambda_d,\xi,\Psi,\mu,\nu,(\epsilon|\epsilon^{(k)}),(\kappa|\kappa^{(k)})). \tag{6.4}$$

Proof. For (1), by Theorem 5.13(1), θ_{n+k} is a constituent of a certain induced representation that has π as its unique LKT constituent. By Theorem 4.8 we know that π' has at least one LKT that is of minimal degree, and it is straightforward to check that it corresponds in $\mathcal{H}$ to a LKT of π_k. So $\pi_k=\theta_{n+k}(\pi')$. The proof of (2) is analogous.

□

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