

**A Calculator Supplement for Calculus**  
*University Calculus: Elements with Early Transcendentals (WMU edition)*  
**Regular Section Edition, TI-89 version**

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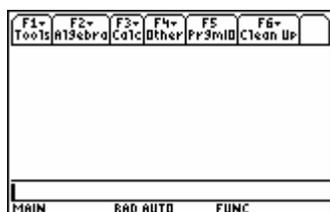
- §1. Symbolic Differential Equations
- §2. Differential Equation Graphing
- §3. Slope Fields and Euler's Method

# Chapter 0

## Overview of the TI-89 (TI-92 Plus, Voyage 200)

This chapter provides a brief introduction to some of the features of this family of graphing/symbolic calculators that relate to pre-calculus material. For the most part, we demonstrate with the TI-89 (including the TI-89 Titanium), which is the most common choice for students at WMU. The TI-92 Plus (no longer made) and the Voyage 200 provide the same capabilities in a larger-sized model with a QWERTY keyboard. If you have not already purchased your calculator, you might want to consider the Voyage 200. For only about \$50 more, you get more memory, several cost applications, a larger keyboard and screen, and the GraphLink cable to connect to your computer. The newest TI-89 Titanium also has more memory and included is a USB connection cable to connect to your computer. See the Texas Instruments Calculator web site for the latest information about what is available for this family. <http://education.ti.com>

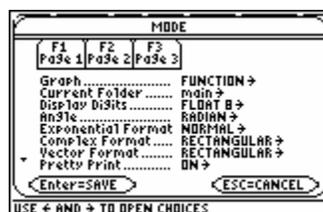
### 1. Settings, Layout, Operating Systems, Desktop



TI-89 Home Screen

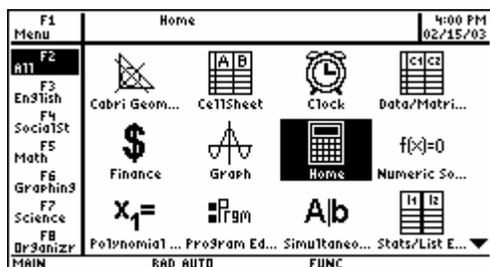


Applications List

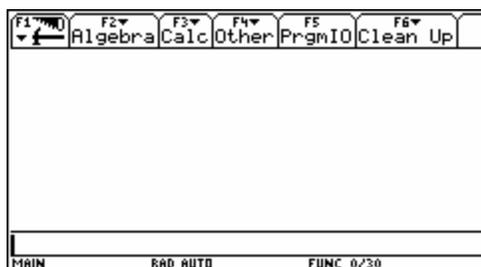


Mode Settings

When you press the ON button (lower left), a TI-89 generally comes on in what is called the HOME screen as in the first figure above. You will do immediate computations in the command line at the bottom of the screen. Results will appear in the middle region, which is called the “history” area, and many commands can be obtained by pressing one of the function keys F1-F8, which are blue keys below the screen. The different “screens” of the calculator are called applications. Press the blue APPS key to bring up the list of applications as in the second figure above. Note that additional applications can be loaded into the calculator, and they will appear under the heading 1:FlashApps. Press the MODE key to bring up the first page (of three) of various settings for the calculator.



TI-89 and Voyage 200 Desktop

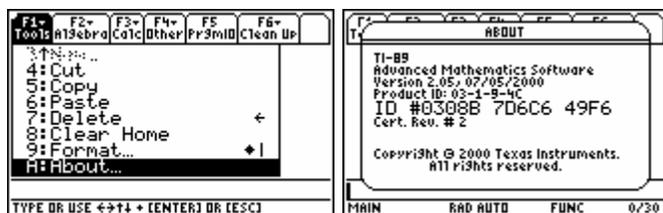


Home Screen

A Voyage 200 and the TI-89 Titanium generally come on with an icon desktop displaying the applications. This can be turned off in a page 3 mode setting, but it includes a possible clock in the corner which is nice. Press ENTER with an icon highlighted to move to that application. As you see in the figure above, the Voyage 200 comes with some additional flash applications (most of which can be added to a TI-89). The APPS key brings you back to the icon desktop.

You adjust the contrast of the screen by holding the green “diamond” key and pressing the “+” key to darken and the “-” key to lighten. If you need to keep increasing the contrast, this is a sign that your battery is low. A low battery signal will also come on as a warning and certain linking operations will not be allowed when the battery is low.

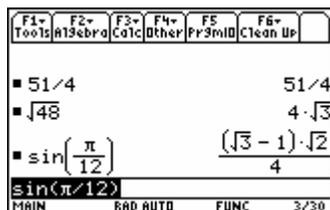
This family of calculators uses flash ROM, enabling all of the code in the calculator to be upgraded. Texas Instruments calls the software running the calculator the operating system or OS. You can check at the TI web site (<http://education.ti.com>) for the newest OS version and for new applications available for downloading. You will need a computer and the GraphLink cable (or the TI-89 USB cable) to download new things from the web site. You can also get a new OS and free applications from another calculator (of the same model). Press F1 A: About to see what version you have.



See <http://education.ti.com> and look for DOWNLOADS to find the latest OS for you calculator. A new OS might fix bugs that have been discovered, add new commands, and help to make some of the new applications mentioned in this document at that location as well.

The ESC key is very important for escaping out of some situation. If you pull up a menu and decide *not* to select anything, press ESC to back out of the menu (or submenu). In many situations, the display will give you the option to ENTER to save or confirm and ESC to cancel out of that process. For example, none of the changes you make in the MODE pages take effect until you press ENTER=save, with ESC=cancel as an option.

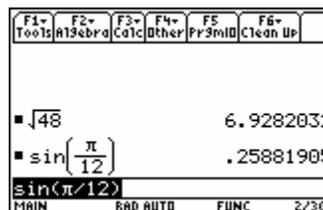
## 2. Home Screen Operations



ENTER in AUTO

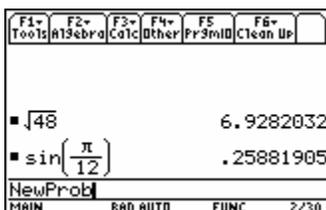


◆ ENTER in AUTO

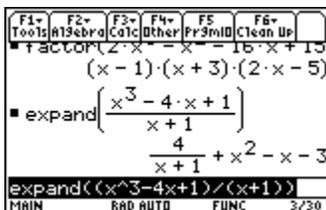


The default Exact/Approx mode setting is AUTO, and this allows many computations to be done exactly. The command F1 8:Clear Home erases the “history” area. When you want a decimal approximation in this mode setting, just press the “green diamond” key before the ENTER key to get “APPROX ≈”, rather than changing the mode setting.

Notice below the command line some of the mode settings are displayed. In the figures above, the MAIN indicates we are in the main folder, RAD indicates radian angle mode, AUTO indicates the exact/approx mode, FUNC indicates function graphing is selected, and the 3/30 in the first two figures indicates that the history contains 3 items out of the 30 that will be remembered (with more or less possible as a format setting).

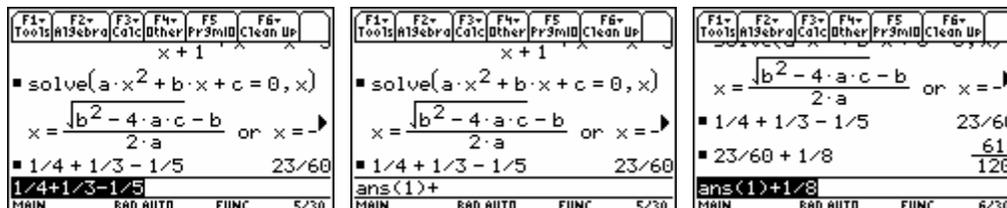


Variable names can be up to 8 characters long and must begin with a letter. Usually we use single letter names for true variables in our algebraic expressions, and we use longer names for programs and other things we wish to keep. The F6 2:New Problem command is a very nice way to start fresh. It clears the history, clears out all variables named by a single letter (as does F6 1:Clear a-z...), and deselects all functions and plots in graphing.

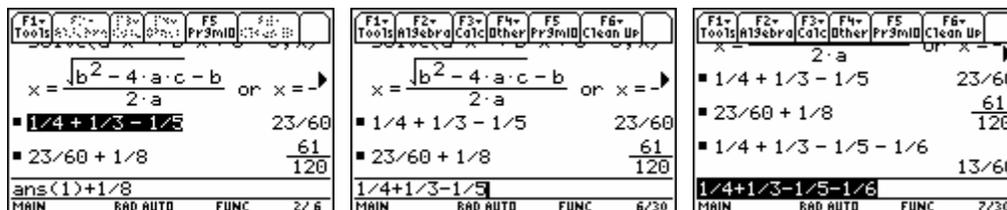


Notice in the first figure above, we have used “implied multiplication” when we typed the coefficients of the polynomial. In the last figure, we needed the multiplication symbol between “a” and “x” and between “b” and “x” because typing “ax” and “bx” would have been interpreted as new variable names.

You notice that after you press ENTER, the last command stays in the command line, but it is all highlighted. If you simply start typing, the new typing replaces the highlighted expression. If, instead, you want to edit the last line, press the left or right cursor keys to move the insertion point to that end and remove the highlighting. Then you can edit the expression. The “back arrow” ← key is a destructive backspace command. Above that key, the command “green diamond” DEL above the back arrow is a destructive forward movement. By default, any typing that you do is inserted where the blinking vertical bar appears in the command line. Pressing 2<sup>nd</sup> INS above the back arrow is a toggle converting between the insert entry mode and the typeover entry mode (blinking space).



After entering a command (as in the first figure above), you can simply press “+” to do a continuation calculation. If you press any command requiring an argument before the command (like “+”), the calculator assumes you wish to use the last answer. In the second figure, we see it has added “ans(1)” in front of the “+” that we typed to indicate this. When you want the last answer (and any one stored in the history) somewhere else in the command line, use ANS (above the negation key). Just edit “ans(1)” to have another number if you want to go back more than one answer.



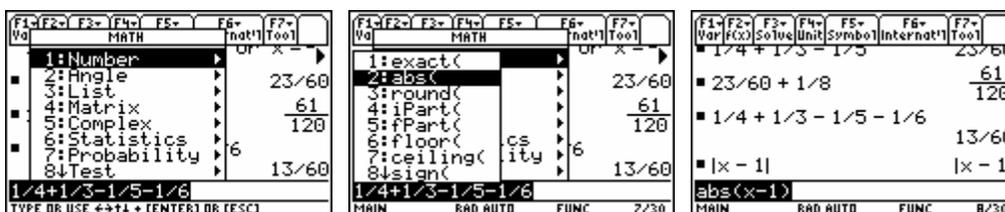
When you use the up cursor, you move up into the history area and highlight results (on the right) and command lines (on the left by a ■). You sometimes must do this to scroll sideways to read all of a long result. Pressing ENTER while something in the history is highlighted causes it to be pasted into the command line as figure one and two above. Thus you should never need to retype anything you can see!! While the cursor is anywhere in the history area, you can press ESC to jump down to the command line without pasting anything there.

Holding the “up arrow” ↑ key and then cursoring in the command line allows you to selectively highlight a part of the command line. Under F1 Tools, you will find that you have Cut, Copy, and Paste much like on a computer.



Some Default TI-89 Custom Menu Items

Since it is a little hard to type long phrases on the TI-89 (with no QWERTY keyboard), this calculator is provided with a default CUSTOM menu providing phrases and symbols commonly used. Pressing 2<sup>nd</sup> CUSTOM toggles this to be available or not. If you wish, you can define your own Custom Menu with anything you use often. Selecting something from the Custom Menu simply pastes that phrase into the command line.



The MATH Menu (above the “5”) has many common mathematical commands. You can also type out the commands one character at a time, but this is slow. As with any menu, you can cursor down to highlight the command (as in the second figure) and press ENTER or you can press the number (or letter) in front of the command.

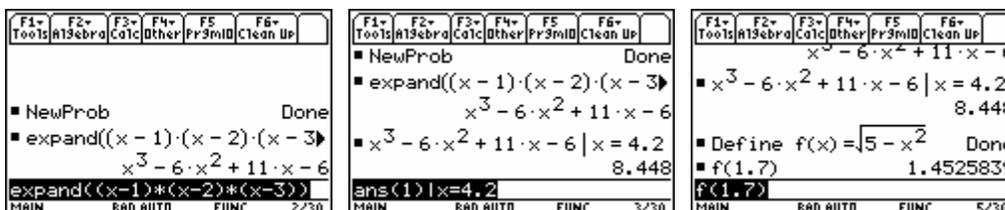


The CATALOG key brings up the catalog of all of the commands in the calculator. Press any key with a letter on it or above it (no need for the ALPHA first) while in the catalog to jump to commands beginning with that letter. Notice that when a command is selected (with the arrow), you get some on-screen help. The very bottom of the screen indicates the syntax required for that command. The syntax items will be separated by commas, and syntax items in square brackets [ ] are optional.





You can store a number in a variable name either with the F4 1:Define command or the STO→ key (which appears on the screen only as the arrow →). Then whenever you use that name, the value is immediately used. The F4 4:DelVar command is used to delete the definitions for these variable names. Note that F6 1:Clear a-z... and 2:NewProblem will not clear these longer variable names. **Suggestion:** Never store a number in the variables  $x$ ,  $y$ ,  $z$ , or  $t$  because we often use these letters for algebra variables. There is too great a chance you will forget to delete them before you want to use them as an undefined variable.

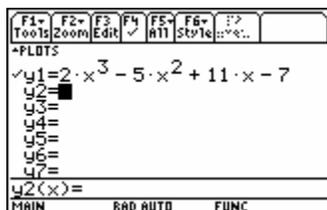


Alternatives to “storing a number in  $x$ ” are better. The “with” command is a vertical bar | key, and it allows a value to be substituted in for a variable temporarily without permanently storing that value in the variable name. When we define a function, then we can simply evaluate the function without storing anything permanently in the variable name.

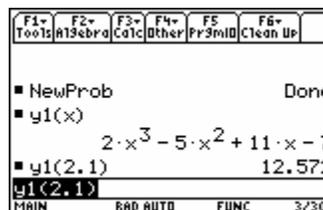
### 3. Functions, Tables, and Graphs



FUNCTION graphing



Y= Editor

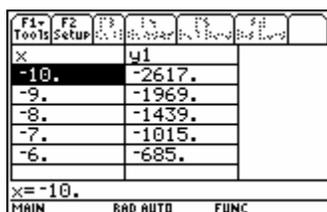


Home

A formula entered into the Y= Editor (which you can get to via APPS 2:Y= Editor or via “green diamond” F1) is then available to you in the Home screen.



TblSet



Table

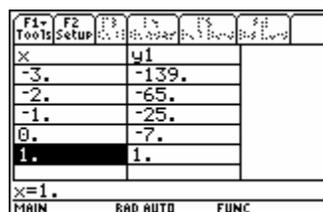


Table (scrolled down)

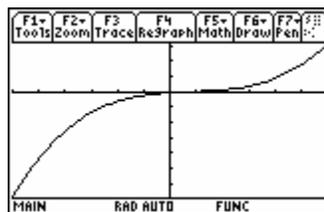
You can start a Table with any value, increment with any  $\Delta t_{bl}$ , and then have the table automatically generated. Then you can scroll (up and down) to see evaluations of the function selected in the Y= Editor. Get in the habit of looking at a table *before* trying to set a viewing window for the function. For example, after looking at the table for this cubic polynomial, we decide on the following reasonable Window.

F1	F2	F3	F4	F5	F6	F7
Tools	Setup	Zoom	Trace	ReGraph	Math	Draw
x		u1				
2.		11.				
3.		35.				
4.		85.				
5.		173.				
6.		311.				
x=6.						
MAIN RAD AUTO FUNC						

Table (scrolled down)

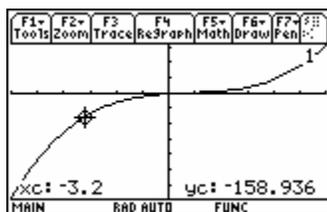
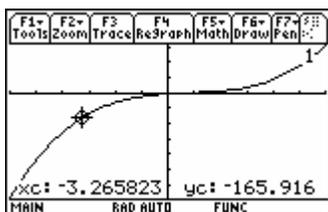
F1	F2
Tools	Zoom
xmin=-6.	
xmax=6.	
xsc1=1.	
ymin=-685.	
ymax=311.	
ysc1=100.	
xres=2	
MAIN RAD AUTO FUNC	

Window



Graph

Note that “xsc1” and “ysc1” give values to the marks on the axes. The setting “xres” equal to 2 causes points to be plotted in only every other column of the graph (thus only 79 columns out of the 159 on the TI-89). Since the points are connected, we don’t notice any “roughness” in the plot. A larger “xres” will plot more rapidly, but might miss important features of the plot. Further when you do F3 Trace, the cursor moves only to the plotted points based upon “xres”.



F1	F2	F3	F4	F5	F6	F7
Tools	Zoom	Trace	ReGraph	Math	Draw	Pen
[Graph with cursor]						
1: Value						
2: Zero						
3: Minimum						
4: Maximum						
5: Intersection						
6: Derivatives						
7: f(x)dx						
8: Inflection						
TYPE OR USE $\leftarrow$ + $\rightarrow$ + $\leftarrow$ + $\rightarrow$ + $\leftarrow$ + $\rightarrow$ OR (ENTER) OR (ESC)						

In the first figure, we Trace with the cursor to near  $x = -3.2$ . While tracing, we can also type  $-3.2$  to see the more exact value there. The F5 Math 1:Value command will also evaluate the function wherever we want on the interval from “xmin” to “xmax”.

F1	F2	F3
Pa3e 1	Pa3e 2	Pa3e 3
MODE		
* Split Screen..... LEFT-RIGHT $\rightarrow$		
* Split 1 App..... GRAPH $\rightarrow$		
* Split 2 App..... $\leftarrow$ TABLE $\rightarrow$		
Number of Graphs: 1 $\rightarrow$		
Style: 1		
Exact/APPROX..... AUTO $\rightarrow$		
Base..... DEC $\rightarrow$		
Enter=SAVE		
ESC=CANCEL		
USE $\leftarrow$ AND $\rightarrow$ TO OPEN CHOICES		

MODE Page 2 Split Screen

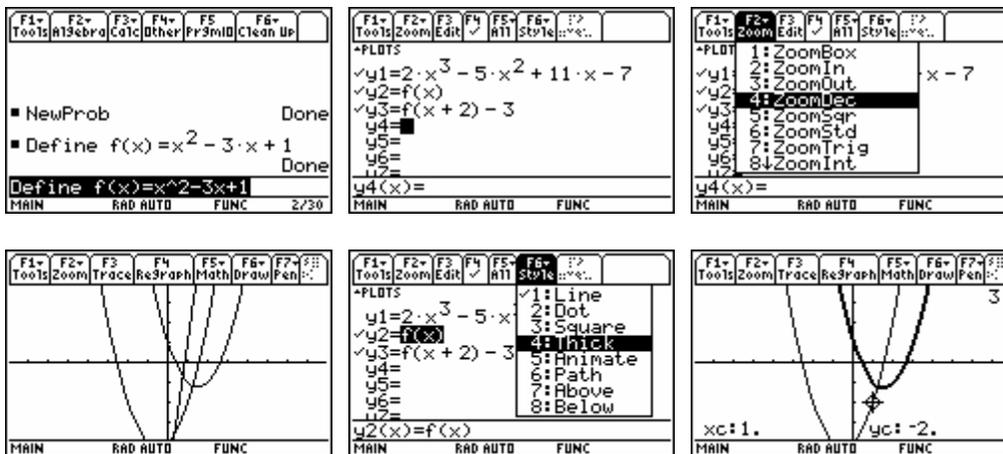
F1	F2	F3	F4	F5	F6	F7
Tools	Zoom	Trace	ReGraph	Math	Draw	Pen
TABLE SETUP						
tblStart..... 2.						
$\Delta t_{bl}$ ..... 1.						
Graph $\leftrightarrow$ Table						
Independent..... 1: OFF						
2: ON						
Enter=SAVE						
ESC=CANCEL						
MAIN RAD AUTO FUNC						

Table Setup

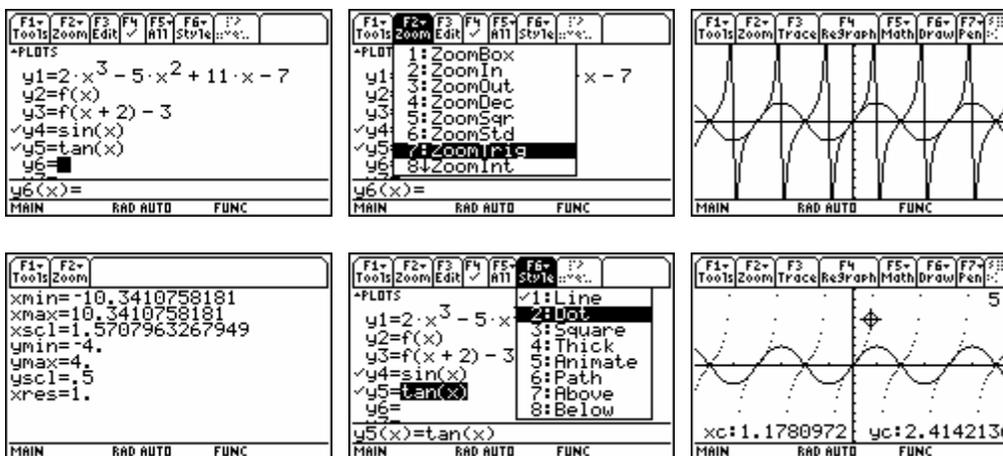
F1	F2	F3	F4	F5	F6	F7
Tools	Zoom	Trace	ReGraph	Math	Draw	Pen
[Graph with cursor]						
x/c: .94736842 y/c: .63405744						
MAIN RAD AUTO FUNC						

Graph/Table

With the split screen, you can have the Graph and the Table appearing on the same screen. The symbol above the APPS key is the toggle to move from one side of the screen to the other (and you must move to the Table side to have the table generated). Here we have the Table tied directly to the points actually plotted in the Graph via the Table Setup choice in the second figure above.



Functions can be defined in the Home screen, but we can only graph a formula appearing in one of the function slots of the Y= Editor. (You can define a function with one of these function slot variable names in the Home screen.) When more than one functions is selected, we may want to change the F6 Style to be able to tell them apart. In the Y= Editor, F4  $\checkmark$  will change whether a function slot is selected or not.

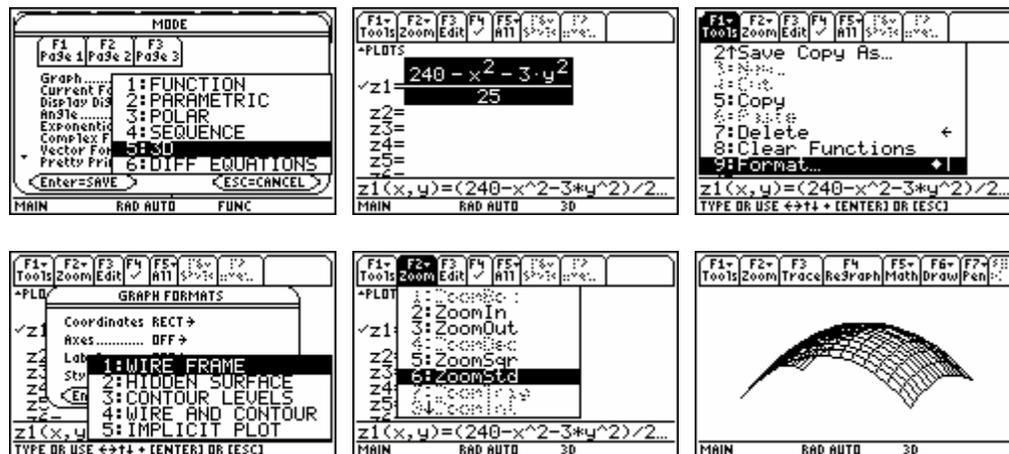


Notice that the default Line Style and “xres = 2” do not do such a great job plotting the tangent function. Generally we wish to avoid the false “near vertical” lines near vertical asymptotes. The Dot Style avoids this. [For OS 3.0 or higher, you can turn on the GRAPH FORMAT “Discontinuity Dection ON” to avoid these false vertical lines.]



The “when” command gives us a way to do piecewise-defined functions, if there are not too many pieces. The Dot Style is also usually best for such piecewise-defined functions, which may not be continuous between the pieces.

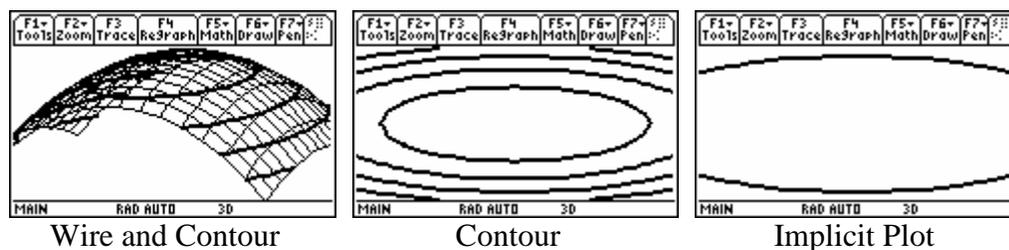
Certainly one of the most interesting features of the TI-89 is the ability to do 3D plotting. While somewhat limited compared to 3D computer plots, this is still an amazing feature for such a low-cost, low-resolution device. One example will be enough to introduce this feature.



Here we have given a function  $z = f(x, y)$  that shows nicely in the ZoomStd viewing “box”, namely

$$-10 \leq x \leq 10, \quad -10 \leq y \leq 10, \quad -10 \leq z \leq 10.$$

Always plot the function first in the WIRE FRAME Style because it is the fastest. Then when you are happy with the viewing “box” and the formula, you can explore the alternative styles.



Wire and Contour

Contour

Implicit Plot

#### 4. Entering Data, Performing a Regression

We show several ways to work with the data and plotting for a linear regression. Consider the following table of data.

$x$	1.2	1.7	2.0	2.6	2.8	3.1
$y$	0.6	1.9	2.1	3.1	3.4	3.8

First we use the Data/Matrix Editor.



DATA	c1	c2	c3
1	1.2	.6	
2	1.7	1.9	
3	2.	2.1	
4	2.6	3.1	

r7c2=3.1

DATA	c1	c2	c3
4	2.6	3.1	
5	2.8	3.4	
6	3.1	3.8	

r7c2=



F2 Plot Setup

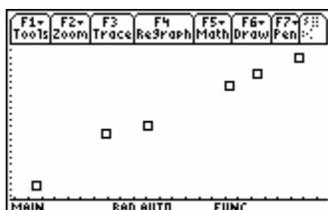
F1 Define



ENTER, then ESC Y=



F2 9:ZoomData

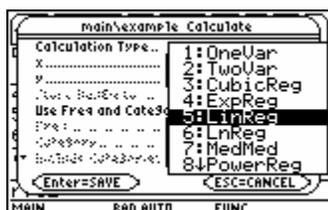


It takes all of this to get the scatter plot of the data. Notice that performing ZoomData after the Plot 1 has been defined gives a nice viewing window for the plot (we are back in Function graphing).

DATA	c1	c2	c3
4	2.6	3.1	
5	2.8	3.4	
6	3.1	3.8	

r7c2=

Return to current data



F5 Calculate



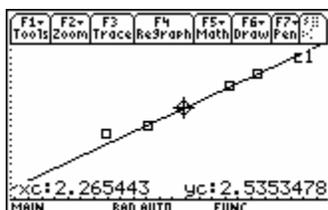
DATA	y=a+bx
a	=1.619898
b	=-1.134439
corr	=.9906
R2	=.981288

r7c2=

Current Stat Vars



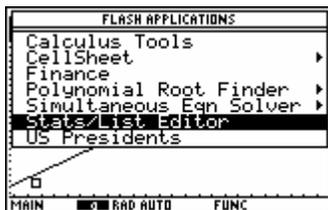
Y= Editor



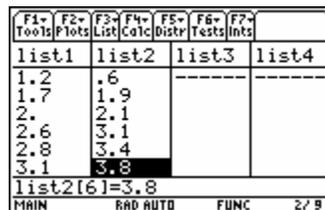
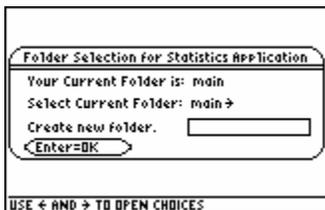
Graph and F3 Trace

Above we see the steps to calculate the linear regression, store the formula in slot y1(x), and then look at the graph with both the scatter plot and the regression line. Most people found this statistical process a little cumbersome compared to the more advanced statistics on the TI-83. TI has also offered a free flash application called the Statistics List Editor with all of the power and

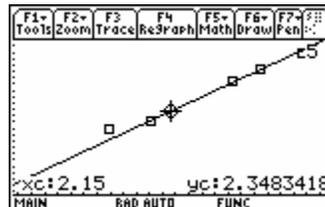
ease of use of the TI-83 (and even additional statistical commands). Most new TI-89's come with this flash application loaded, and anyone can download it from the web site.



APPS 1:FlashApps



Type data in lists

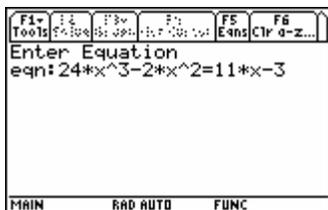


Statistics, including linear regression, can also be done in the CellSheet flash application, which gives a mini-spreadsheet for this family of calculators. The CellSheet costs extra for the TI-89 but comes loaded in a Voyage 200.

## 5. Solving Equations

We can solve some equations exactly with the symbolic command F2 1:solve( in the Home screen. We can also solve equations numerically in the Numeric Solver application. We can also graph both side of an equation, and seek an intersection point. We demonstrate an example of each method.





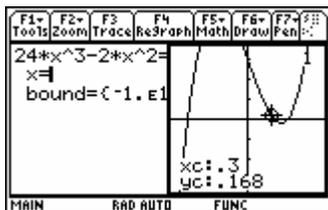
Enter Equation



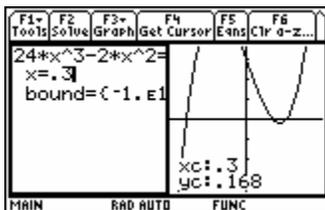
Ready to Solve



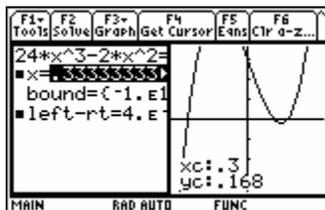
Graph "Left - Right"



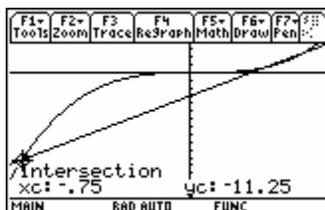
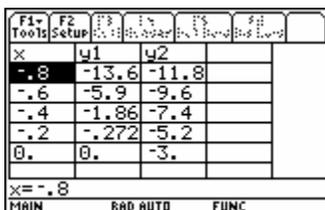
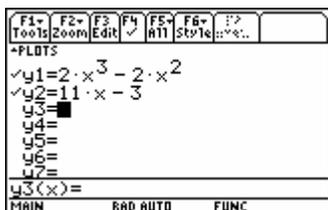
Trace for approximate



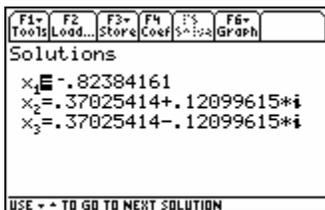
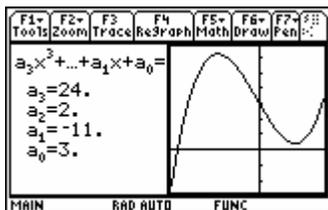
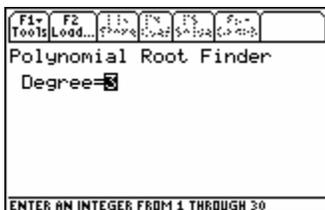
Seed solver, F2 Solve



Numeric Solution



There is also a free flash application that finds numerical real and complex zeros of polynomials.

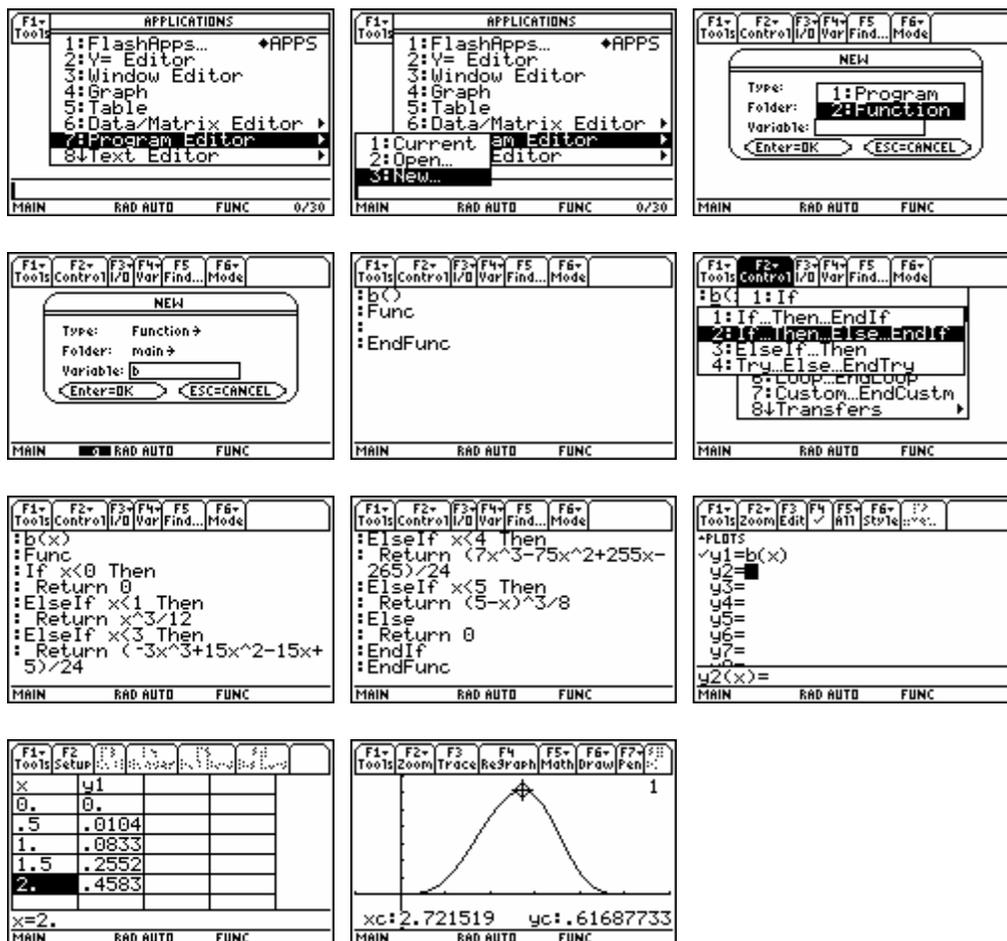


## 6. Programming

The early graphing calculators had relatively few features. Thus there was a need to write short programs to do many of the activities in pre-calculus and calculus. Now as I look back at the calculator programs that I wrote 10 or 15 years ago, I find that most of the things I tried to do are now easily provided to the user in some command or application. Still there is occasionally something we want to do repeatedly that is not yet provided. Here we simply look at an example of a function-type program that returns a value. This provides a more understandable

way to implement a piecewise-defined function with many pieces. Consider the following function (a B-spline).

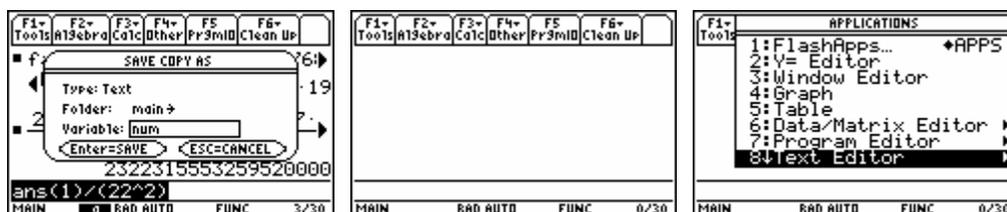
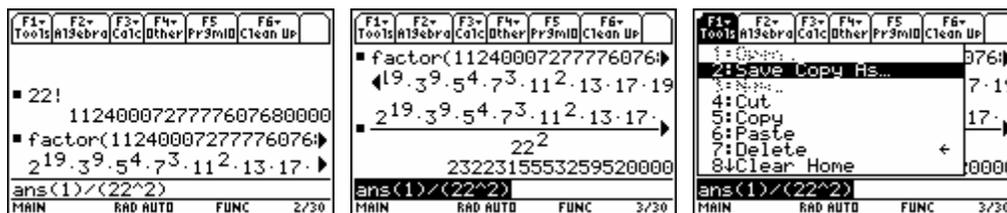
$$b(x) = \begin{cases} \frac{1}{12}x^3, & \text{if } 0 \leq x < 1 \\ \frac{1}{24}(-3x^3 + 15x^2 - 15x + 5), & \text{if } 1 \leq x < 3 \\ \frac{1}{24}(7x^3 - 75x^2 + 255x - 265), & \text{if } 3 \leq x < 4 \\ \frac{1}{8}(5-x)^3, & \text{if } 4 \leq x < 5 \\ 0, & \text{else} \end{cases}$$



A function-type program returns a value (or an algebraic object). Thus it can be used in the Home screen, in the Y= Editor, and in other places just as you would use a built-in function. A program-type program will display its output on the I/O screen while it is running. Generally you cannot use the results outside of the program-type program.

## 7. Saving Your Work in a Text File

You may do a sequence of command lines in the Home screen that you would like to save for later use. Perhaps you have not finished, but you need to do something else with the calculator now. Perhaps the sequence is appropriate to use again and again (somewhat like a program). We show here how to save the command lines represented in the history.



Give name to text

F1 8:ClearHome



Notice that the actual value was inserted wherever we used the ANS key or a continuation computation. This text file can be edited, perhaps to put “ans(1)” into these places. With the cursor within some line in the text, press F4 Execute to have that command line (preceded with the “C:”) executed back in the Home screen. Most of the time, we select under F3 View 1:Script view to split the screen vertically so that we can see both the text file and the Home screen result.

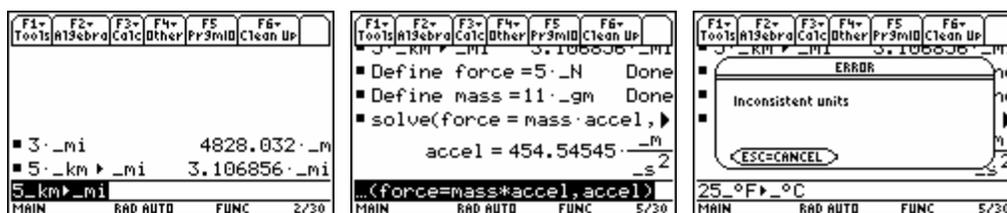
## 8. Units

Calculations for science and engineering always involve units. Mathematics instructors are often guilty of ignoring units (really they just assume they are correct). The calculator has several features that help you work with units and standard constants. First, there is a Page 3 Mode setting for the default unit system. The first two choices are the metric system (SI for System International) and the English system. Above the “3” key is the Units command to bring up the units menu. They have tried to give you units in the order most often needed (not alphabetical), so explore what is given.

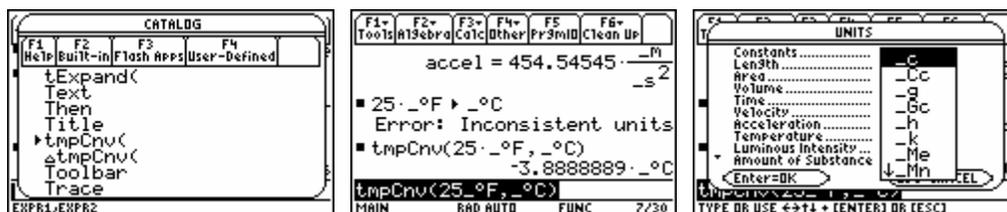


You type a number and then attach the unit after it. It is easier to get the unit designation from the menu. You can also type the symbols, but notice that all unit designations begin with an “underbar”. In the third figure above we have attached miles to 3. The calculator automatically converts the result to the mode-specified unit system, here to meters.

You can also convert from one unit to another in the same category. Simply type the desired number, attach the original unit, press the “bold arrow” above the MODE key, and attach the new unit desired.



If you assign units to objects in an algebraic expression, as in the second figure above, the “solved” variable will be given the appropriate units (in the mode-selected system). For some strange reason, temperature conversion does not work in the same way. Look in the CATALOG for the command tmpCnv().



The first line of the units menu gives many common constants from science and engineering. For example “\_Gc” is Newton’s universal gravitation constant which in the SI system equals  $6.67259E-11$  in  $m^3/(kg \cdot s^2)$ .

## 9. Connecting to a Computer, Other Applications

All TI graphing calculators come with a short cable to connect one calculator to another calculator. A GraphLink cable is for connecting the calculator to a computer. Older GraphLink cables (grey and black) plug into the COM port of a PC or the modem port of a Macintosh. The newer GraphLink cables (silver) use a USB connection (for both PC’s and Macintoshes). You will need such a cable to upgrade the OS or to download new applications. You can also save

your work (text files, programs, functions) on the computer. All of the figures in this document have been obtained using the GraphLink connection. The Voyage 200 comes with the GraphLink cable and software. TI-89 users may want to purchase a GraphLink cable (about \$20). The newest TI-89 includes a mini USB connection and a cable to connect to the USB port of a computer. The latest computer software for connection with the GraphLink cable can always be downloaded for free from the TI web site (<http://education.ti.com>). The Arts and Sciences Computing Labs on the third floor of Rood Hall have GraphLink cables for both PC's and Macintoshes and the required software on the server. On the new engineering campus, the CEAS computer lab has the required software on the server and you may check out a USB GraphLink cable from the desk.

We have already mentioned two flash applications that are free, the Statistics List Editor and the Polynomial Root Finder. Other free flash applications available now for the TI-89 are Calculus Tools, Finance, Simultaneous Equation Solver, Symbolic Math Guide, Study Cards, and US Presidents. Cost flash applications include CellSheet (mini spread sheet \$15), Equation Writer (\$15), EE200 (circuit analysis \$20), ME\*Pro (mechanical engineering \$50), EE\*Pro (electrical engineering \$50), The Geometer's Sketchpad (\$30), and Cabri Geometry (\$30). Some of these cost applications come preloaded in the Voyage 200. Recently they have added a free Organizer (address/phone and calendar/appointment book) for this family of calculators. See <http://education.ti.com> in the area for DOWNLOADS for your calculator model to find out what is available.

# Chapter 1

## Functions and Limits

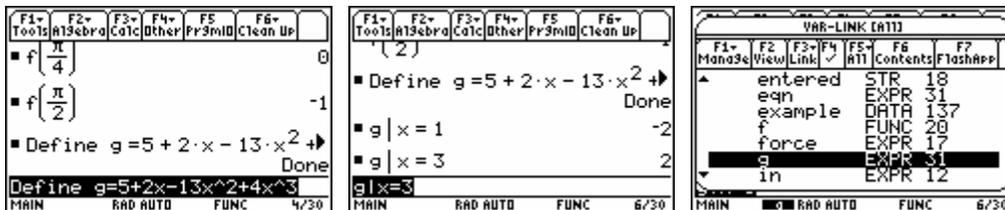
Function is a data type in the TI-89. There are many things that we can do with certain functions. This is loosely coordinated with Chapter 1 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

### 1. Entering and Storing Functions

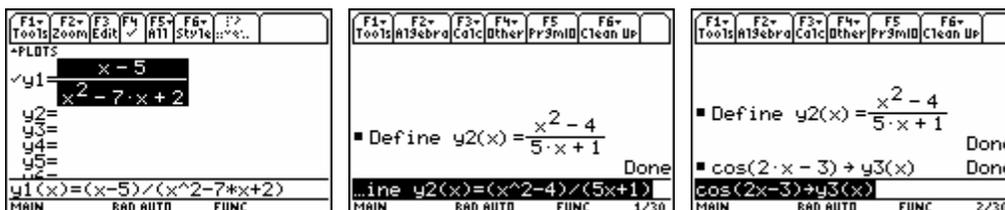
In the Home screen we can store a formula for a function using the Define command or the STO key. We can use any variable name (up to eight characters beginning with a letter) as the name for the function.



Custom menu items



Notice in the first row of figures that we define a function  $f(x)$ , which we can evaluate using function notation. In the second row of figures, we name an expression instead. To evaluate an expression, use the vertical bar to temporarily assign a value to  $x$ . Pressing the VAR-LINK keystroke brings up a list of all the variables defined, the data type, and the bytes of storage required. While a variable in the VAR-LINK list is highlighted, you can press F6 Contents to see one screen of the definition or program (which is usually enough to remember what you have stored here). If you press ENTER while the name is highlighted, the name will be pasted into the command line. If you press the destructive back arrow while the name is highlighted, you can delete the object from memory.



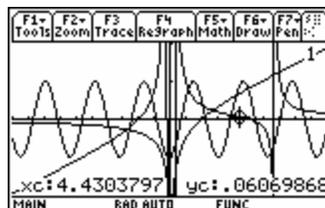
You can store a formula in the Y= Editor in the three ways demonstrated above. All three functions will end up showing in the Y= Editor screen and will be selected (with the check mark) since they are new. Again, I highly recommend that you look at a table of values for these functions before trying to select a viewing window.

x	y1	y2	y3
-10.	-.087	-1.96	-.533
-9.	-.096	-1.75	-.548
-8.	-.107	-1.54	.9887
-7.	-.12	-1.32	-.275
-6.	-.138	-1.1	-.76

x=-10.  
MAIN RAD AUTO FUNC

xmin=-10.
xmax=10.
xsc1=1.
ymax=2.
ysc1=1.
xres=1.

MAIN RAD AUTO FUNC



## 2. Composition of Functions and Inverse Functions

There is no special symbol for the composition  $f \circ g$ , but it can be accomplished via  $f(g(x))$  in the Home screen. Where possible, the result will be simplified.

Define f(x)=2·x <sup>2</sup> -x+1	Done
Define g(x)=5·x+4	Done
Define g(x)=5x+4	Done

MAIN RAD AUTO PAR 3/30

Define f(x)=2·x <sup>2</sup> -x+1	Done
Define g(x)=5·x+4	Done
f(g(x))	50·x <sup>2</sup> +75·x+29
g(f(x))	10·x <sup>2</sup> -5·x+9
g(f(x))	

MAIN RAD AUTO PAR 5/30

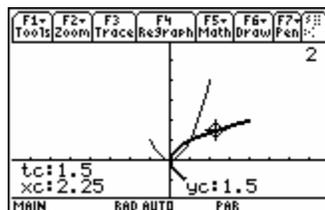
For any real-valued function  $f(t)$ , plotting the parametric equations  $x = t$  and  $y = f(t)$  gives the graph of the function  $f$ , and plotting  $x = f(t)$  and  $y = t$  gives the graph of the inverse relation. When the function  $f$  happens to be one-to-one and have an inverse function  $f^{-1}$ , the plot of the inverse relation will actually be the graph of the inverse function. In Parametric graphing, we may be able to restrict the domain (via the choice of  $tmin$  and  $tmax$ ) in order to make the plot one-to-one. Notice that you can trace on either curve.

1:Line	
2:Dot	
3:Square	
4:Thick	
5:Animate	
6:Path	
7:Shave	
8:Glow	

MAIN RAD AUTO PAR

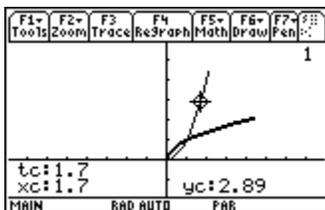
tmin=-1.
tmax=2.
tstep=.1
xmin=-7.9
xmax=7.9
xsc1=1.
ymin=-1.8
ymax=5.8
ysc1=1.1

MAIN RAD AUTO PAR

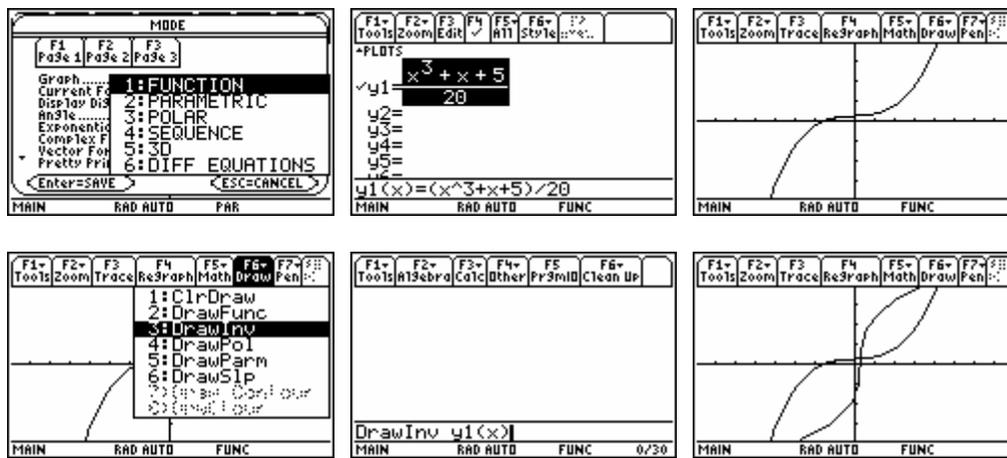


tmin=0.
tmax=2.11
tstep=.1
xmin=-7.9
xmax=7.9
xsc1=1.
ymin=-1.8
ymax=5.8
ysc1=1.

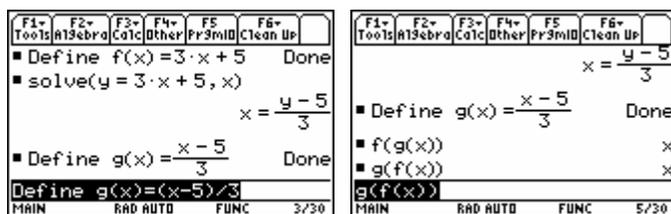
MAIN RAD AUTO PAR



In Function graphing mode, the calculator provides a command to have the inverse relation as a “drawn” object. However “drawn” objects cannot be traced, and they disappear when the graph is re-plotted for any reason. It is also hard to restrict the domain in Function graphing mode.



For very simple functions (little more than linear polynomials), the solve command in the Home screen can enable us to find the formula for the inverse function.

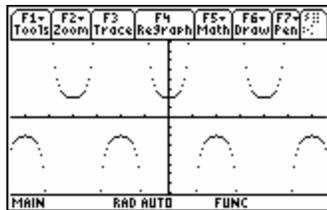


### 3. Trigonometric Functions

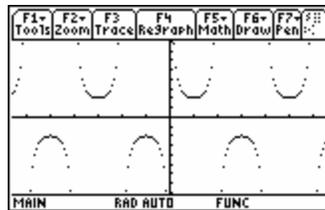
The calculator provides the trigonometric functions for the sine, cosine, and tangent. It also provides an inverse function for each (on a suitably restricted domain). The other trigonometric function can be computed from these, but it might be nice to have them defined permanently. If we use variable names longer than one letter, then the commands in the F6 Clean Up menu like NewProb will not delete these functions.



ZoomTrig  $y = \cot(x)$



ZoomTrig  $y = \sec(x)$



ZoomTrig  $y = \csc(x)$

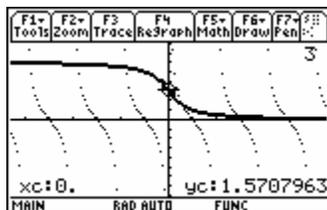
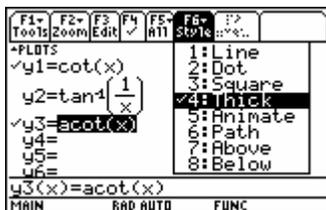
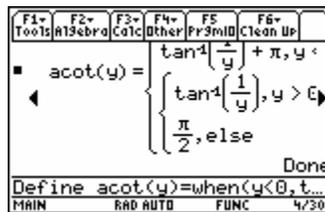
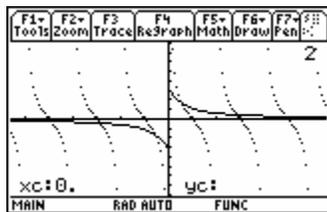
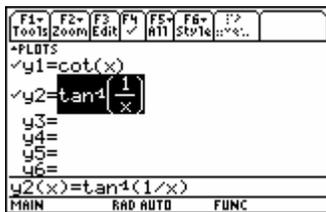
We can also compute the inverse functions for the cotangent, secant, and cosecant functions in terms of the inverse trigonometric functions provided. There is simply a question of the most desirable domain and range. For example, if we desire the inverse cotangent, we consider the following algebra.

$$y = \cot(x) = \frac{1}{\tan(x)}$$

$$\tan(x) = \frac{1}{y}$$

$$x = \tan^{-1}\left(\frac{1}{y}\right) \text{ implying that } \cot^{-1}(y) = \tan^{-1}\left(\frac{1}{y}\right)$$

Most people do not like this choice of range for the inverse cotangent (and it leaves the problem of what to do with  $\cot^{-1}(0)$ ).



As in many computer languages, we use “acot” as the variable name for the inverse cotangent or arccotangent, and then “asec” and “acsc” for the remaining inverse functions. Make the following definitions to have these functions.

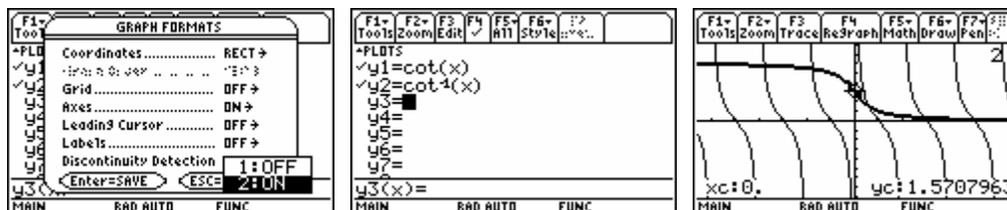
**Define  $\text{acot}(y) = \text{when}(y < 0, \tan^{-1}(1/y) + \pi, \text{when}(y > 0, \tan^{-1}(1/y), \pi/2))$**

**Define  $\text{asec}(y) = \cos^{-1}(1/y)$**

**Define  $\text{acsc}(y) = \sin^{-1}(1/y)$**

When you want to use these functions, you can either type the name or you can get it from your VAR-LINK list of all of your variables as in the last figure above.

By the way, all of the above steps are not needed if you will simply upgrade your OS to at least version 2.08 where the “other” trigonometric functions and their inverses have been included. Since the variable names “cot”, “sec”, and “csc” become reserved words in the newer versions of the operating system, you will not be allowed to keep programs around with these names. Further for OS 3.0 and later (which is only available for the TI-89 Titanium and the Voyage 200), you can turn on the Graph Formatting option for “discontinuity detection” to avoid the vertical asymptotes rather than needing to use the “dot” style for graphing the discontinuous trigonometric functions.



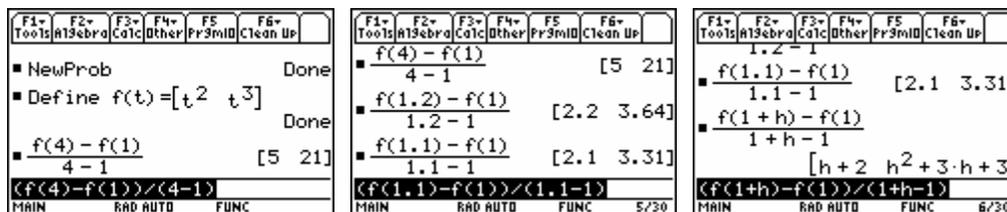
TI-89 Titanium OS 3.10, cotangent and inverse cotangent provided in the catalog

#### 4. Average Rate of Change

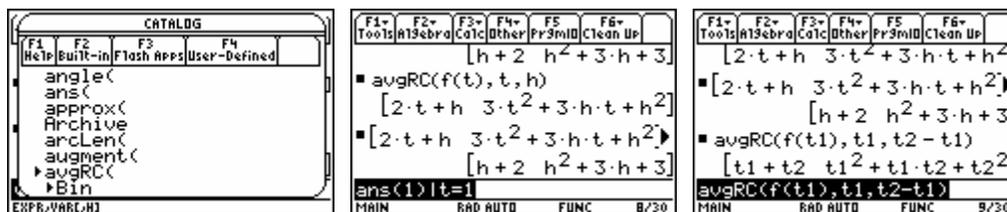
The concept of average velocity and average rate of change for a function  $\vec{f} : \mathbb{R} \rightarrow \mathbb{R}^n$  involve the formula

$$\frac{\vec{f}(t_2) - \vec{f}(t_1)}{t_2 - t_1}$$

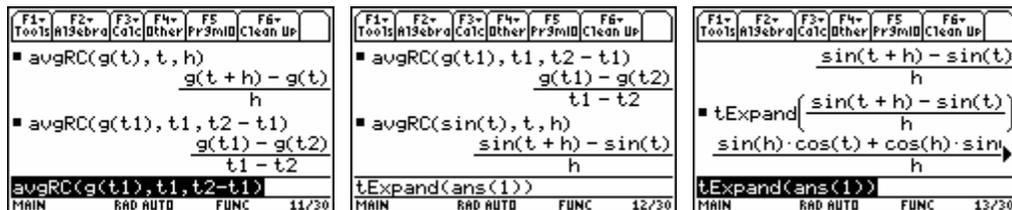
For a given vector-valued function, we can do numerical and symbolic computations directly with this formula.



After using the formula directly a few times, you might seek a shortcut. This is such an important concept, that the average rate of change is given as a command. The easiest way to find this command is to use the CATALOG (pressing the key with “A” above it to move to the part of the catalog with commands beginning with “A”).



As with many commands which we are just starting to use, we can make sure of what the command is doing by applying the command to undefined variables.

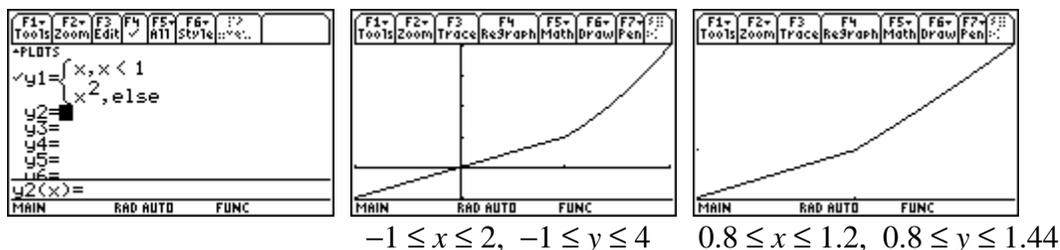


## 5. Using Graphs and Tables to Investigate Limits

The limit concept is first intuitively approached by looking at graphs. Consider the piecewise-defined function

$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \leq x \end{cases}$$

where the most interesting question is what happens when  $x$  approaches 1.



Thus there is graphical evidence that  $\lim_{x \rightarrow 1} f(x) = 1$  and that this function is continuous. We can also explore this limit numerically in a table. For limits, it is best to select the “ASK” mode for the Table Setting involving the independent variable. Clear the Table of old results, and enter in  $x$ -values approaching 1 (both above and below).



The most interesting limits arise from taking the average rate of change over a smaller and smaller interval. E.g.

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{where} \quad f(x) = \frac{1}{x-1} \quad \text{and} \quad x_0 = 2.$$

F1- Tools	F2- Zoom	F3- Edit	F4- AT1	F5- Styls	F6- :v:..
+PLOTS					
$y1 = \begin{cases} x, & x < 1 \\ x^2, & \text{else} \end{cases}$					
$y2 = \frac{1}{2+x-1} - \frac{1}{2-1}$					
$y3 = \frac{1}{x}$					
y2(x) =					
MAIN      RAD AUTO      FUNC					

F1- Tools	F2- Setup	F3- :Header	F4- Del Row	F5- Ins Row	F6- Row
x					
y2					
-.02      -1.0204082					
.1      -.90909091					
.005      -.99502488					
-6.E-4      -1.0006004					
4.E-6      -.999996					
y2(x) = -.999996					
MAIN      RAD AUTO      FUNC					

The numerical evidence seems to suggest that the limit here is about  $-1$ .

## 6. Symbolic Limits

In the Home screen, we can actually ask the calculator to find symbolic limits. It knows the patterns and rules to find many limits exactly. Appropriately the limit command is in the F3 Calculus menu (as well as in the CATALOG).

F1- Tools	F2- Algebra	F3- Calc	F4- Other	F5- Pr3mid	F6- Clean Up
NewProb      Done					
$\lim_{h \rightarrow 0} \left( \frac{\sin(t+h) - \sin(t)}{h} \right)$					
cos(t)					
...(\sin(t+h)-sin(t))/h,h,0					
MAIN      RAD AUTO      FUNC      2/30					

F1- Tools	F2- Algebra	F3- Calc	F4- Other	F5- Pr3mid	F6- Clean Up
h → 0					
cos(t)					
$\lim_{x \rightarrow 0^+} \text{sign}(x) = 1$					
$\lim_{x \rightarrow 0^-} \text{sign}(x) = -1$					
limit(sign(x),x,0,-1)					
MAIN      RAD AUTO      FUNC      4/30					

F1- Tools	F2- Algebra	F3- Calc	F4- Other	F5- Pr3mid	F6- Clean Up
$\lim_{x \rightarrow 0^-} \text{sign}(x) = -1$					
$\text{sign}(0) = \text{sign}(0)$					
$\lim_{h \rightarrow 0^+} \left( \frac{\sqrt{h}}{h} \right) = \infty$					
limit(f(h)/h,h,0,1)					
MAIN      RAD AUTO      FUNC      6/30					

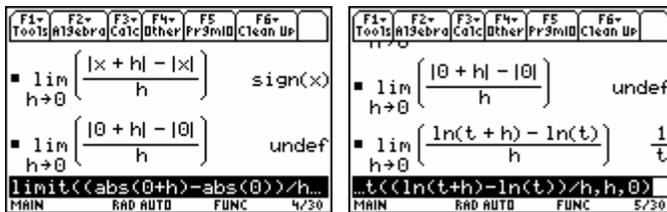
The optional last argument for the limit command gives one-sided limits (with the limit from the left if the number is negative and with the limit from the right if the number is positive).

## Chapter 2 Differentiation

We consider in this chapter symbolic and numerical computations related to the differentiation rules. We also explore implicit functions and implicit differentiation a little more than the text does. This is loosely coordinated with Chapter 2 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

### 1. The Derivative Definition

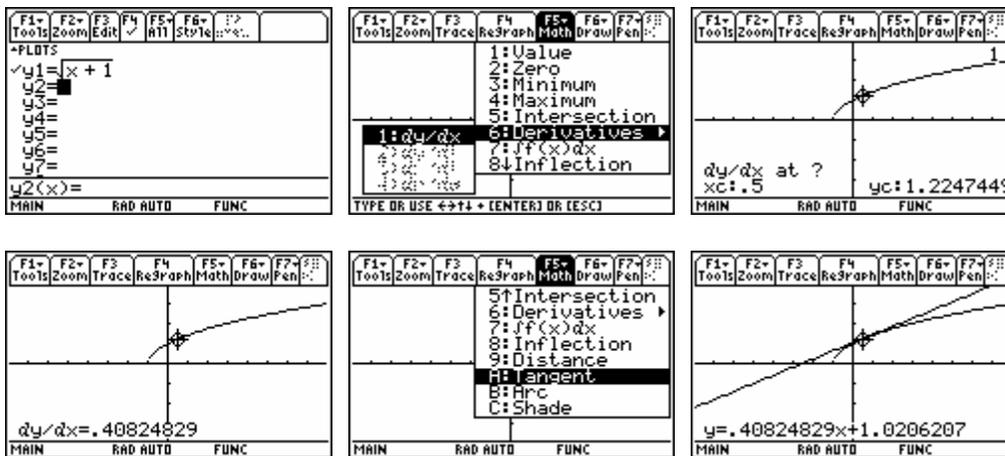
Highly interesting to us now are limits involving the definition of the derivative.



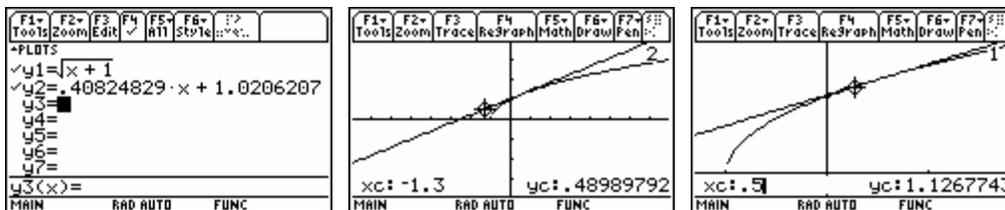
The history indicates that a limit does not exist by the notation “undef”.

### 2. Graphical Derivatives

While looking at the graph of a function, we can evaluate the derivative at any point on the graph or have a tangent line drawn. We consider first real-valued functions.



The tangent line produced by the F5 MATH command is a drawn object on the plot. Thus you cannot trace along the tangent line, and this line will disappear when you make any change to the graph (zoom, scroll, change the functions selected, or change the window). You would need to type the tangent line as a Y= function to get an active graph including it.



after a ZoomBox

### 3. Numerical and Symbolic Derivatives

From the derivative definition, we know that a derivative (if it exists) can be approximated by the following.

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

When  $h = r > 0$ , this is called a *forward difference approximation*  $\frac{f(x_0 + r) - f(x_0)}{r}$ , and when

$h = -r < 0$ , this is called a *backward difference approximation*

$$\frac{f(x_0 - r) - f(x_0)}{-r} = \frac{f(x_0) - f(x_0 - r)}{r}$$

The command “averRG(f(x),x,h)” gives us a convenient way to compute these. In general, one of these approximations will tend to be larger than the derivative and the other will tend to be smaller. Thus it is reasonable to expect the average of the two to be a more accurate approximation.

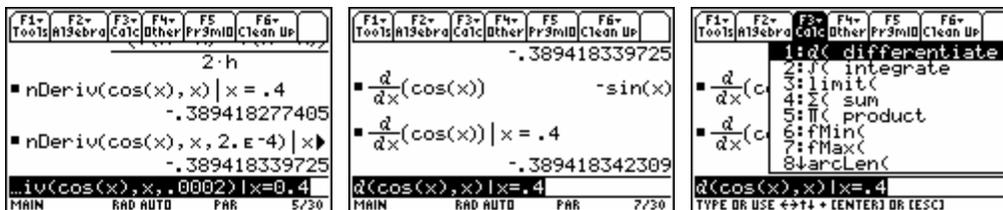
*Centered Difference Approximation*

$$\frac{1}{2} \frac{f(x_0 + h) - f(x_0)}{h} + \frac{1}{2} \frac{f(x_0) - f(x_0 - h)}{h} = \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

This numerical approximation is provided by the F3 Calc command A : nDeriv( Note that all of these derivative approximations can always be computed, even when the derivative does not exist (or even later when there is no rule to find the derivative).



Default  $h = 0.001$



The symbolic derivative operation is provided in the F3 Calc menu by the command 1 :  $d$ ( differentiate. Note that you can also get this command on the keyboard (2<sup>nd</sup> “8”) and that this is a special “slanted” character on the command line, which is different from the letter “d”. We will explore the symbolic derivative command more in the next chapter.

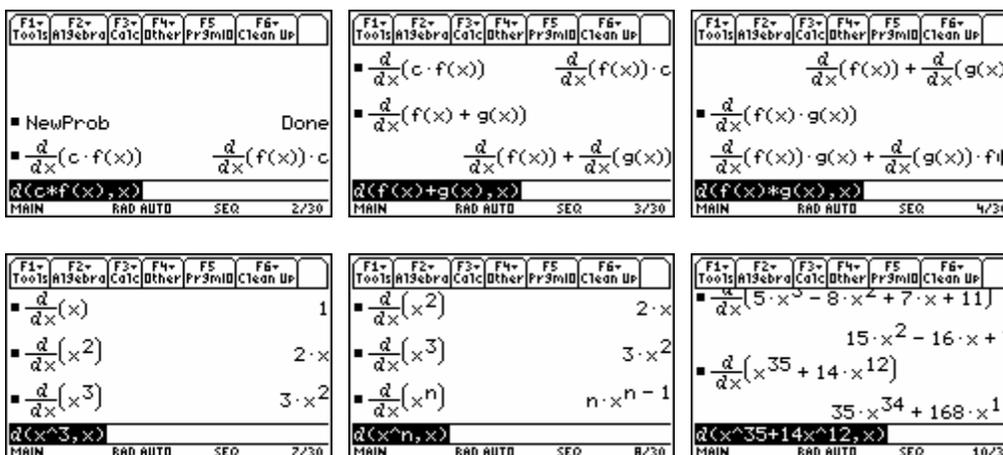
### 3. Exploring Differentiation Rules with the Calculator

**Warning:** *The fact that the TI-89 “knows” the differentiation rules does not relieve you of the responsibility to learn them for yourself.* Once you become fairly proficient at differentiation by hand, you can write down the derivative much faster than you can type in the function and have the calculator do it. Most instructors will want to test your knowledge of the rules of differentiation in the absence of the tool. Use the tool only as an aid while you are learning the rules.

We explore various ways to see that the calculator knows the following rules:

**Theorem** *Suppose that  $c$  is a real number, that  $f$  and  $g$  are real-valued functions so that  $f'$  and  $g'$  are defined at  $x$ , and that  $n$  is a positive integer. Then*

- (a)  $(c f)'(x) = c f'(x)$ . (Multiple rule)
- (b)  $(f + g)'(x) = f'(x) + g'(x)$ . (Sum rule)
- (c)  $(f g)'(x) = f'(x) g(x) + f(x) g'(x)$ . (Product rule)
- (d)  $\frac{d(x^n)}{dx} = n x^{n-1}$



When we ask to the calculator to differentiate something undefined, it tends to indicate what it will do as far as it can. Effectively the “rules” above allow us to differentiate any polynomial. With a little practice, you can write down the derivative of a polynomial much faster than you can type in the original polynomial. You can use the calculator to check while you are first learning, but as soon as possible, you will want to loose your dependency upon the tool to do simple derivatives.

Certainly the calculator knows and can implement the quotient rule and the chain rule. It can actually “show you” the abstract quotient rule (using undefined functions) but it needs to know the “outer function” to proceed with the chain rule.

**Theorem** *If  $f$  and  $g$  are a real-valued functions defined on appropriate subsets of  $\mathbb{R}$  and differentiable at  $t$ , then*

(a)  $D\left(\frac{f}{g}\right)(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$  (Quotient Rule)

(b)  $D(f \circ g)(x) = D(f(g(x))) = f'(g(x))g'(x)$  (Chain Rule)

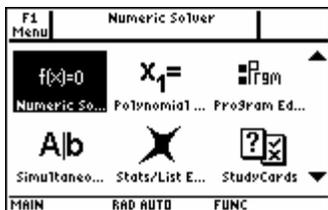
▪ NewProb	Done
▪ $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$	$\frac{\frac{d}{dx}(f(x))}{g(x)} - \frac{\frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}$
▪ <code>comDenom(ans(1))</code>	$\frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}$
▪ $\frac{d}{dx}(f(g(x)))$	$\frac{d}{dx}(f(g(x)))$
▪ $\frac{d}{dx}(\sin(g(x)))$	$\cos(g(x)) \cdot \frac{d}{dx}(g(x))$

## 5. Implicit Functions and Plotting

Up until now, you have tended to only work with functions for which you have a simple explicit formula (such as can be easily typed into the Y= Editor). We now start to work with functions which may not have such a simply-found formula. Consider the set of all pairs  $(x, y)$  in the plane satisfying the equation

$$x^3 + y^3 = 6xy.$$

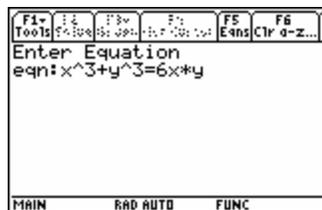
You can easily verify that the pair  $(3, 3)$  is in this set. We wish to claim that if you specify an  $x$ -value near 3, then there is a corresponding  $y$ -value (also near 3) that still solves the equation. We can use the numerical solver to see how this works for a few values.



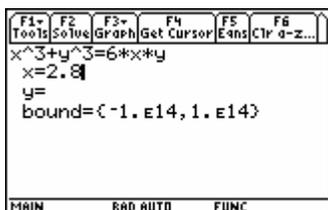
APPS Numeric Solver  
(APPS Desktop ON)



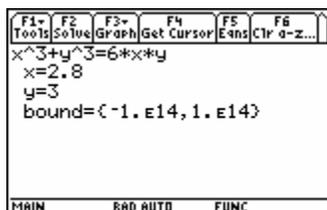
APPS Numeric Solver  
(APPS Desktop OFF)



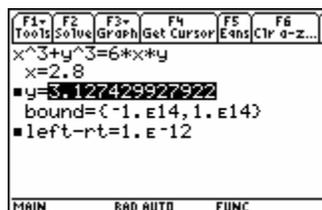
Enter equation



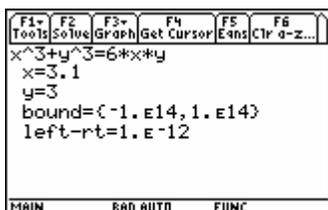
Enter desired  $x$ -value



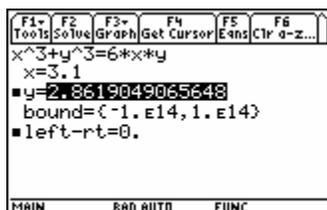
Enter “seed”  $y$ -value



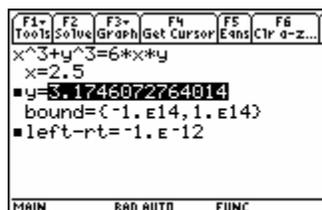
Press F2 Solve



Enter  $x$ -value and “seed”



Press F3 Solve



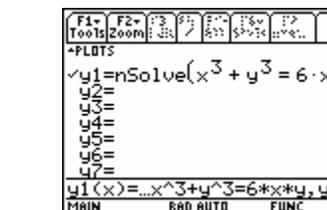
Once more

We say that the equation  $x^3 + y^3 = 6xy$  implicitly defined a function near the solution pair (3, 3).

What we want to do now is to get a plot of this function on the calculator. The trick is to make use of a command line version of this numeric solver, which can be found in the catalog.

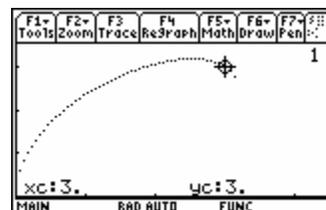


Command in CATALOG



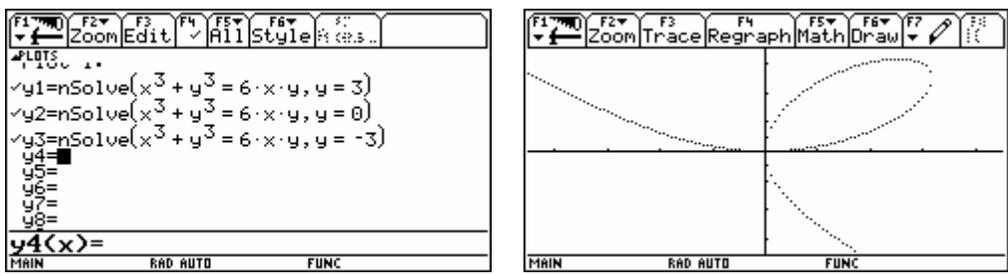
“Seed” is last argument as “ $y=3$ ”  $0 \leq x \leq 4.5, 0 \leq y \leq 3.5$ ,

Press TRACE, type  $x = 3$



It is strongly suggested that you do not make your initial viewing window too large. Also since it takes a very long time to do this plotting, select  $xres = 2$  or  $3$  to reduce the number of points plotted (and the time). [You must turn “Discontinuity detection” OFF in OS 3.0 or higher to get this  $xres$  setting option.] Finally select the DOT style because this function might not be defined for the whole  $x$ -range given (the one above is not), and the LINE style might “drop down” to another implicit function defined by the same equation.

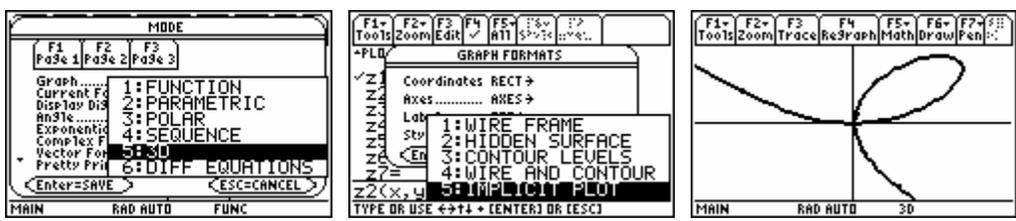
To get a larger plot of the solution set for this equation, we expand the viewing window a little and put in some additional numeric solver commands with different seeds.



Voyage 200 screen, Y= Editor  $-4.5 \leq x \leq 4.5, -3.5 \leq y \leq 3.5, xres = 2$  (plots slowly!)

From the plot, we can guess that for  $x = 3$  there will actually be three different solutions for possible  $y$ -values which solve the equation. We can go back to the interactive numeric solver and try these alternate seeds to find that  $y = 3$  or about  $1.8541019662496$  or  $-4.8541019662499$ .

The TI-89 has implicit plotting as a style in 3D graphing mode, but you cannot easily add the tangent line or trace along the curve (as we will want to do in the next section), so the above seems better for a Calculus 1 class.



Select 3D graphing  $z1=x^3+y^3-6x*y$  F1 Format Style  $-4.5 \leq x \leq 4.5, -3.5 \leq y \leq 3.5$

### 6. Implicit Differentiation

Once you get a rough understanding of implicitly-defined functions, then you can ask for how we can find the derivative for such a function. It turns out that this usually ends up to be implicitly-defined as well. For illustration purposes, we go through the steps for what is called implicit differentiation for  $x^3 + y^3 = 6xy$ .

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

$$(3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{3(2y - x^2)}{3(y^2 - 2x)} = \frac{2y - x^2}{y^2 - 2x}$$

In particular, since we know that  $(3, 3)$  solves the original equation (giving us the “top” function in our plot), then we can compute the derivative there.

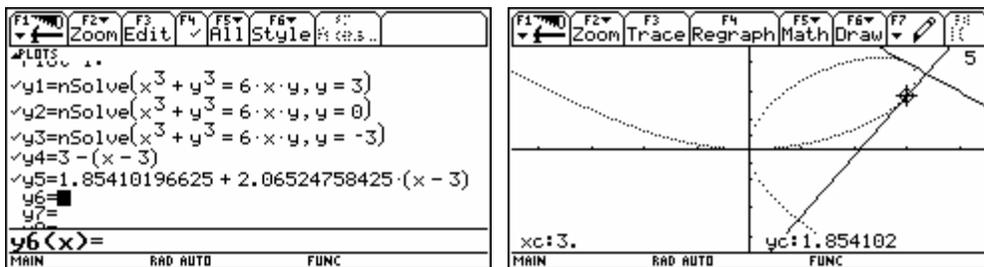
$$\left. \frac{dy}{dx} \right|_{x=3, y=3} = \frac{2(3) - (3^2)}{(3^2) - 2(3)} = -1$$

In a similar fashion, we can find the derivatives at the other solutions corresponding to  $x = 3$ .

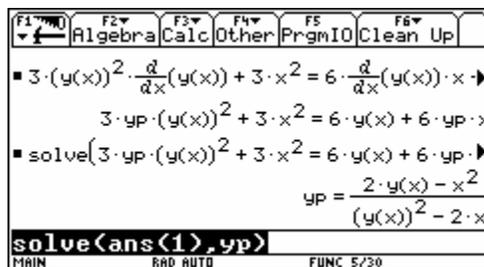
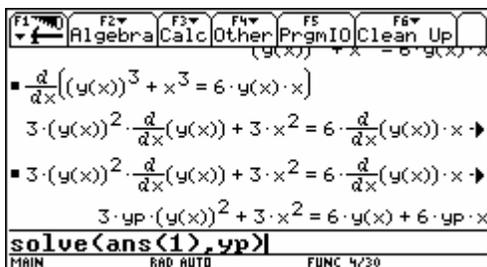
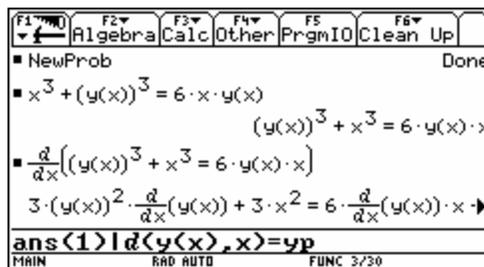
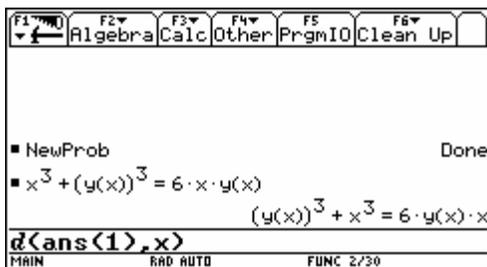
$$\left. \frac{dy}{dx} \right|_{x=3, y=1.8541019662496} = \frac{2(1.8541019662496) - (3^2)}{((1.8541019662496)^2) - 2(3)} = 2.06524758425$$

$$\left. \frac{dy}{dx} \right|_{x=3, y=-4.8541019662499} = \frac{2(-4.8541019662499) - (3^2)}{((-4.8541019662499)^2) - 2(3)} = -1.06524758425$$

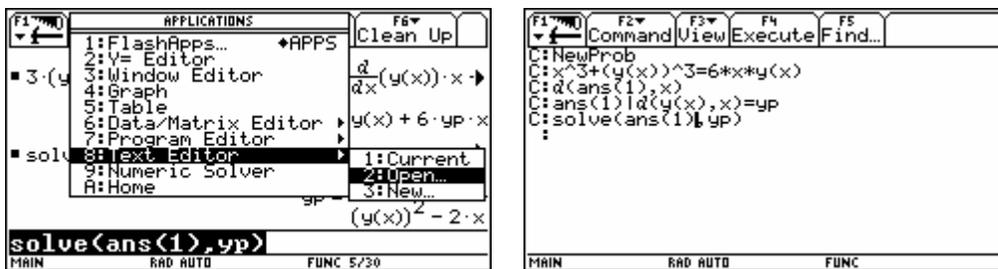
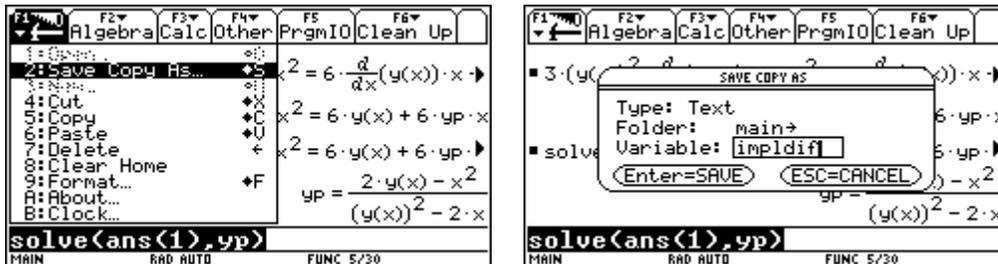
We now add the tangent lines (for the “top” function and the “middle” function) to the plot we have of this curve.



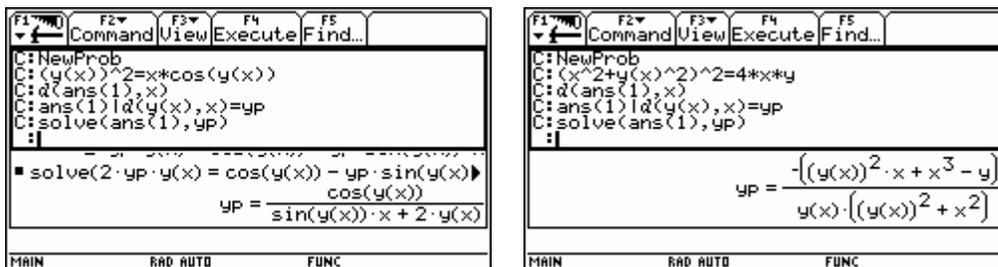
Although it might be more trouble than it is worth, it is possible to get the calculator to do symbolic implicit differentiation. At least you can use it to confirm some of your hand work as you are first learning to do implicit differentiation. To accomplish this, we need to convince the calculator that  $y$  is some unknown function of  $x$ . We do this by replacing the single letter  $y$  by the notation  $y(x)$ . The result of differentiating such an equation has the derivative of this unknown function  $y(x)$  in it. Unfortunately we can “solve” for such an unknown derivative in the result. However we can substitute a new variable name (we use  $yp$  below) for this derivative. Then we solve for  $yp$ .



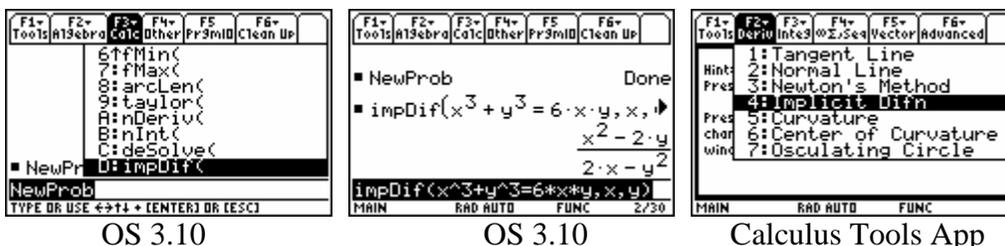
A tricky process like the above steps for implicit differentiation might be something that you want to save in a text file. Notice that everywhere that we used the previous answer, the actual answer has been inserted in the command as it appears in the history. You will have to edit the text file to put the command ANS(1) back in. Then you can simply edit the original equation and run the text file as a script to perform another implicit differentiation.



To work a new problem, (1) edit the original equation (second line of the text file), (2) move the cursor up to the first line, (3) press F3 View 1:Script view to split the screen to see both the text file at the top and the home screen at the bottom, and (4) press F4 Execute four times to complete the implicit differentiation.



Note that OS 3.10 and later now includes a direct command for implicit differentiation. For older calculators, you can also find implicit differentiation in the free Calc Tools application.



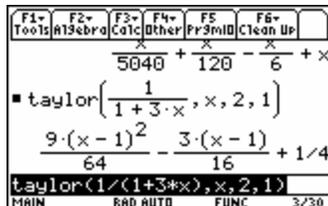
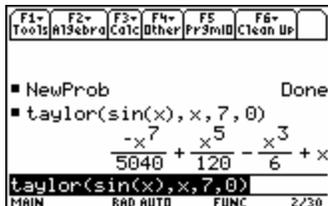
OS 3.10

OS 3.10

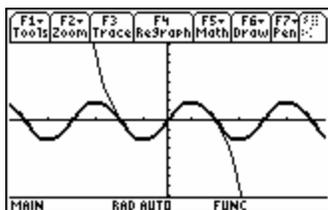
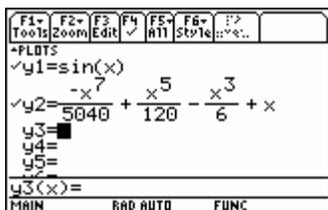
Calculus Tools App

## 7. Taylor Polynomials

There is a nice command to generate a Taylor polynomial of any order for a function centered at any point.



To improve the speed of plotting, generate the polynomials in the home screen, copy and paste the results in the Y= Editor for graphing. If you put the taylor command in the Y= Editor, the polynomial is re-generated again for every point plotted (making plotting extremely slow).



F2 Zoom 7:ZoomTrig

## Chapter 3

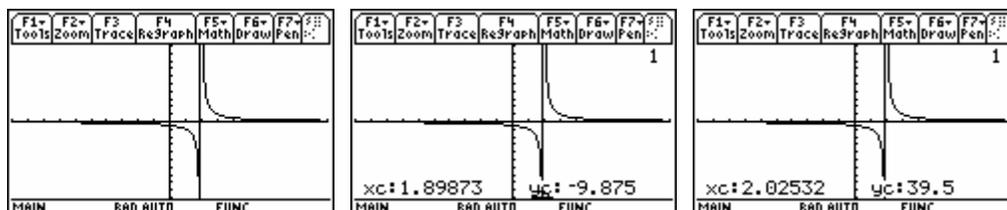
### Applications of Derivatives

For the activities in this chapter, the various graphing capabilities of the calculator will be used. This is loosely coordinated with Chapter 3 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

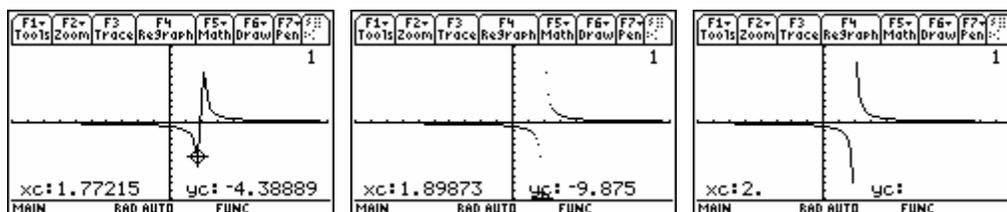
#### 1. Zooming to See Asymptotes

Asymptotes can roughly be described as observing how the graph of a function behaves as something goes to infinity. Since the viewing window is always finite in every direction, calculator and computer graphs have a difficult time accurately representing either vertical or horizontal asymptotes. In particular, the two pieces of a vertical asymptote (where the function is not continuous) may be connected by a near vertical line segment which makes the plot appear continuous. Remember that in the default style (line), the calculator plot evaluates the function at a finite number of points, plots these points in the graph, and then connects the points (in order) with line segments. If the two plotted points lie on different sides of a vertical asymptote, then we would like to remove the line segment.

Consider  $f(x) = \frac{1}{x-2}$  for viewing window  $-10 \leq x \leq 10$ ,  $-10 \leq y \leq 10$  (ZoomStd).



If we press F3 Trace and move with the cursor keys to the plotted point with the largest  $x$  less than 2, we find the plotted point  $(1.89873, -9.875)$  near the bottom of the viewing window. Moving with the right cursor key once to the smallest  $x$  greater than 2, we find the plotted point  $(2.02532, 39.5)$  well above the viewing window. The near vertical line in the plot attempts to connect these two points. All of the above plots have  $xres = 1$ . Making  $xres$  larger than 1 can give a plot looking even worse with the connecting line segment looking less vertical (see below). [Discontinuity detection in OS 3.10 eliminates the need to use  $xres$  and dot style here.]



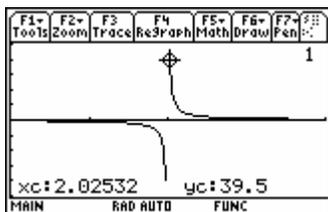
$xres = 3$  Style Line

$xres = 1$  Style Dot

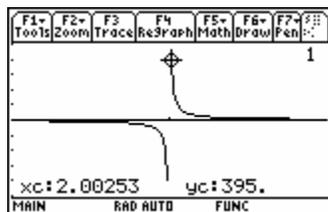
$-8 \leq x \leq 12, -10 \leq y \leq 10$

Notice above how the Style Dot removes all of the line segments between plotted points on the graph. I usually find this less satisfying (but it certainly removes the “false” near vertical line segment we do not want as well as all others that we would really still like to have). If the plot actually attempts to plot at  $x = 2$ , as it will do in the window  $-8 \leq x \leq 12, -10 \leq y \leq 10$ , then it find a place where the function is undefined (see above). No connecting line segments will be drawn across such a place where the function is known to be undefined.

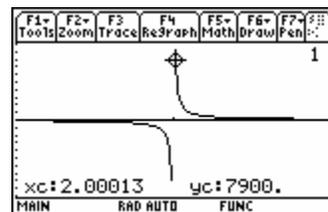
The most convincing way to visualize a vertical asymptote is to zoom in near the  $x$ -value where this happens at the same zooming out to see larger and larger  $y$ -values. In a similar fashion, we need to zoom out along the  $x$ -axis and zoom in along the  $y$ -axis to better understand a horizontal asymptote.



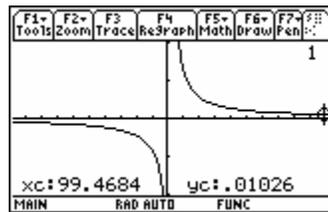
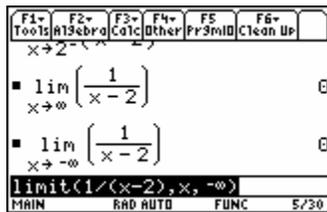
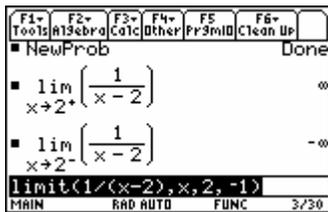
$$0 \leq x \leq 4, -50 \leq y \leq 50$$



$$1.8 \leq x \leq 2.2, -500 \leq y \leq 500$$

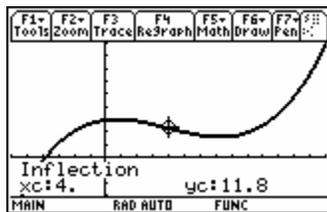
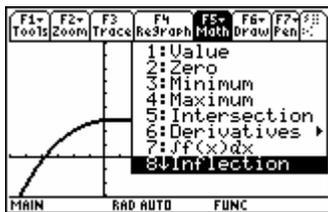


$$1.99 \leq x \leq 2.01, -10,000 \leq y \leq 10,000$$

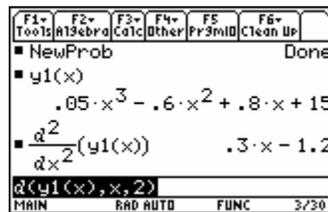


$$-98 \leq x \leq 102, -0.2 \leq y \leq 0.2$$

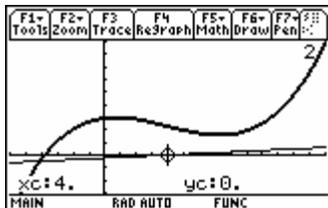
## 2. Points of Inflection



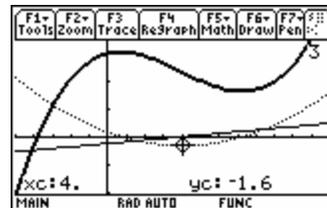
$$-6 \leq x \leq 14, -15 \leq y \leq 45$$



We can find points of inflection numerically and graphically in the graph screen, and we can work with the formulas symbolically in the home screen to solve exactly for where the second derivative is zero.



$$-6 \leq x \leq 14, -15 \leq y \leq 45$$

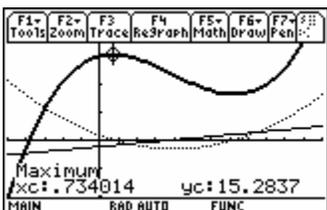
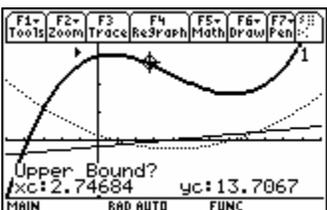
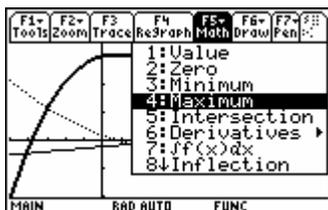


$$-5 \leq x \leq 12, -10 \leq y \leq 17$$

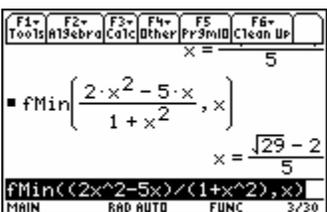
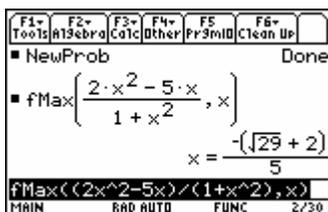
Further we can plot the function together with its derivative and/or its second derivative to help visualize the sign of the second derivative near the  $x$ -value giving a point of inflection on the graph is related to concavity.

### 3. Maxima and Minima

The graphical and numerical work with maxima and minima can take place in the graph screen.

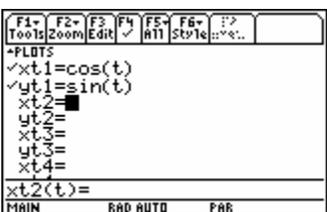


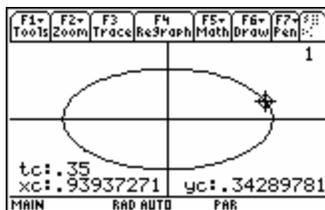
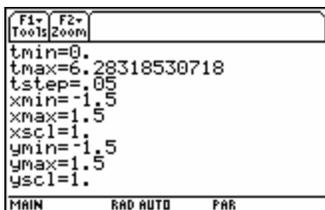
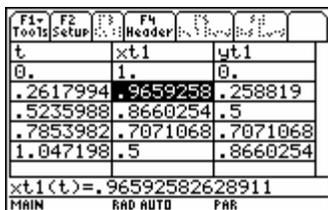
Symbolic work can take place in the home screen.



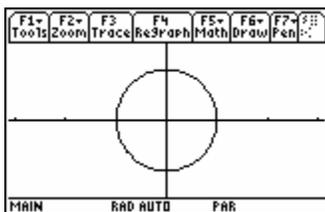
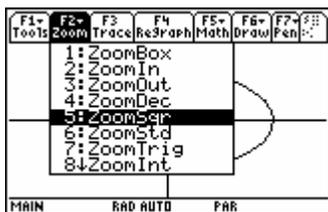
### 4. Parametrization of the Plane

For functions from  $\mathbb{R}$  to  $\mathbb{R}^2$ , we can use Parametric graphing to show the *image*. In Parametric graphing, the graphing variable is  $t$ .

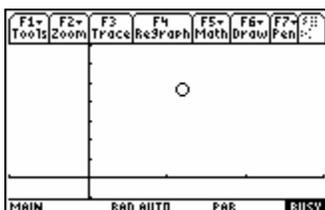
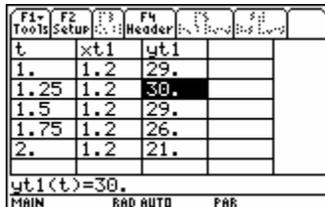
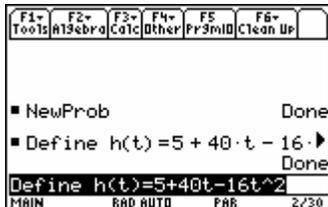




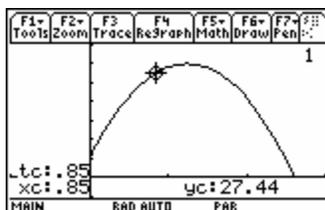
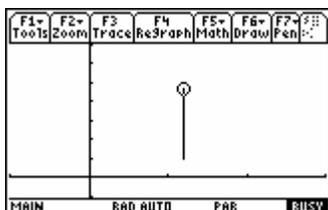
Often when plotting a two-dimensional image, we might be happier using a plot with equally-scaled axes. The F2 Zoom 5:ZoomSqr command will widen the range of either the  $x$ -axis or the  $y$ -axis so that we see everything we saw before only in window with equally-scaled axes. Then this plot will look like a circle.



We can also use parametric graphing to simulate some motion problems. Suppose that  $h(t) = 5 + 40t - 16t^2$  describes the height of a ball (in feet) thrown straight up at time  $t = 0$  after  $t$  seconds. We can provide an animation of the motion, the image of  $h$ , and the graph of  $h$ . Notice we have set  $tmax$  below to correspond with the time when the ball hits the ground.



Animation (part way up)



Path (partially drawn)

Graph of  $h$

## 5. Newton's Method

A very common task is to solve the equation  $f(x) = 0$ . We call the solutions the *zeros* or the *roots* of the function  $f$ . We have already considered in the introductory chapter how to solve equations symbolically in the home screen, numerically in the numeric solver application, and numerically and graphically in the graph screen. The method explored here is a numerical method that has a long history. It shows a very nice application of the concept of the derivative and the concept of continuity. While the built-in numeric solver is a little more sophisticated than these, it works in a similar fashion.

We certainly could implement Newton's method in a program (and you are encouraged to do so if you chose). This is more of an educational activity than a truly needed process (since the numeric solver does this very easily). Thus we will look at how to do this in the home screen, and then we will save our work in a text file for later reference. Key to the success of the method is a *good* starting value  $c_0$  (perhaps you should look at a graph of  $f$  if this is not a homework or test question with a specified starting value). The iterative formula for the next approximation to

the zero is  $c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)}$ .



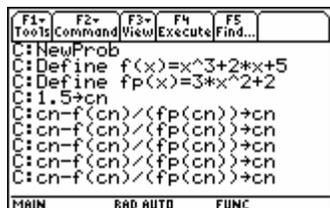
Just press ENTER

Define the function with one variable name and the derivative of the function with another variable name. If you need to do so, you could let the calculator find the derivative symbolically. Make sure the Exact/Approx Mode is set to "AUTO" and that you type the starting value using a decimal point so that you get floating-point numbers (rather than exact results). Usually we will also want to see lots of digits, so the Display Digits Mode set to "FLOAT" is nice. After you have typed the command line " $cn - f(cn)/fp(cn) \rightarrow cn$ " once, you can simply press ENTER repeatedly to have the same command redone (each time using the most recently stored value for the variable "cn"). When things converge, you will eventually get virtually no change in the result. (*No change* does not guarantee that you have an exact zero, but it does imply that the "correction" in the next step of Newton's method is so small that we cannot see a change.)

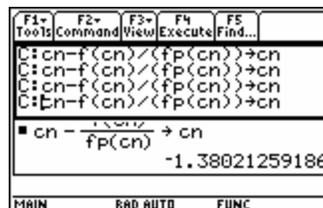


We save the work in the history section by the command F1 2: Save Copy As. The dialog box will ask you to provide a name for the resulting text file. Here we give it the name “newton”. This process will save the work, even if you clear the history area.

Press the APPS key and select the Text Editor. Choose to open the text we named “newton”. In this Text Editor, we can edit the file to change the starting value or to change the function and its derivative. Press F3 View to choose a Split view so that we can see both this text file (at the top) and the home screen (at the bottom). Move the cursor to some command line (with a beginning C:) and press F4 Execute to paste that command into the home screen and have it executed.



Respond with ESC



Repeatedly press F4

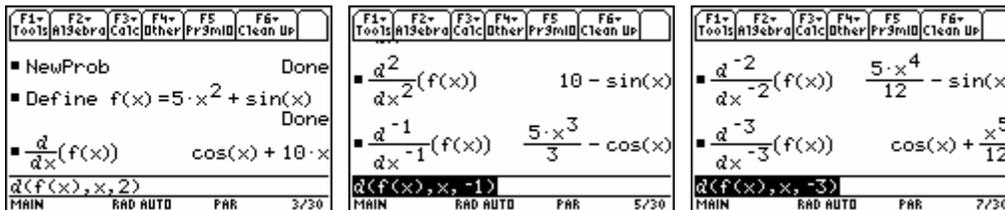
We finish this section by noting that some of the features and characteristics of Newton’s method are present in the numeric solver available as an application. If you type a “good” starting value into the variable in the numeric solver before pressing F2 Solve, you effectively tell the internal algorithm to start its iteration with the given value. This can help it to converge more rapidly, much as in Newton’s method. There is also the option to specify an interval (the default bound ={-1.E14,1.E14} is effectively no bound at all). The internal algorithm stays within the interval given.

# Chapter 4 Integration

For the activities in this chapter, the various graphing, numerical, and symbolic capabilities of the calculator will be used. This is loosely coordinated with Chapter 4 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

## 1. Two Notations for Antiderivatives

Interestingly, there are two different ways that we can get the TI-89 to give a symbolic antiderivative. The first way we present here relates naturally to the command for finding higher order derivatives. Recall  $d(f(x),x,2)$  is the command for getting the second derivative. The last argument in this command is the order of the derivative desired. (If you leave off this last argument, you get a first derivative by default.) Simply use order  $-1$  to get the antiderivative. Other negative integers give repeated antidifferentiation. Notice the interesting (and logical) notation for an antiderivative in the history area. Also note that we get only one antiderivative (and not the family of all possible antiderivatives).



The textbook notation for an antiderivative is actually the most common. It relates to the relationship we will soon learn between definite integrals and antiderivatives. This command notation (called integration) is also available. The commands to differentiate and integrate appear on the keyboard (as 2<sup>nd</sup> functions above 8 and 7) and in the F3 Calculus menu.



Using the integration command with only the first two arguments gives an antiderivative (not the family of all possible antiderivatives). Giving an undefined variable name (as we did above with "c") will give an antiderivative with this undefined variable name added to the result. This provides a way for us to represent the whole family of all antiderivatives. Later in the chapter, you will learn the significance of further arguments in the integration command ([LO,UP] in the catalog help line above).

## 2. Riemann Sums and the Sigma Notation

Definite integrals are defined by means of a limit of Riemann sums, and one of the ways to approximate a definite integral is to use some kind of Riemann sum. There are several ways to compute Riemann sums on the TI-89/Voyage 200 family of calculators.

A general Riemann sum approximating  $\int_a^b f(x)dx$  is defined in terms of a partition  $\{x_i\}$  of the interval  $[a, b]$  into subintervals  $[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$  and in terms of a selection of evaluation points  $\{s_i\}$  with  $s_i \in [x_{i-1}, x_i]$  for  $i = 1, 2, \dots, n$ . Namely,

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(s_i)(x_i - x_{i-1}).$$

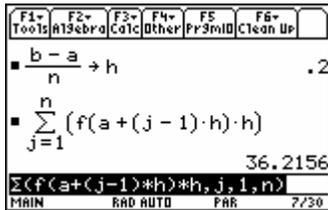
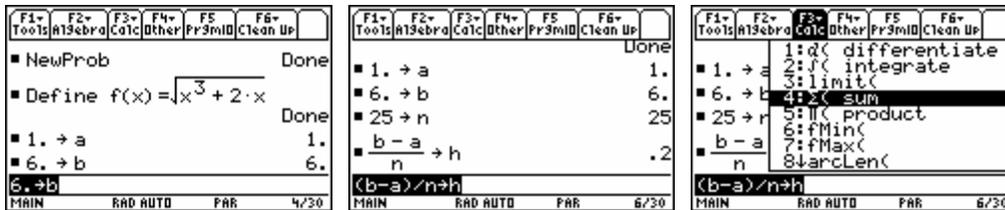
Each term in the sum can be interpreted as the area of a rectangle with width  $(x_i - x_{i-1})$  and “signed” height  $f(s_i)$ . Special cases include selecting  $s_i = x_{i-1}$  (left endpoints),  $s_i = x_i$  (right endpoints), or  $s_i = \frac{x_{i-1} + x_i}{2}$  (midpoints). Often the partition points  $\{x_i\}$  are equally spaced across the interval, giving  $(x_i - x_{i-1}) = \Delta x$  for all  $i = 1, 2, \dots, n$ . First we show how to compute a general Riemann sum (no special assumptions) using lists.

Start by storing the formula for  $f(x)$ . Store the partition points  $\{x_i\}$  in a list, say I1. (Note that variable names must begin with a letter, and so the first character in the name of this list is the letter “I”, not the number “1”.) Store the selection points  $\{s_i\}$  in another list, say I2. Then take advantage of the list operations to compute the Riemann sum in the HOME screen. Applying a function to a list gives a list of evaluations. The command  $\Delta$ List computes a list of the differences (i.e.  $(x_i - x_{i-1})$ ). Multiplying two lists of equal length, gives a list where like terms in the two lists have been multiplied. Finally the sum command sums the terms in a list.

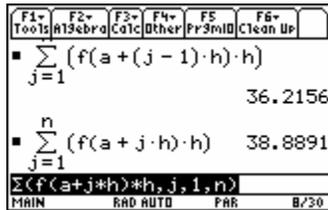
The screenshots illustrate the following steps:

- Define the function:** `Define f(x)=sin(x^2)` is entered and stored.
- Store partition points:** The list `{0, .2, .3, .5, .75, .9}` is stored in list I1.
- Store evaluation points:** The list `{.1, .22, .4, .6, .8}` is stored in list I2.
- Compute differences:** The command `ΔList(I1)` is used to create a list of subinterval widths: `{.2, .1, .2, .25, .15}`.
- Compute function values:** The command `f(I2)` is used to evaluate the function at the selected points, resulting in `{.01, .048381, .159318}`.
- Compute the sum:** The command `sum(ΔList(I1)*f(I2))` is used to sum the products of the widths and function values, yielding the final result `.297158`.

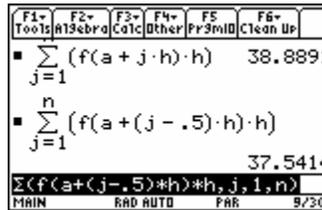
If you desire an equally spaced partition with special evaluation points (left endpoints, right endpoints, midpoints), there are simpler ways to have the calculator compute these.



Left endpoints

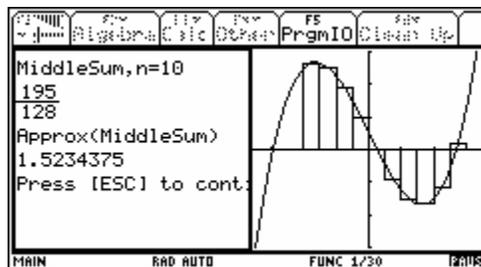
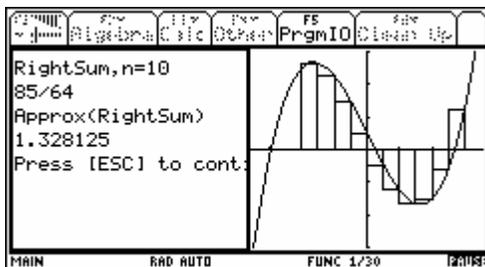
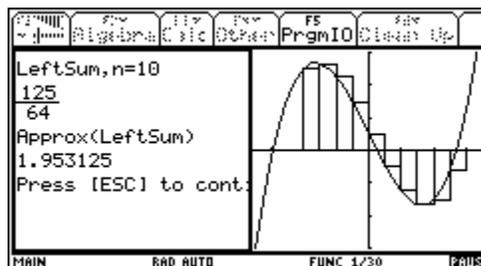
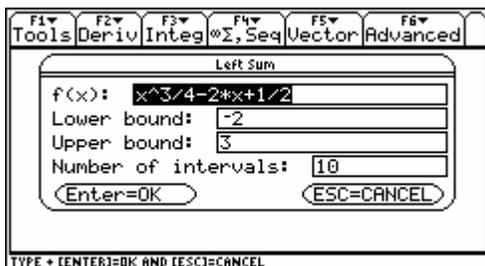


Right endpoints



Midpoints

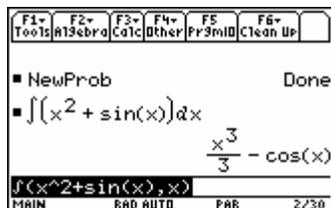
The free Flash App called Calculus Tools has nice implementations of these Riemann sums as well.



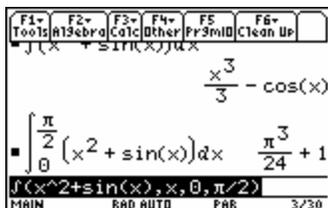
### 3. Symbolic Integrals

The Fundamental Theorem of Calculus shows how definite integrals and antiderivatives are related. Thus another name used for an antiderivative is *indefinite integral*. We have already

seen this integration command used to find antiderivatives in the first section of this chapter. The same command will find definite integrals if we specify the interval of integration. Make sure the Exact/Approximate mode is set to AUTO or EXACT so that we can see exact integration answers.

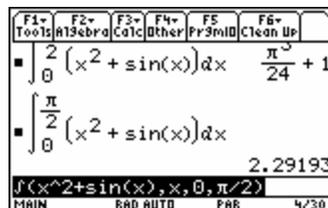


Indefinite Integral



Exact Definite Integral

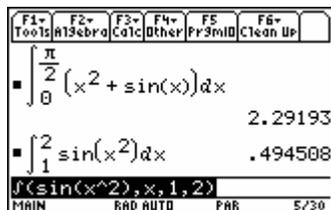
[Press ENTER.]



Approximate Definite Integral

[Press  $\blacklozenge$  ENTER.]

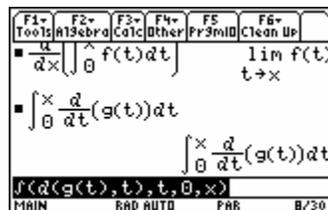
Thus you can use the calculator to check your hand work using the Fundamental Theorem of calculus. Usually the calculator can do a symbolic problem if you can do it by hand. If the integrand in a definite integral does not have an elementary antiderivative (or the calculator cannot find one), the AUTO mode will automatically switch to a numerical approximation (using a method a little more accurate and efficient than Riemann sums).



[Press ENTER.]



Using unknown  $f$



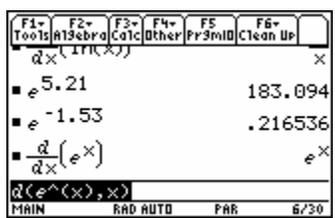
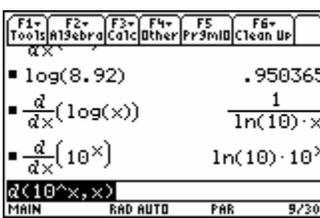
Using unknown  $g$

## Chapter 5 Techniques of Integration

Most of the activities in this chapter involve functions where you really need the calculator to evaluate the functions. This is loosely coordinated with Chapter 5 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

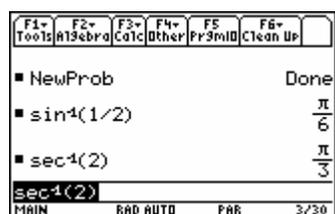
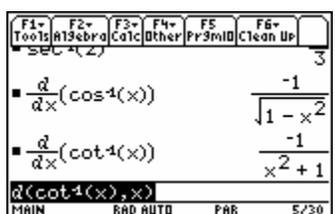
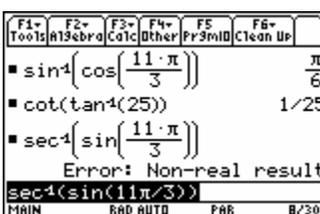
### 1. Logarithmic and Exponential Functions

The TI-89/Voyage 200 offers the natural logarithm function and the natural exponential function on the keyboard. Many people do not know that the common logarithm function is also available in the CATALOG (or you can just type log). [In the newer OS versions, you can even get logarithms to other bases by typing “log(x,b)”, where  $b > 0, b \neq 1$  is the desired base. However you seldom have a need for this feature in Calculus.]

		
Base $e$	Base $e$	Base 10

### 2. Inverse Trigonometric Functions

If you have upgraded your OS to at least 2.08 or higher, then you have not only all of the trigonometric functions, but also all of the inverse trigonometric functions as well. The ones not printed on the keyboard can be found in the CATALOG.

		
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Notice that you will get the error message “Non-real result” if you get outside of the domain of some of the inverse trigonometric functions (assuming your MODE setting for Complex Format is REAL). All of the trigonometric functions have generalizations to domains of complex numbers, and the calculator is programmed to handle when the Complex Format is selected to something other than REAL. You may use these functions using complex numbers in a later mathematics or physics or engineering course.

### 3. Calculator Substitutions

You may be able to get the calculator to do some of the algebraic work involved in completing a substitution.

Calculator screen showing the definition of  $g = \frac{\sqrt{x^2 - 9}}{x} \cdot dx$  and the substitution  $x = 3 \cdot \sec(\theta)$  and  $dx = \frac{d}{d\theta}(3 \cdot \sec(\theta)) \cdot d\theta$ . The screen also shows the expression  $3 \cdot \tan(\theta) \cdot \left| \frac{\sin(\theta)}{\cos(\theta)} \right| \cdot d\theta$ .

Calculator screen showing the definition of  $k = \frac{1}{4 \cdot \sin(x) - 3 \cdot \cos(x)} \cdot dx$  and the substitution  $x = 2 \cdot \tan^2(u)$  and  $dx = \frac{d}{du}(2 \cdot \tan^2(u)) \cdot du$ . The screen also shows the expression  $(u^2 + 1) \cdot (3 \cdot \cos(2 \cdot \tan^2(u)) - 4 \cdot \sin(2 \cdot \tan^2(u))) \cdot (-2 \cdot du)$ .

Calculator screen showing the expansion of the expression  $(u^2 + 1) \cdot (3 \cdot \cos(2 \cdot \tan^2(u)) - 4 \cdot \sin(2 \cdot \tan^2(u))) \cdot (-2 \cdot du)$  into  $3 \cdot u^2 + 8 \cdot u - 3$ .

Calculator screen showing the integration of the expression  $\int \left( \frac{1}{4 \cdot \sin(x) - 3 \cdot \cos(x)} \right) dx$  resulting in  $-\ln \left( \frac{3 \cdot \cos(x) + \sin(x) + 3}{-\cos(x) + 3 \cdot \sin(x) - 1} \right) + C$ .

### 4. Algebra with Rational Functions

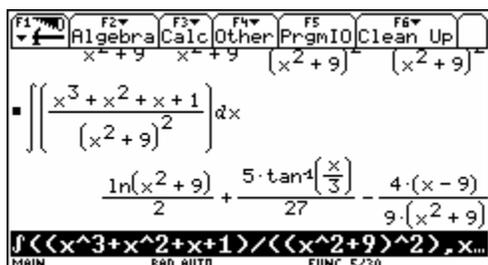
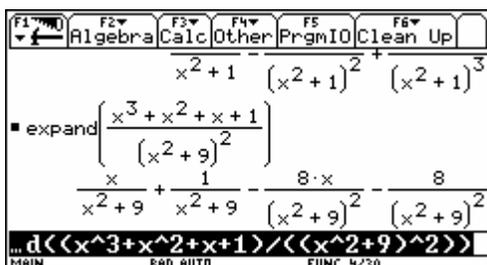
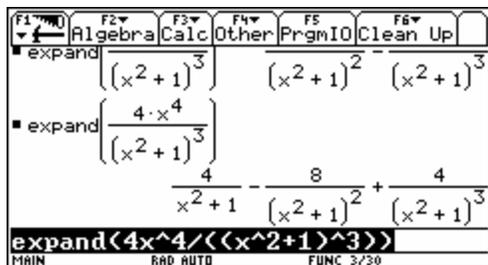
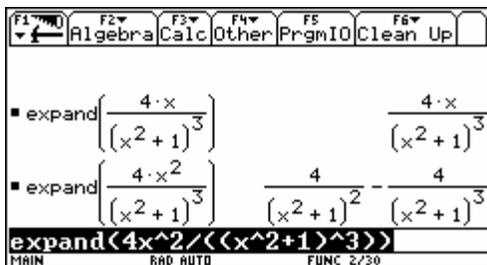
When integrating a rational function, the first step is to perform “long division” of the numerator by the denominator to convert the task into a problem of integrating a polynomial plus a rational function with the degree of the numerator strictly less than the denominator. Then you use partial fractions on the last piece. The F2 Algebra commands 3:expand( and 7:propFrac( may do some this algebraic work.

Calculator screen showing the expansion of the rational function  $\frac{5 \cdot x^4 + 2 \cdot x}{x^3 - 6 \cdot x^2 + 3 \cdot x + 10}$  into  $\frac{1}{6 \cdot (x + 1)} - \frac{28}{3 \cdot (x - 2)} + \frac{1045}{6 \cdot (x - 5)} + 5 \cdot x + 30$ .

Calculator screen showing the partial fraction decomposition of the rational function  $\frac{5 \cdot x^4 + 2 \cdot x}{x^3 - 6 \cdot x^2 + 3 \cdot x + 10}$  into  $\frac{1}{6 \cdot (x + 1)} - \frac{28}{3 \cdot (x - 2)} + \frac{1045}{6 \cdot (x - 5)} + 5 \cdot x + 30$ .

Calculator screen showing the integration of the expanded rational function  $\int \left( \frac{5 \cdot x^4 + 2 \cdot x}{x^3 - 6 \cdot x^2 + 3 \cdot x + 10} \right) dx$  resulting in  $\frac{\ln(x + 1)}{6} - \frac{28 \cdot \ln(x - 2)}{3} + \frac{1045 \cdot \ln(x - 5)}{6} + 5 \cdot \frac{x^2}{2} + 30 \cdot x + C$ .

Calculator screen showing the integration of the partial fraction decomposition of the rational function  $\int \left( \frac{5 \cdot x^4 + 2 \cdot x}{x^3 - 6 \cdot x^2 + 3 \cdot x + 10} \right) dx$  resulting in  $\frac{3 \cdot \ln(x - 2)}{3} + \frac{1045 \cdot \ln(x - 5)}{6} + \frac{5 \cdot x^2}{2} + 30 \cdot x + C$ .

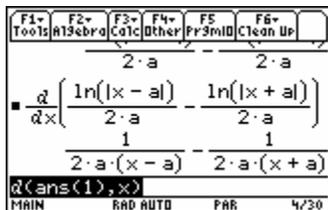
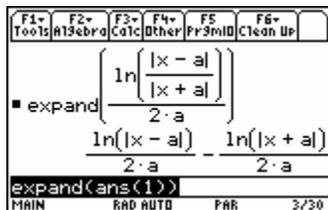
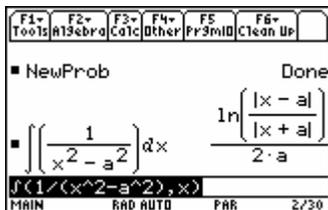


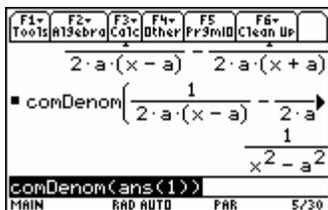
Since the calculator can do symbolic integration so easily, there seems to be no need for partial fractions and then hand integration of the pieces. However this technique of expanding a rational function into its partial fraction decomposition has other uses besides integration, and you should learn about it. For example, electrical engineers using the Laplace transform will find the technique of partial fractions useful for finding inverse transforms.

## 5. Comparing Symbolic Integration to Tables of Integrals

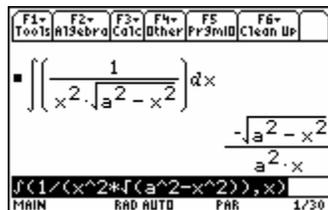
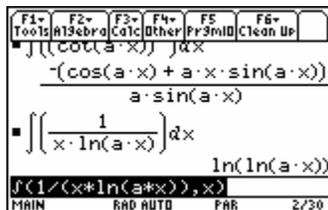
Our text presents a short table of integral (and many others do as well). Before computer algebra systems were available, books were printed with very extensive tables of known indefinite and definite integrals. Now is more common to rely on the expertise of a computer algebra system (or CAS for short). The TI-89/Voyage 200 OS can do essentially all of the integrals that can be handled by using the table in the text. It is useful to compare how the results might look a little different.

Different computer algebra systems can give different looking results for integration problems (and these results may look very different from a table). Consider  $\int \frac{dx}{x^2 - a^2}$ . Here we try it on the TI-89. Notice we can always differentiate the result to check.





Here we try a few more.



Some of the formulas in a table do not give a final answer, but rather reduce the task to another in the table or a simpler integral of the same type. For example, many tables give this reduction formula for powers of the sine function.

$$\int \sin^m u \, du = -\frac{1}{m} \sin^{m-1} u \cos u + \frac{m-1}{m} \int \sin^{m-2} u \, du$$

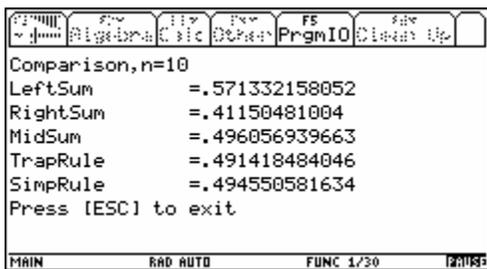
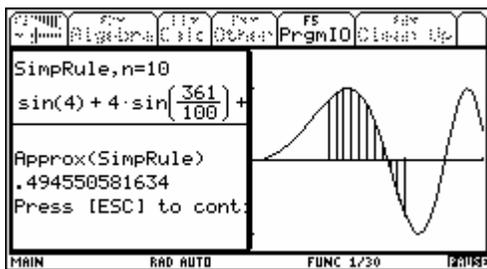
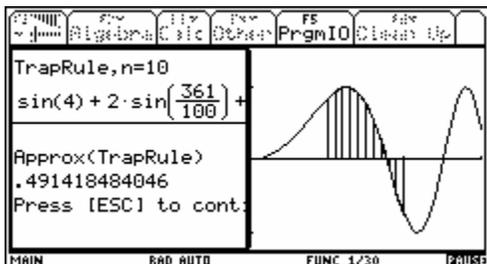
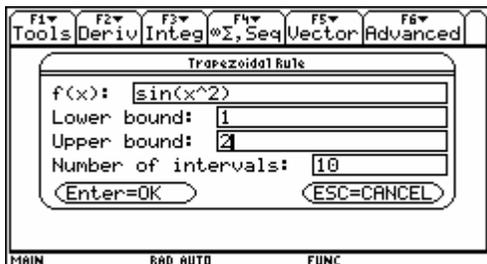
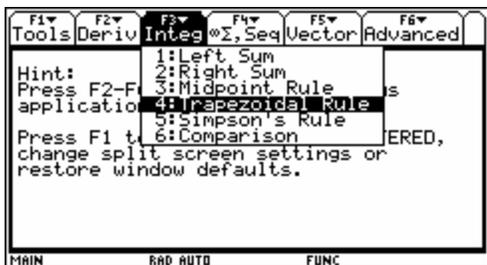
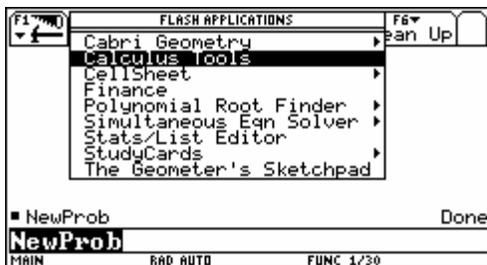
Used repeatedly, we eventually can complete the task of integrating any power (with the help of some other formulas in the table). The best that we can do to “test out” the formula on the calculator is to try specific integers, such as  $n = 4$  and  $5$ . Because the results are two long to fit in the screen without scrolling, we reproduce them here nicely typeset.

$$\int ((\sin(x))^4) dx = \left( \frac{-(\sin(x))^3}{4} - \frac{3 \cdot \sin(x)}{8} \right) \cdot \cos(x) + \frac{3 \cdot x}{8}$$

$$\int ((\sin(x))^5) dx = \left( \frac{-(\sin(x))^4}{5} - \frac{4 \cdot (\sin(x))^2}{15} - 8/15 \right) \cdot \cos(x)$$

## 6. Trapezoidal Rule and Simpson’s Rule

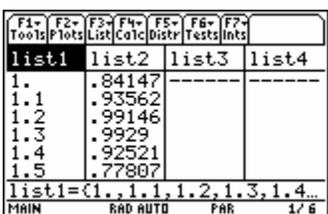
If you happen to have the free Flash App called Calculus Tools, then it includes an implementation of the Trapezoidal Rule and Simpson’s Rule for evaluating a definite integral. In the text, these are called the Composite Trapezoidal Rule (Theorem 3) and the Composite Simpson’s Rule (Theorem 5).



If you do not have this flash application, then it is easy to write short programs to do these computations.

It is possible to use these ideas to approximate antiderivatives numerically. This corresponds to one step of the trapezoidal rule (Theorem 1) and one step of Simpson's rule (Theorem 4). Perhaps it will be easiest if we think of a situation where the integrand  $f(x)$  is known only by a table of discrete values. This is often the case for scientific data. If you happen to have a formula for the function  $f(x)$ , then you can easily create a discrete list of this form.

For example, consider  $f(x) = \sin(x^2)$  for  $1 \leq x \leq 2$ . The choice of  $n = 10$  we made above corresponds to  $h = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$ . The command `seq(1+j*0.1,j,0,10)` will generate a list of the desired  $x$ -values on this interval  $[1, 2]$ . Taking  $f$  of this list will give the desired evaluations.



View in Stats/List Editor (free Flash App)

This information will most often be displayed in a book or on your paper in a table.

$x$	$f(x)$
1.0	0.841471
1.1	0.935616
1.2	0.991458
1.3	0.992904
1.4	0.925212
1.5	0.778073
1.6	0.549355
1.7	0.248947
1.8	-0.098249
1.9	-0.451466
2.0	-0.756802

We are interested in approximating  $\int_1^x f(t) dt$  for the same set of  $x$ -values as used in the table above, using only the numbers available in the table. This is easy to do for  $x = 1$  because

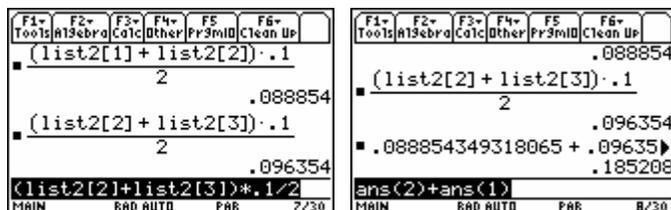
$$\int_1^1 f(t) dt = 0. \text{ For } x = 1.1, \text{ we use one trapezoid to approximate } \int_1^{1.1} f(t) dt \approx \frac{f(1) + f(1.1)}{2} (0.1)$$

and get 0.088854. In a similar fashion, we can use another trapezoid to approximate

$$\int_{1.1}^{1.2} f(t) dt \approx \frac{f(1.1) + f(1.2)}{2} (0.1) \approx 0.096354. \text{ Thus to get}$$

$$\int_1^{1.2} f(t) dt \approx 0.088854 + 0.096354 \approx 0.185208.$$

All of this can be done in the home screen as illustrated below. If the number  $n$  is small, these step-by-step calculations are reasonable to do.



Obviously what we want to do is to automate this process a little. The command

$$\text{seq}((\text{list2}[j] + \text{list2}[j+1]) * 0.1 / 2, j, 1, 10)$$

will generate a sequence of numbers representing the individual trapezoid approximations

$$\int_{x_j}^{x_{j+1}} f(t) dt \approx \frac{f(x_j) + f(x_{j+1})}{2} (0.1).$$

The MATH menu, List submenu command  $\text{cumSum}(\text{above list})$  will give the cumulative sum we desire to approximate

$$\int_1^{x_{j+1}} f(t) dt \approx \sum_{k=1}^j \frac{f(x_k) + f(x_{k+1})}{2} (0.1).$$

Finally we append the initial zero to the list to get a final list of the same length as the original lists. This command is `augment({0}, above list)`

$x$	$f(x)$	$\int_1^x f(t) dt$ (Trapezoidal)
1.0	0.841471	0
1.1	0.935616	0.088854
1.2	0.991458	0.185208
1.3	0.992904	0.284426
1.4	0.925212	0.380332
1.5	0.778073	0.465496
1.6	0.549355	0.531868
1.7	0.248947	0.571783
1.8	-0.098249	0.579318
1.9	-0.451466	0.551832
2.0	-0.756802	0.491418

Notice that the final answer in the last column is the Trapezoidal Rule computation from the Calculus Tools choice we did at the very beginning.

Implementing Simpson rule step-by-step is similar. The one different feature here is that our new column is not complete. The approximation we use is

$$\int_{x_j}^{x_{j+2}} f(t) dt \approx \frac{f(x_j) + 4f(x_{j+1}) + f(x_{j+2})}{3} (h),$$

giving us only partial definite integrals over “double” subintervals.

```
augment(cumSum(seq((list2[j]+4*list2[j+1]+list2[j+2])*0.1/2,j,1,10,2)))
```

Notice that we use an additional argument in the sequence command to indicate that the index  $j$  goes from 1 to 10 in steps of 2.

$x$	$f(x)$	$\int_1^x f(t) dt$ (Simpson)
1.0	0.841471	0
1.1	0.935616	
1.2	0.991458	0.185846
1.3	0.992904	
1.4	0.925212	0.382123
1.5	0.778073	
1.6	0.549355	0.535018
1.7	0.248947	
1.8	-0.098249	0.583248
1.9	-0.451466	
2.0	-0.756802	0.494551

Again notice that the value at the bottom of the last column is the Simpson rule approximation from Calculus Tools that we did at the very beginning of this section.

You may want to put these steps into a short program or save the history as a text file to more quickly do the step-by-step trapezoidal rule and the step-by-step Simpson's rule repeatedly for new problems.

The discrete computations in this section work very nicely in a spreadsheet. If you have the CellSheet (it comes on the Voyage 200), you can experiment with alternative ways to accomplish the step-by-step trapezoidal rule and step-by-step Simpson's rule in the spreadsheet that is available on this device.

## 7. Other Numerical Integration

There are more efficient and accurate algorithms for numerical integration than the composite trapezoidal rule and the composite Simpson's rule. One feature often put in these more sophisticated algorithms is an adaptive process to estimate the error made for the particular problem given. The algorithm then adjusts the step-size  $h$  as it works along the interval to achieve a desired accuracy. TI uses such an adaptive numerical integration routine internally, and it works to try to achieve 6 significant digits of accuracy on this family of calculators. Thus when display digits is set to the default FLOAT 6 mode, you can generally expect all of the displayed digits in a numerical integration computation to be correct. If you display more digits than 6, you cannot assume all the additional digits are correct.

F1	F2	F3	F4	F5	F6
Tools	Algebra	Calc	Other	Pr3mID	Clean Up
$f(x) \quad \sin(x^2)$ $nInt(f(x), x, 1, 2) \quad .494508$ $nInt(f(x), x, 1, 2)$					

F1	F2	F3	F4	F5	F6
Tools	Algebra	Calc	Other	Pr3mID	Clean Up
$f(x) \quad \sin(x^2)$ $nInt(f(x), x, 1, 2) \quad .494508$ $\int_1^2 f(x) dx \quad .494508$ $\int(f(x), x, 1, 2)$					

It is generally better to use the internal numerical integration (either the nInt( command or the regular integration command with a  $\blacklozenge$ ENTER) rather than the composite trapezoidal rule or the composite Simpson's rule. The best advice is to use one of these simple numerical integration routines only when the specific routine is specified. Generally in a textbook problem or a test question, the instruction will tell you not only the method (trapezoidal or Simpson) but also the step size.

## 8. Symbolic Infinite Limits

The TI-89 can handle many limits involving infinity. It is not a good idea to jump to conclusions about the value for infinite limits based upon numerical evidence. However the same theorems you are learning about how to handle infinite limits can be implemented in the symbolic part of the calculator. Notice that you can find "∞" on the keyboard, and infinity is a possible output if you have the Exact/Approx mode set to either AUTO or EXACT.

<p> <math>\lim_{x \rightarrow \infty} \left( \frac{\ln(x)}{x} \right)</math> 0  <code>limit(ln(x)/x,x,∞)</code> </p>	<p> <math>\lim_{x \rightarrow \infty} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right)</math> 1  <code>limit((e^x+e^-x)/(e^x-e^-x),x,∞)</code> </p>	<p> <math>\lim_{x \rightarrow 0^+} (\sin(x) \cdot \ln(x))</math> 0  <math>\lim_{x \rightarrow 0^+} \left( \frac{1 - \cos(x)}{x^2} \right)</math> 1/2  <code>limit(sin(x)*ln(x),x,0+)</code>  <code>limit((1-cos(x))/x^2,x,0+)</code> </p>
<p> <math>\lim_{x \rightarrow 0^+} \left( \frac{1 - \cos(x)}{x^3} \right)</math> ∞  <math>\lim_{x \rightarrow 0^+} (x \cdot \sin(x))</math> undef  <code>limit((1-cos(x))/x^3,x,0+)</code>  <code>limit(x*sin(x),x,0+)</code> </p>	<p> <math>\lim_{x \rightarrow \infty} (x \cdot \sin(x))</math> undef  <math>\lim_{x \rightarrow \infty} \left( \left( 1 + \frac{1}{x} \right)^{5 \cdot x} \right)</math> e^5  <code>limit(x*sin(x),x,∞)</code>  <code>limit((1+1/x)^(5*x),x,∞)</code> </p>	<p> <math>\lim_{x \rightarrow 0^+} (\cot(2 \cdot x) \cdot \tan^4(x))</math> 1/2  <math>\lim_{x \rightarrow \infty} (\sinh(x) - x)</math> ∞  <code>limit(cot(2*x)*tan^4(x),x,0+)</code>  <code>limit(sinh(x)-x,x,∞)</code> </p>

## 9. Improper Integrals

Since the evaluation of improper integrals generally involves a symbolic integration followed by some evaluation of an infinite limit, it is not a surprise that the TI-89 can do many of these.

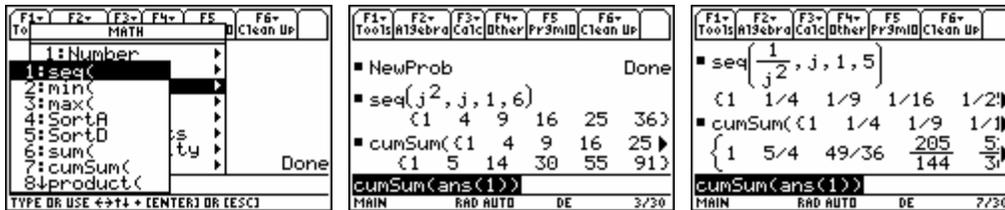
<p> <math>\int_1^{\infty} \left( \frac{1}{x^{4/3}} \right) dx</math> 3  <code>int(1/x^(4/3),x,1,∞)</code> </p>	<p> <math>\int_{-\infty}^0 (e^x) dx</math> 1  <math>\int_{-2}^{\infty} \left( \frac{1}{(x+2)^{5/4}} \right) dx</math> ∞  <code>int(e^x,x,-∞,0)</code>  <code>int(1/((x+2)^(5/4)),x,-2,∞)</code> </p>	<p> <math>\int_{-2}^{\infty} \left( \frac{1}{(x+2)^{5/4}} \right) dx</math> ∞  <math>\int_0^{\pi/2} \sec(x) dx</math> ∞  <code>int(1/((x+2)^(5/4)),x,-2,∞)</code>  <code>int(sec(x),x,0,pi/2)</code> </p>
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## Chapter 7 Sequences and Series

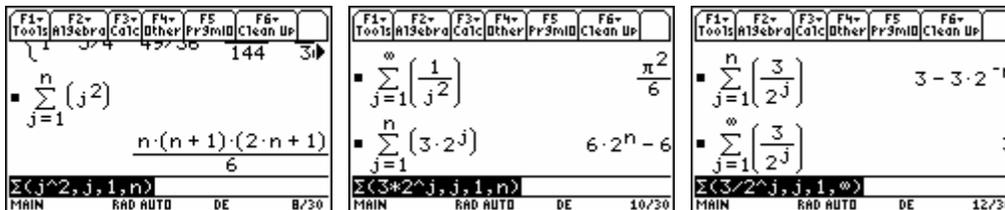
The activities in this chapter present symbolic, numerical, and graphical activities involving sequences and series. This is loosely coordinated with Chapter 7 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

### 1. Sequences and Series

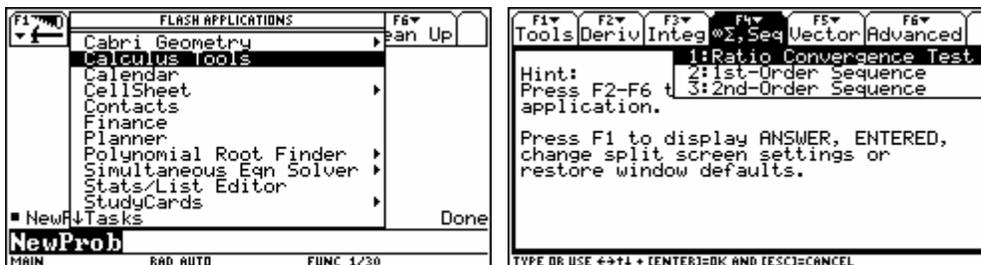
It is possible to work with *finite* sequences and series using lists. This may be enough to recognize patterns.

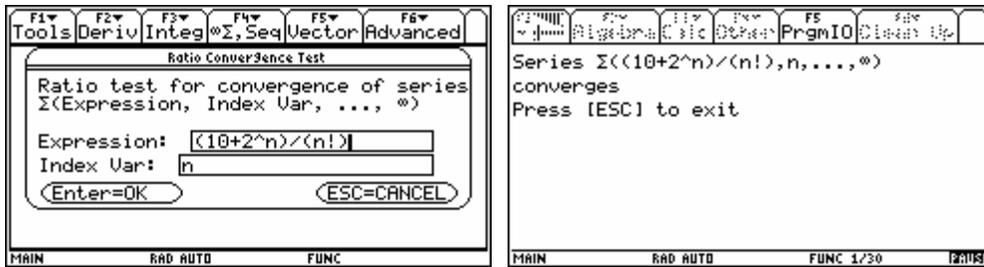


The F3 Calc command 4: $\Sigma$ ( sum allows some symbolic results. Some special infinite series can be evaluated.



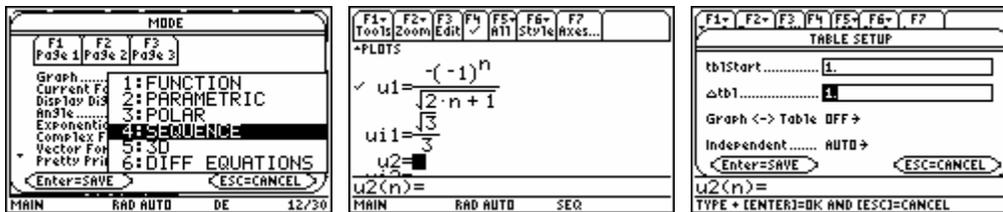
The ability of the calculator to evaluate symbolic limits means it will be useful for some of the limits needed in the ratio test for power series. This has been further facilitated in the Calculus Tools flash application (free from <http://education.ti.com>). You will not get much credit using only the application (because your instructor will want to see that you understand the ratio test), but you can check your work with this flash application.



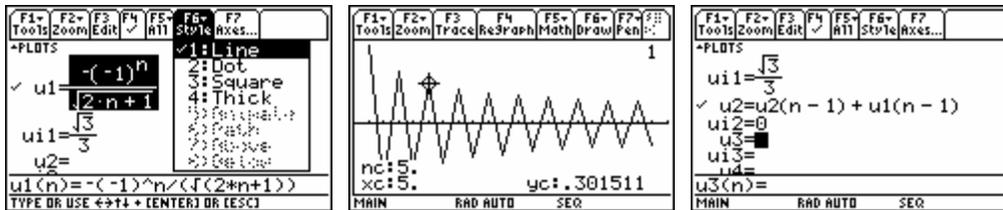
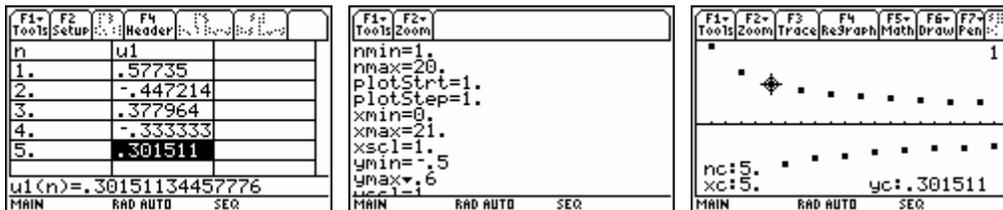


## 2. Sequence Graphing

The sequence graphing mode allows still further ways to study and understand sequences and series.

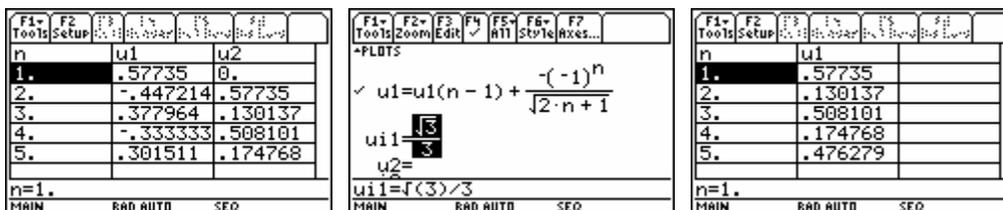


F6 Style 3: Square (default)



F6 Style 1: Line

Second sequence is a partial sum of the first, off by one index.



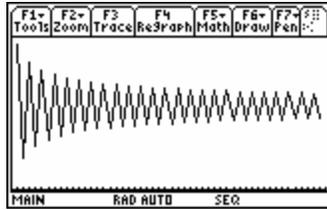
Alternate way to get only partial sums, correct index.

```

F1- F2-
Tools Zoom
nmin=1.
nmax=50.
plotStrt=1.
plotStep=1.
xmin=0.
xmax=51.
xsc1=1.
ymin=0.
ymax=.6
yrc1=1
MAIN RAD AUTO SEQ

```

Alternate way with Line Style



Evaluation of alternate way in HOME screen

F1- Tools	F2- Algebra	F3- Calc	F4- Other	F5- Pr3rdID	F6- Clean Up
■ u1(50)					.282803
■ u1(51)					.381336
■ u1(500)					.316513
■ u1(501)					.348088
u1(501)					

MAIN RAD AUTO SEQ 4/30

## Chapter 8

### Polar Coordinates and Conics

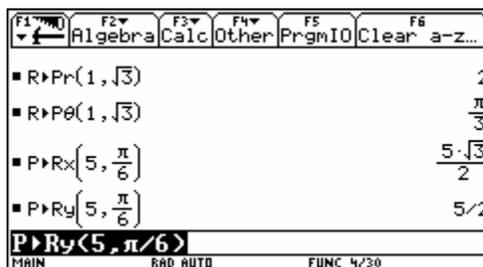
The activities in this chapter show polar coordinates and polar graphing. This is loosely coordinated with Chapter 8 of the text *University Calculus: Elements with Early Transcendentals* (WMU edition), by Hass, Weir, and Thomas, Pearson Custom Publishing.

#### 1. Polar Coordinates

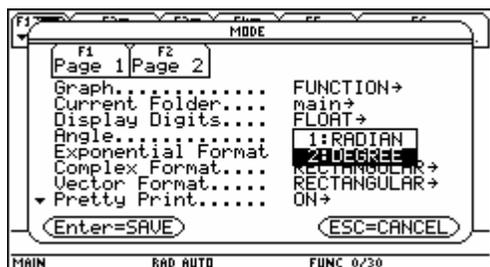
Since the TI-92, this family of graphing calculators (including the TI-89 and the Voyage 200 that replaced the TI-92) has offered some nice features relating to polar coordinates. In the MATH menu, the 2:Angle submenu, you will find several commands for converting between rectangular and polar coordinates. *Note:* The “angles” will automatically be in the Angle MODE (either radians or degrees).



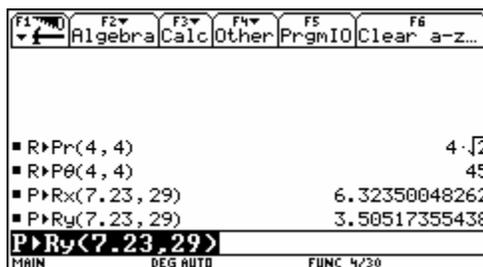
Old TI-92, in RAD Angle MODE



Conversions (in radians)



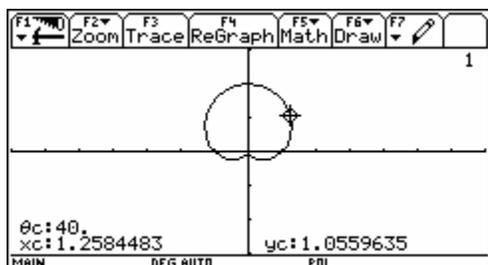
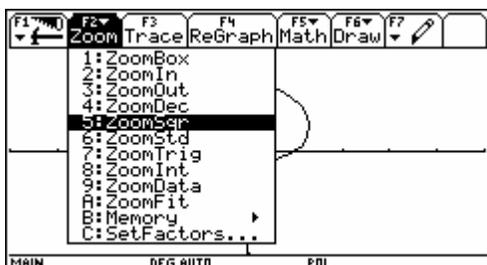
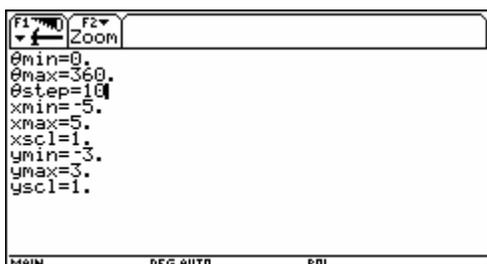
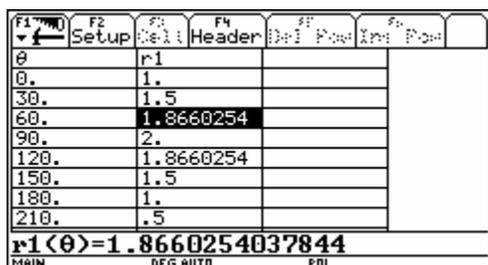
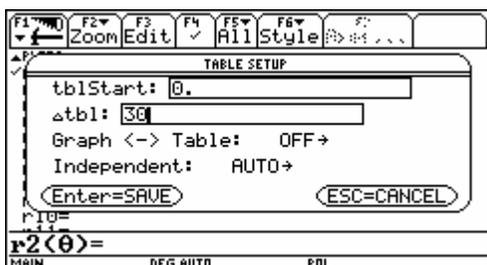
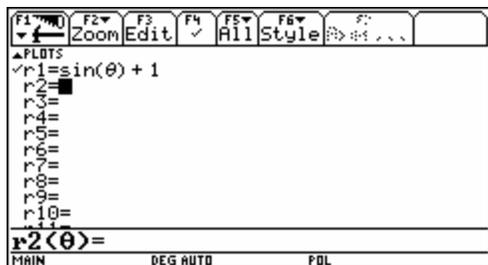
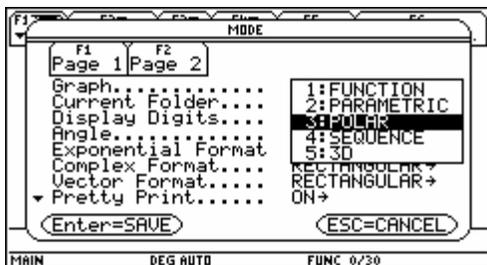
Switching to DEG Angle MODE



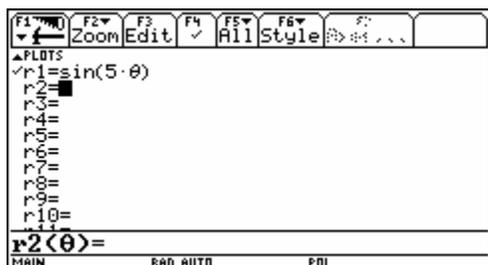
Conversions (in degrees)

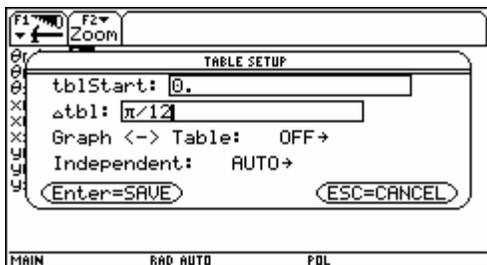
#### 2. Graphing in Polar Coordinates

There is a polar equation graphing mode that allows the graphing of polar equations that can be solved for  $r$ , i.e. where  $r$  can be expressed in a formula in terms of  $\theta$ . We demonstrate below with several examples. For the first example, note that the calculator is in DEG Angle MODE (which is usually not what we want in a calculus course). Note also that for polar graphing, we will typically want to have equally-scaled axes (which we get with the ZoomSqr command from the ZOOM menu).



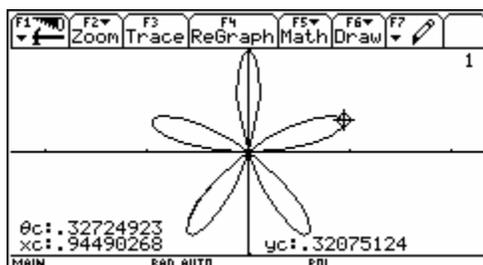
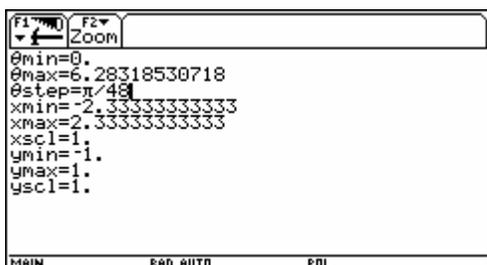
In calculus, we are much more likely to want to work in RAD Angle MODE. Here are a few examples.



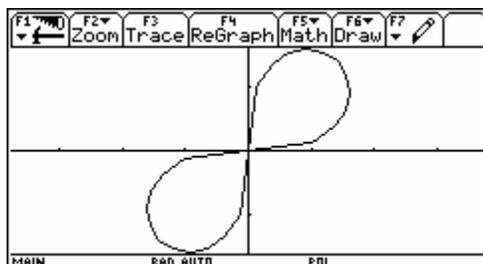
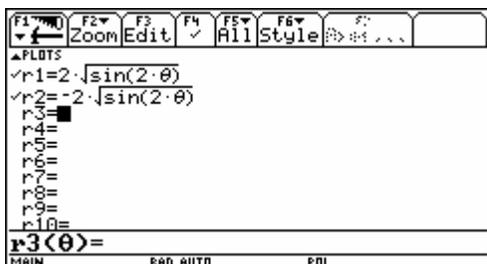


$\theta$	r1
0.	0.
.261799388	.965925826
.523598776	.5
.785398163	<b>-.70710678</b>
1.04719755	-.8660254
1.30899694	.258819045
1.57079633	1.
1.83259571	.258819045

**r1(θ) = -.70710678118496**



For the polar equation  $r^2 = 4 \sin(2\theta)$ , we plot the two possible solutions for  $r$ .



$\theta_{\min} = 0$ ,  $\theta_{\max} = \pi$ ,  $x_{\min} = -3.7699$ ,  $x_{\max} = 3.76099$ ,  $y_{\min} = -1.6119$ ,  $y_{\max} = 1.6119$

## Chapter 15 Elementary Differential Equations

The activities in this chapter present some additional things that we can now do with differential equations on this family of calculators. This is loosely coordinated with Chapter 15 of the text *University Calculus: Elements with Early Transcendentals (WMU edition)*, by Hass, Weir, and Thomas, Pearson Custom Publishing.

### 1. Symbolic Differential Equations

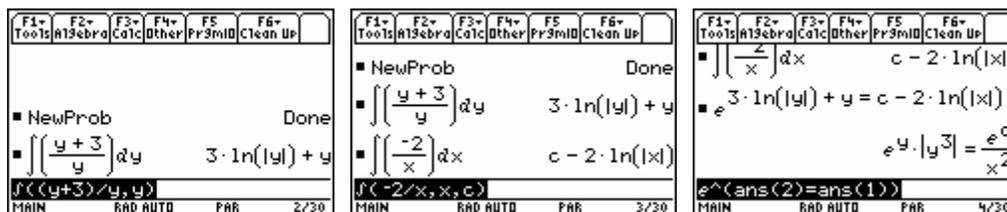
Many calculus texts present one method for solving some first-order differential equations, namely the “method of separation of variables.” Assuming you can do the “separation” algebraically by hand, the task reduces to two indefinite integration problems. Obviously the symbolic integration on the TI-89 can be helpful.

Consider  $2y + (xy + 3x)\frac{dy}{dx} = 0$ . Working by hand we find

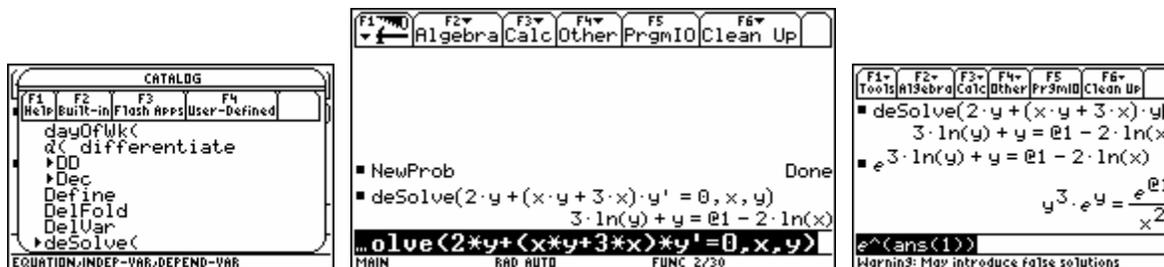
$$x(y+3)\frac{dy}{dx} = -2y$$

$$\frac{y+3}{y}dy = -\frac{2}{x}dx$$

with the last line showing the separation of the two variables. Integrating both sides can be done by hand or by the calculator.

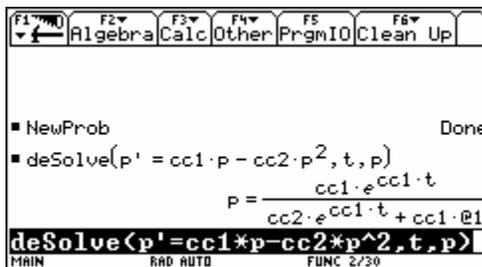
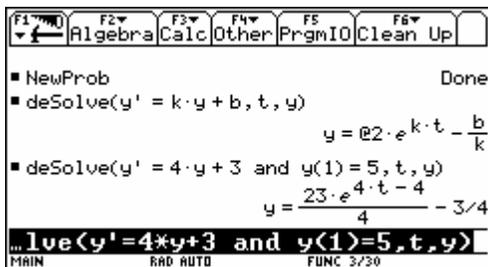


Letting  $k = \pm e^c$  and dropping the absolute value, we would generally express the solution as  $x^2 y^3 e^y = k$ , which implicitly defines  $y$  as a function of  $x$ . The calculator can directly solve some differential equations, including the example above in a similar fashion. The command `deSolve` can be found in the F3 Calc menu or in the catalog.

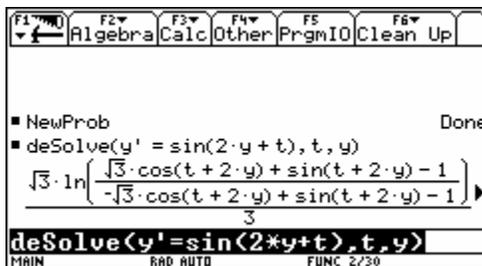
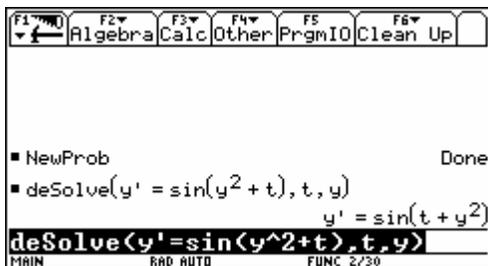


Notice the arbitrary constant in the general solution is represented by the variable “@1” in the output. Here the calculator is not as careful with absolute values as we were above.

We can also ask the calculator to give us the symbolic solution for the general differential equation, namely  $y'(t) = k y(t) + b$ . The command deSolve can be used to solve for a particular solution for such an equation with an initial condition. Notice below when we solve the general logistic differential equation, we cannot use the variable names “c1” and “c2” because they are reserved variables use by the system (for the columns in a data set or matrix). We use “cc1” and “cc2” instead.



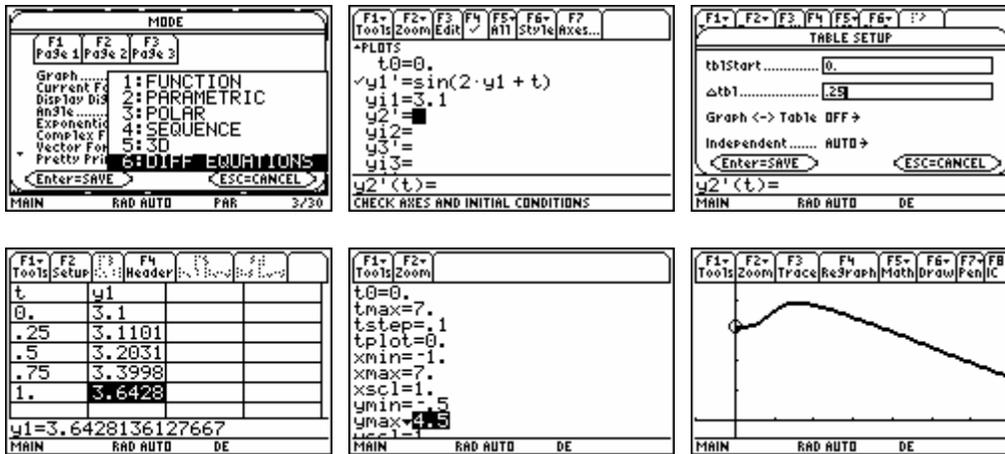
The command deSolve can solve some second-order differential equations as well. If it cannot solve a differential equation, it will just repeat the problem back as a result. Even when it can solve a problem with an implicit solution (such as  $y' = \sin(2y + t)$  below), the solution may not be all that useful.



## 2. Differential Equation Graphing

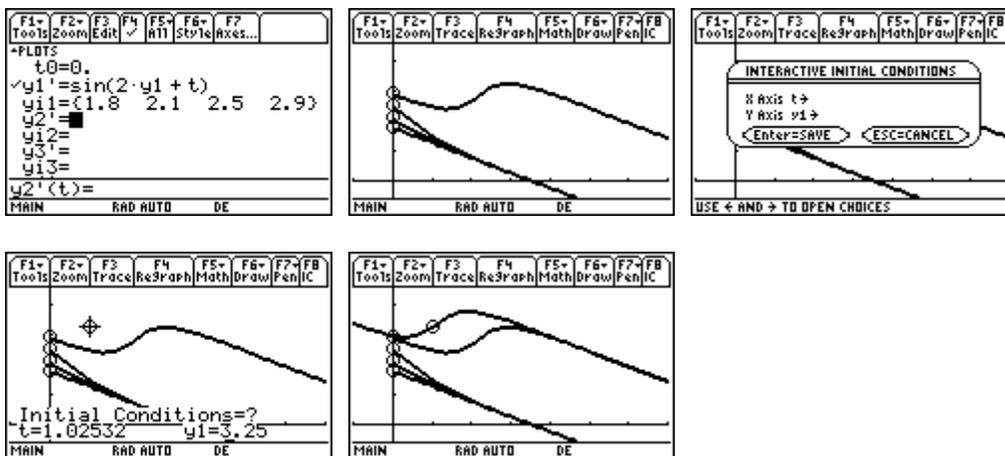
There is a differential equation graphing mode that allows the numerical solution for first-order differential equations with one or more initial conditions. We demonstrate below for a single equation first.

Consider  $y' = \sin(2y + t)$ ,  $y(0) = 3.1$



F1 9:Format Fields FLDOFF

We can look at other solutions to this differential equations, either by entering in a list of initial values in the Y= edit screen or by pressing F8 IC to select further initial conditions interactively in the graph screen.



In a later differential equation course, you will study higher order differential equations and systems of first-order equations where the other features of this family of calculators will be used nicely.

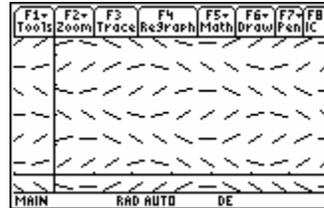
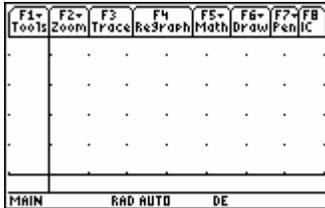
### 3. Slope Fields and Euler's Method

Many calculus texts now present the graphical understanding of first-order differential equations by considering a slope field for the general solution in some viewing window. We present the general idea of a slope field below and introduce the simplest numerical method for solving a first-order differential equation with an initial condition, called Euler's method, which is closely related to a slope field.

Consider a general first-order differential equation of the form

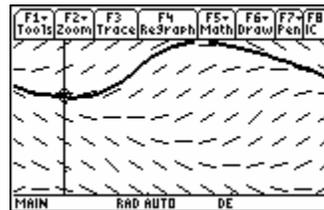
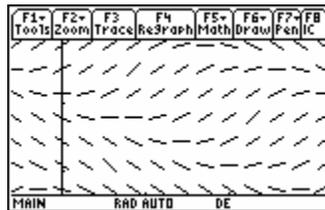
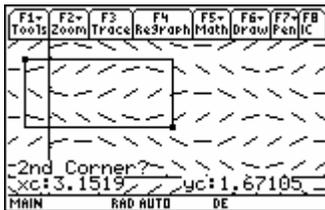
$$\frac{dy}{dt}(t) = f(t, y(t)).$$

Suppose the graph of the solution passes through the point  $(a, b)$ , i.e. when  $t = a$  then  $y(a) = b$ . We know from the differential equation that the solution must have  $y'(a) = f(a, b)$ . Given a desired viewing window, we construct a grid of points in the rectangle. We treat the coordinates of each point in our grid as the pair  $(a, b)$ , and we draw a short line segment through the grid point with the slope  $y'(a) = f(a, b)$  that the solution must have (if it goes through that grid point).



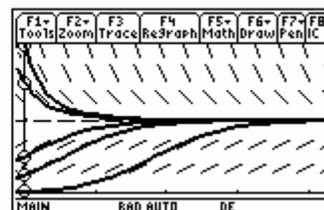
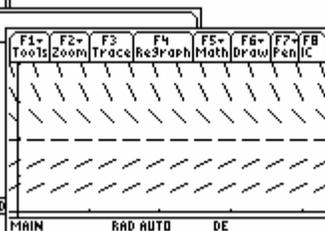
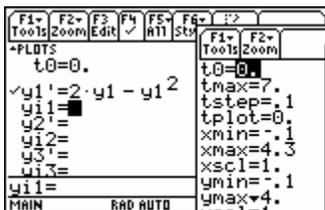
Sample Grid (Grid ON) DE with no initial condition F1 9:Format Field SLPFLD (default grid, not as before)

For example, one of the grid points used in the last image seems to be about  $(0.87, 2.79)$ . Thus that line segment is drawn with slope approximately  $\sin(2 \cdot 2.79 + 0.87) \approx 0.166$



F2 Zoom ZoomBox More detail Add initial condition  $y(0) = 3.1$

Here we reproduce the slope field for  $P'(t) = 2P(t) - (P(t))^2$ .



$y'(0) = \{0.05, 0.5, 1.0, 3.0, 4.0\}$

TI uses an adaptive Runge-Kutta routine to numerically approximate the solutions in these differential equation plots by default. This routine works to try to get plotted points with an accuracy of diftol, a parameter you can set in the Window screen. The parameter fldres in the Window screen sets the grid spacing for the slope field. The defaults which often work nicely are diftol = 0.001 and fldres = 14 (giving a 14 by 7 array of slope lines in the slope field).

The Graph Format screen allows you to select this adaptive method (RK) and a much simpler (less accurate) method (EULER). Euler's method is now taught in many calculus texts, and it is a required topic in the Advanced Placement Calculus course commonly taught in high schools. This simple method is based upon the fact that knowing the slope enables us to compute a tangent line approximation to the solution, which will be fairly accurate if we do not move the  $t$ -value too far from the known point.

To implement Euler's method for  $\frac{dy}{dt}(t) = f(t, y(t))$ ,  $y(t_0) = y_0$ , with step size  $h$ , we first compute the slope  $m_0 = f(t_0, y_0)$ . The line tangent to the graph of the solution we want at the point  $(t_0, y_0)$  will have the equation  $y = y_0 + m_0(t - t_0)$ . Thus for a small step  $h$ , the value predicted on the tangent line, namely  $y = y_0 + m_0((t_0 + h) - t_0) = y_0 + m_0h$ , will be close to the solution  $y(t_0 + h)$ . We have started with  $(a, b) = (t_0, y_0)$ , used the slope and a short segment of the tangent line, and arrived at the point  $(t_0 + h, y_0 + m_0h)$  which we label  $(t_1, y_1)$ . Then we treat  $(t_1, y_1)$  as  $(a, b)$ , compute a new slope  $m_1 = f(t_1, y_1)$ , and move another step  $h$  along the new tangent line to estimate  $y(t_2) = y(t_0 + 2h) \approx y_1 + m_1h$ . Continuing in this fashion, we recursively define sequences with

$t_0$  and  $y_0$  given

$$t_{i+1} = t_i + h, \quad y_{i+1} = y_i + f(t_i, y_i)h$$

where  $y(t_i) \approx y_i$ .

For example, with  $y' = \sin(2y + t)$ ,  $y(0) = 3.1$  and  $h = 0.1$ , we get

$$t_0 = 0, \quad y_0 = 3.1$$

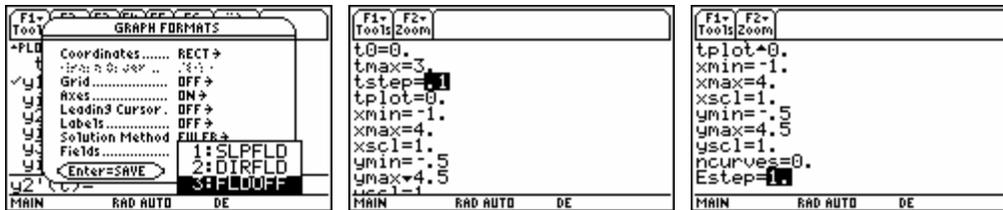
$$t_1 = 0.1, \quad y_1 = 3.1 + \sin(2 \cdot 3.1 + 0) \cdot 0.1 \approx 3.09169$$

$$t_2 = 0.2, \quad y_2 = 3.09169 + \sin(2 \cdot 3.09169 + 0.1) \cdot 0.1 \approx 3.09171$$

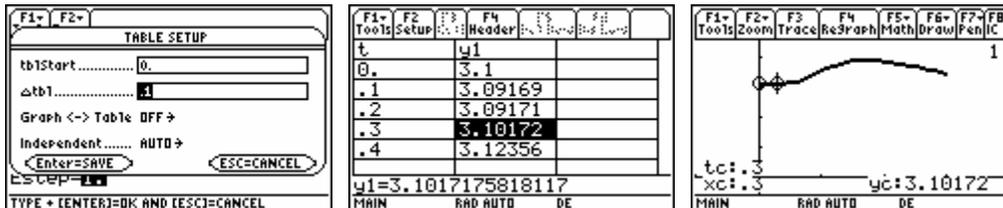
$$t_3 = 0.3, \quad y_3 = 3.09171 + \sin(2 \cdot 3.09171 + 0.2) \cdot 0.1 \approx 3.10172$$

It is possible to have the calculator do these computations, display them in a table, and plot them as an approximate solution.





With Estep = 1, tstep =  $h$



In general, Euler's method needs a very small step size  $h$  to be accurate. Thus you might not want to plot all of the values that you are computing. Choosing Estep = 5 and tstep = 0.1 will effectively use  $h = 0.1/5 = 0.02$ , even though the graph still only plots every 0.1. Thus Estep = 5 causes 5 Euler steps between each plotted point. Since you will be using Euler's method only when someone asks you to do so, I would expect you to always use Estep = 1 with the plotted points the desired answer (which can also be displayed in a table). If the method is not specified by your assignment, the select the RK method instead, picking diftol to be the desired tolerance.

What the some textbooks call "Euler's method" is really just a special case of the more general method we have presented here when it happens that the differential equation does not explicitly depend upon  $y$ . Thus solving  $F(t) = \int_c^t f(u)du$  for  $t = c, c+h, c+2h, \dots$  is equivalent to solving  $y'(t) = f(t)$ ,  $y(c) = 0$  using  $t_0 = c, t_1 = c+h, t_2 = c+2h, \dots$ . This special case is also equivalent to looking at partial sums of the Riemann sum approximation for the definite integral with all of the evaluation points selected as left endpoints on an equally spaced partition.