PRACTICE GATEWAY EXAM

1. Find the following limits:

(a) $\lim_{x\to 0^+} \dfrac{\sin|x|}{x}$

(b) $\lim_{x\to 2} \dfrac{\sqrt{x+7}-3}{\sqrt{x+2}-2}$

(c) $\lim_{x\to 0} \dfrac{2\cos(x+\pi/3)-1}{x}$

(d) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

(e) $\lim_{n\to\infty} \dfrac{1}{n}\left(1^{15}+2^{15}+\cdots+n^{15}\right)$.

2. Find the derivatives of the following functions:

(a) $y = x^3 \sin 2x + x^{-2}$

(b) $y = \dfrac{x+\tan x}{x-\tan x}$

(c) $y = \sin(\sqrt{\sin x})$

(d) $y = \int_0^{\sin x} \sqrt{1-t^2}\, dt.$

3. (a) Compute $\sum_{k=300}^{500} k^2$

(b) For what m is $\sum_{k=m}^{30} k = 444$?

4. Evaluate the integrals:

(a) $\int x^2 \cos(x^3-5)\, dx$

(b) $\int_1^2 2x\sqrt{x^2-1}\, dx$

(c) $\int \cos^3(1-3x)\sin(1-3x)\, dx$

(d) $\int_0^1 \sqrt{1-x^2}\, dx.$

HINTS AND RESULTS

1. (a) Hint: $x \to 0^+$ implies that $x > 0$. Answer: 1.

(b) Hint: multiply both the numerator and the denominator by $(\sqrt{x+7}+3)(\sqrt{x+2}+2)$. Answer: $2/3$.

(c) Hint: $1 = 2\cos(\pi/3)$ and think of derivatives. Answer: $-\sqrt{3}$.

(d) Hint: $1 - \cos x = 2\sin^2 \dfrac{x}{2}$. Answer: $1/2$.

(e) Hint: This is a Riemann sum. Answer: $1/16$.

2. (a) Solution: $y' = 6x^2 \cos 2x - 2x^{-3}$.

(b) Solution: $y' = \dfrac{2(x\sec^2 x - \tan x)}{(x - \tan x)^2}$

(c) Solution: $y' = \dfrac{\cos(\sqrt{\sin x})\cos x}{2\sqrt{\sin x}}$

(d) Hint: FTC. Solution: $y' = \cos^2 x$.

3. (a) Hint: $\displaystyle\sum_{k=300}^{500} k^2 = \sum_{k=1}^{500} k^2 - \sum_{k=1}^{299} k^2$. Answer: $32,836,700$.

(b) Answer: $m = 7$.

4. (a) Substitution: $u = x^3 - 5$. Answer: $\dfrac{1}{3}\sin(x^3 - 5) + C$

(b) Substitution: $u = x^2 - 1$. Answer: $2\sqrt{3}$

(c) Substitution: $u = \cos(1 - 3x)$. Answer: $-\dfrac{1}{12}\cos^4(1 - 3x)$

(d) Hint: Draw the picture. Answer: $\pi/4$.