Honors Math/Physics Integrating Seminar
Worksheet on the First Kepler Law

1. (a) In the previous worksheet we have seen that $\mathbf{p}(t) \times \mathbf{v}(t) = \mathbf{c}$. Write $\mathbf{p}(t) = r(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ is a vector of magnitude 1. Using properties of the cross product derive a formula for $\mathbf{c}$ in terms of $\mathbf{u}(t)$, $\mathbf{u}'(t)$, and $r(t)$ (no $r'(t)$, please):

(1) $$\mathbf{c} =$$

(b) By Newton's Second Law: $\mathbf{F} = m\mathbf{a}(t)$, and by Newton's Law of Gravitation $\mathbf{F} = -\dfrac{GMm}{||\mathbf{p}(t)||^3}\mathbf{p}(t)$. Combine these with (1) to obtain an expression for $\mathbf{a}(t) \times \mathbf{c}$ in terms of $\mathbf{u}(t)$ and $\mathbf{u}'(t)$:

(2) $$\mathbf{a}(t) \times \mathbf{c} =$$

(c) Use the property 1(c) and (5) (Cross Product Worksheet I) to obtain the formula for $\mathbf{a}(t) \times \mathbf{c}$ in terms of $\mathbf{u}'(t)$:

(3) $$\mathbf{a}(t) \times \mathbf{c} =$$

(d) Integrate both sides of (3) to obtain the formula for $\mathbf{v}(t) \times \mathbf{c}$. [Do not forget the constant of integration $\mathbf{A} = (A_1, A_2, A_3)$!]

(4) $$\mathbf{v}(t) \times \mathbf{c} =$$

2. (a) We will choose the coordinate system so that $\mathbf{c}$ has the direction of positive z-axis which means that the planet moves in the xy-plane. In addition, we will set the positive x-axis in the direction of $\mathbf{A}$. We will denote by $\theta(t)$ the angle between the x-axis and $\mathbf{p}(t)$ measured counterclockwise. Using (4) write an expression for the scalar quantity $\mathbf{p}(t) \cdot (\mathbf{v}(t) \times \mathbf{c})$ and solve it for $r = r(t)$:

(5) $$r =$$

(b) Show that for any 3 vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we have $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

(c) Use (b) to obtain that

(6) $$r = \frac{e\,d}{1 + e\cos\theta}$$

for some constants d and e.

3. (a) We will show that the equation (6) represents an ellipse with a focus in the origin. Start by drawing a point (x, y) in the plane, and label x, y, r, and θ in your picture.

(b) Using the picture express r and θ in terms of x and y:

$$r =$$

$$\theta =$$

(c) Using formulas in (b) write (6) in the form $\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1$ and express a, b, and h in terms of d and e.

$$a =$$

$$b =$$

$$h =$$

(d) Now that we know that the planet follows an ellipse let us show that the sun is in a focus. Compute $c = \sqrt{a^2 - b^2}$ and show that it equals $-h$.