COMPOSITION OPERATORS WITH A MINIMAL COMMUTANT

MIGUEL LACRUZ, FERNANDO LEÓN-SAAVEDRA, SRDJAN PETROVIC, AND LUIS RODRÍGUEZ-PIAZZA

Abstract. Let $C_\varphi$ be a composition operator on the Hardy space $H^2$, induced by a linear fractional self-map $\varphi$ of the unit disk. We consider the question whether the commutant of $C_\varphi$ is minimal, in the sense that it reduces to the weak closure of the unital algebra generated by $C_\varphi$. We show that this happens in exactly three cases: when $\varphi$ is either a non-periodic elliptic automorphism, or a parabolic non-automorphism, or a loxodromic self-map of the unit disk. Also, we consider the case of a composition operator induced by a univalent, analytic self-map $\varphi$ of the unit disk that fixes the origin and that is not necessarily a linear fractional map, but in exchange its Königs's domain is bounded and strictly starlike with respect to the origin, and we show that the operator $C_\varphi$ has a minimal commutant. Furthermore, we provide two examples of univalent, analytic self-maps $\varphi$ of the unit disk such that $C_\varphi$ is compact but it fails to have a minimal commutant.

Contents

Introduction 1
1. Some general results about operators with a minimal commutant 4
2. Composition operators induced by parabolic and hyperbolic automorphisms 5
3. The commutant of a composition operator induced by an elliptic automorphism 7
4. The commutant of a composition operator induced by a parabolic non-automorphism 8
5. The commutant of a composition operator induced by a hyperbolic non-automorphism 11
6. The commutant of a composition operator induced by a loxodromic mapping 12
7. Compact composition operators with univalent symbol and non-minimal commutant 18
Appendix: The Sobolev space as a Banach algebra with an approximate identity 24
References 26

Introduction

Let $B(H)$ stand for the algebra of all bounded linear operators on a Hilbert space $H$ and let $A \in B(H)$. Recall that the commutant of $A$ is defined as the family of all operators that commute with $A$, that is,

$$\{A\}' = \{X \in B(H): AX =XA\}.$$  

It is a standard fact that $\{A\}'$ is a subalgebra of $B(H)$ that is closed in the weak operator topology $\sigma$. We shall denote by $\text{alg}(A)$ the unital algebra generated in $B(H)$ by the operator $A$, that is,

$$\text{alg}(A) = \{p(A): p \text{ is a polynomial}\}.$$
It is clear that \( \text{alg} (A)^\sigma \) is a commutative algebra with the property that \( \text{alg} (A)^\sigma \subseteq \{ A \}' \). We say that an operator \( A \) has a minimal commutant provided that \( \text{alg} (A)^\sigma = \{ A \}' \).

It is the purpose of this paper to investigate the last equality in the case when \( A \) is a composition operator \( C_\varphi \), acting on \( H^2(\mathbb{D}) \), the Hardy space of the unit disk, and \( \varphi \) is a linear fractional transformation. We will show that the equality holds precisely when the symbol is either a non-periodic elliptic automorphism, or a parabolic non-automorphism, or a loxodromic self-map of the unit disk (see definitions below).

Recall that the Hardy space \( H^2(\mathbb{D}) \) is the Hilbert space of all analytic functions \( f \) on the unit disk \( \mathbb{D} \) that have a Taylor series expansion around the origin of the form

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{with} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty,
\]

endowed with the norm

\[
\|f\|_2 := \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.
\]

It is a well-known fact that if \( f \in H^2(\mathbb{D}) \) then the radial limits

\[
f(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})
\]

exist almost everywhere, and moreover,

\[
\|f\|_2 = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta \right)^{1/2}.
\]

Every analytic mapping \( \varphi : \mathbb{D} \to \mathbb{D} \) defines a linear composition operator \( C_\varphi \) on the space of all analytic functions on the unit disk by the formula \( C_\varphi f = f \circ \varphi \). The analytic map \( \varphi \) is usually called the symbol of \( C_\varphi \). It is a consequence of Littlewood’s subordination principle [30] that every composition operator \( C_\varphi \) restricts to a bounded linear operator on \( H^2(\mathbb{D}) \).

Recall that a linear fractional map \( \varphi \) on the Riemann sphere \( \hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \) is any map of the form

\[
\varphi(z) = \frac{az + b}{cz + d},
\]

where the coefficients \( a, b, c, d \in \mathbb{C} \) satisfy the condition \( ad - bc \neq 0 \), just to ensure that \( \varphi \) is not constant, and where the standard conventions are applied to the point at infinity. We are interested here in those linear fractional maps that take the unit disk into itself. Every such map has either one or two fixed points. Those with one fixed point are called parabolic. It can be shown that such a fixed point must be on the unit circle. In the remaining cases, the linear fractional map \( \varphi \) has two fixed points, one of which must be in the closed unit disk. If it is on \( \partial \mathbb{D} \) then \( \varphi \) is said to be hyperbolic. If it is in the open unit disk, the other one has to lie in the complement in the Riemann sphere of the closed unit disk. Here we distinguish between two different situations. If \( \varphi \) is an automorphism, then \( \varphi \) is said to be elliptic. Otherwise, \( \varphi \) is said to be loxodromic. We refer the reader to the more detailed presentation in the article by Shapiro [31]. Throughout the paper we will use the facts from [31] about the canonical models for linear fractional maps and the spectral properties of the associated composition operators.

The problem of determining whether an operator \( A \) has a minimal commutant is not new. Sarason [29] showed that the Volterra operator has a minimal commutant. Erdos [10] obtained a simple, direct proof of this result. Shields and Wallen [34] showed that if \( A \) is an injective unilateral shift or if \( A \) is the discrete Cesàro operator then \( A \) has a minimal commutant. Shields [33] also proved that if \( A \in \mathcal{B}(H) \) is a non invertible, injective bilateral weighted shift then \( A \) has a minimal commutant.
Cowen and MacCluer [6] raised the question: when is the commutant of $C_\varphi$ equal to the weakly closed algebra generated by those multiplications and composition operators that commute with $C_\varphi$. Cl oud showed in [4, 5] that this is the case when $\varphi$ is an elliptic automorphism of the unit disk and $\varphi(0) = 0$. He also proved in [5] that, in this situation, the only bounded analytic functions that induce multiplication operators in the commutant of $C_\varphi$ are the constant functions. Hence, it follows from his work that when $\varphi$ is a non-periodic elliptic automorphism of the unit disk, the corresponding composition operator $C_\varphi$ has a minimal commutant. We present in Section 3 a direct proof of this result.

Worner [35] showed that, if $\varphi$ is a hyperbolic non-automorphism, the composition operator $C_\varphi$ fails to have a minimal commutant. She exhibited two classes of operators that commute with $C_\varphi$ but do not belong to the weak closure of the unit algebra generated by $C_\varphi$. First, multiplication operators induced by non-constant, bounded analytic functions, and second, composition operators $C_\psi$, where $\psi$ is a symbol such that $\varphi = \psi \circ \psi$. We present a simplified proof of her result in Section 5. She also described a procedure for constructing composition operators that fail to have a minimal commutant, and she pointed out that one of the reasons for this is that the symbol fails to be univalent. She finally asked the question whether a compact composition operator with univalent symbol must have a minimal commutant. We provide in Section 7 a negative answer to this question.

Linchuck [20, 21] provided a description of the commutant of a composition operator $C_\varphi$ in the cases that $\varphi$ is a hyperbolic or a parabolic automorphism. Although his characterization of the commutants of such composition operators is quite involved, it could perhaps be used to construct examples of operators in the commutant of $C_\varphi$ but not in the weak closure of the unit algebra generated by $C_\varphi$.

The paper is organized as follows.

In Section 1 we provide four general results. The first one is a procedure for constructing multiplication operators that belong to $\{C_\varphi\}'$ but not to $\overline{\text{alg} (C_\varphi)}'$. This procedure was described earlier by Worner [35, Theorem 9], under the more restrictive hypotheses that $\varphi$ is not an automorphism and all its iteration sequences are interpolating. The other three results will establish that the property of having a minimal commutant is closed under the restrictions to reducing subspaces, under similarities and the operation of taking adjoints.

In Section 2 we consider the case when $\varphi$ is a parabolic or a hyperbolic automorphism. We will show that in either case the commutant $\{C_\varphi\}'$ is not minimal. We present two different proofs of these results in order to shed more light on the structure of $\{C_\varphi\}'$.

In Section 3 we consider the case of an elliptic self-map $\varphi$ of the unit disk and we prove first, that in the periodic case, $\{C_\varphi\}'$ fails to be commutative, so that $\overline{\text{alg} (C_\varphi)}' \neq \{C_\varphi\}'$, and second, that in the non-periodic case, we have $\overline{\text{alg} (C_\varphi)}' = \{C_\varphi\}'$. Our proof of the latter result relies on a classical extension theorem of Rudin and Carleson.

In Section 4 we consider the case of a composition operator induced by a parabolic non-automorphism $\varphi$ of the unit disc, and we prove that its commutant is minimal. The proof is based on a remarkable result of Montes-Rodríguez, Ponce-Escudero and Shkarin [24], that the adjoint $C_\varphi^*$ is similar to a multiplication operator by a cyclic element on the Sobolev space $W^{1,2}(0, +\infty)$. Our contribution is the proof that the operator of multiplication by a cyclic element in an abstract Banach algebra with an approximate identity has a minimal commutant. The fact that the Sobolev space is a non-unital Banach algebra with an approximate identity is well-known. However, we are not aware of a simple proof, so we offer one in the Appendix to this paper.

In Section 5 we consider a composition operator $C_\varphi$ induced by a hyperbolic non-automorphism $\varphi$ of the unit disk and we show that its commutant is not minimal. Worner [35] has obtained the same result by studying the case when the iterates of $\varphi$ form an interpolating sequence. Our proof does not require the notion of an interpolating sequence and it is simpler.

In Section 6 we consider the case of a composition operator induced by a univalent, analytic self-map $\varphi$ of the unit disk that fixes the origin and that is not necessarily a linear fractional map, but in exchange
its Königs’s domain is bounded and strictly starlike with respect to the origin, and we show that the operator $C_ϕ$ has a minimal commutant. The loxodromic case becomes a particular instance of this result since the range of the Königs function is an open disk that contains the origin.

In Section 7 we construct an example of a univalent, analytic self-map $ϕ$ of the unit disk such that $C_ϕ$ is compact but fails to have a minimal commutant. This answers a question raised by Worner [35].

Finally, in the Appendix we prove the facts about the Sobolev space $W^{1,2}[0, +∞)$ that were used in Section 4. They can be found in the book [1], where they are stated in a much more general way. Consequently, their proofs are more complicated and we felt that the proofs of these simpler statements would help the exposition.

1. Some general results about operators with a minimal commutant

We present in this section some general results that are interesting for our purposes. The first one deals with those multiplication operators that commute with a composition operator induced by an analytic self-map $ϕ$ of the unit disk, not necessarily a linear fractional map. This result had been obtained earlier by Worner [35, Theorem 9] under the additional assumptions that $ϕ$ is not an automorphism and all its iteration sequences are interpolating.

**Theorem 1.1.** Let $ϕ$ be a non-constant, analytic self-map of the unit disk, let $b ∈ H^∞(D)$ such that $b ∙ ϕ = b$, and consider the multiplication operator $M_b ∈ B(H^2(D))$ defined by the expression $M_b f = b ∙ f$. Then $M_b ∈ \{C_ϕ\}'$, and if in addition $b$ is non-constant, then $M_b ∉ \text{alg}(C_ϕ)^\sigma$.

**Proof.** For any $f ∈ H^2(D)$, we have $C_ϕ M_b f = (b ∙ ϕ) ∙ (f ∙ ϕ) = b ∙ (f ∙ ϕ) = M_b C_ϕ f$, so $M_b ∈ \{C_ϕ\}'$. Next, if $b$ is non-constant then $b$ and $b^2$ are linearly independent, so that there exists $g ∈ H^2(D)$ such that $⟨b, g⟩ = 0$ and $⟨b^2, g⟩ = 1$. Let us define a linear functional $Λ: B(H^2(D)) → \mathbb{C}$ by the expression

$$Λ(X) = ⟨Xb, g⟩.$$ 

It is clear that $Λ$ is continuous in the weak operator topology, and therefore, the subspace $\text{ker} Λ$ is closed in the weak operator topology. We have $Λ(p(C_ϕ)) = ⟨p(C_ϕ)b, g⟩ = p(1)⟨b, g⟩ = 0$, for every polynomial $p$, and therefore $\text{alg}(C_ϕ)^\sigma ⊆ \text{ker} Λ$. On the other hand, $Λ(M_b) = ⟨M_bb, g⟩ = ⟨b^2, g⟩ = 1$, so that $M_b ∉ \text{ker} Λ$, and it follows that $M_b ∉ \text{alg}(C_ϕ)^\sigma$, as we wanted. □

**Remark 1.2.** Cloud [5, Theorem 3 and Theorem 4] showed that, if $ϕ$ is an analytic self-map of the unit disk that is not a periodic elliptic automorphism and fixes a point in the unit disk, then the only bounded analytic functions $b ∈ H^∞(D)$ that satisfy $b ∙ ϕ = b$ are the constant functions. His proof is a nice application of the Denjoy-Wolff theorem.

Our next result ensures that if a direct sum of Hilbert space operators has a minimal commutant then both summands inherit that property.

**Lemma 1.3.** Let us consider a direct sum $H = H_1 ⊕ H_2$, let $A_j ∈ B(H_j)$ for $j = 1, 2$, and let $A = A_1 ⊕ A_2$.

If $A$ has a minimal commutant, then so do both $A_1$ and $A_2$.

**Proof.** Let $X_j ∈ \{A_j\}'$, for $j = 1, 2$, and let $X = X_1 ⊕ X_2$. It is easy to see that $AX = XA$. Since $A$ has a minimal commutant, there is a net of polynomials $(p_d)_{d ∈ D}$ such that $p_d(A) → X$ as $d ∈ D$ in the weak operator topology. Then, it is clear that $p_d(A) = p_d(A_1) ⊕ p_d(A_2)$, and it follows that $p_d(A_j) → X_j$ as $d ∈ D$ in the weak operator topology, for $j = 1, 2$, as we wanted. □

The next two results in this section ensure that the property for a Hilbert space operator of having a minimal commutant is preserved under similarities and under the operation of taking adjoints.

**Lemma 1.4.** If $A ∈ B(H)$ has a minimal commutant, $S$ is an invertible operator and $B = S^{-1}AS$, then $B$ has a minimal commutant, too.
then there exists \( \lambda \in C \) such that \( \lambda f \in \sigma(A) \) and \( \|\lambda f\| = \|f\| \). Finally, \( \|Xf\| = |\lambda| \cdot \|f\| \), and since \( f \neq 0 \) it follows that \( |\lambda| \leq \|X\| \).

\section{Composition operators induced by parabolic and hyperbolic automorphisms}

Our goal in this section is to show that, when \( \varphi \) is either a parabolic or a hyperbolic automorphism, the commutant of \( C_\varphi \) is not minimal. We present two different proofs. The first one is based on the idea of a strongly compact algebra, and it is an immediate consequence of a theorem by Shapiro [31], while the second one is an application of Theorem 1.1.

We say that a subalgebra \( A \subseteq B(H) \) is strongly compact provided that its unit ball \( \{ A \in A : \|A\| \leq 1 \} \) is precompact in the strong operator topology. This notion was introduced by Lomonosov [22] in connection with the invariant subspace problem for essentially normal operators.

An example of a strongly compact algebra is the commutant of a compact operator with dense range. Another example is the commutant of the adjoint of the shift operator on the Hardy space.

Marsalli [25] provided a classification of operator algebras and characterized those self-adjoint subalgebras of operators on Hilbert space that are strongly compact.

Lomonosov and the first and fourth authors [16] established a number of results and constructed many examples and counter-examples concerning strongly compact algebras.

The first and fourth authors [17] studied the class of strongly compact, normal operators, obtaining results closely related to those of Froelich and Marsalli [12], and they provided further examples and counter-examples of strongly compact algebras. They also considered the issue of strong compactness for algebras of analytic functions [18] and they strengthened earlier results of Froelich and Marsalli [12].

Romero de la Rosa and the first author [19] provided a local spectral condition for strong compactness and they applied it to the class of bilateral weighted shifts.

Shapiro [31] studied the strong compactness of the algebras \( \text{alg}(C_\varphi) \) and \( (C_\varphi)' \), where \( C_\varphi \) is a composition operator induced on \( H^2(D) \) by a linear fractional self-map \( \varphi \) of the unit disk, completing earlier work of Fernández-Valles and the first author [11].

Marsalli [25] provided a sufficient condition for strong compactness that is interesting for our purposes and that can be stated as follows.

\begin{proof}
Let \( Y \in \{ B \}' \) and let \( X = SYS^{-1} \). Then \( X \in \{ A \}' \), and since \( A \) has a minimal commutant, there is a net \( (p_d)_{d \in D} \) of polynomials such that \( p_d(A) \to X \) as \( d \in D \) in the weak operator topology. It follows that \( p_d(B) = S^{-1}p_d(A)S \) converges in the weak operator topology to \( Y \), as we wanted. \qed
\end{proof}

\begin{lemma}
An operator \( A \in B(H) \) has a minimal commutant if and only if so does its adjoint \( A^* \).
\end{lemma}

\begin{proof}
Clearly, it suffices to establish the implication that if \( A \) has a minimal commutant, then so does \( A^* \). Let \( X \in \{ A^* \}' \). We have \( X^* \in \{ A \}' \), so there exists a net of polynomials \( (p_d)_{d \in D} \) such that \( p_d(A) \to X^* \) as \( d \in D \) in the weak operator topology. Since the operation of taking adjoints is continuous with respect to the weak operator topology, it follows that \( p_d(A^*) \to X \) as \( d \in D \) in the weak operator topology. Finally, consider the net of polynomials \( (q_d)_{d \in D} \) defined by the expression

\[ q_d(z) = \overline{p_d(z)}. \]

We get \( q_d(A^*) = p_d(A)^* \to X \) as \( d \in D \) in the weak operator topology, whence \( X \in \overline{\text{alg}(A^*)} \), and the proof is complete. \qed
\end{proof}

We end this section with a folk result about the commutant of an operator with a simple eigenvalue, that is easy to prove and is very handy for the study of operators with a minimal commutant.

\begin{lemma}
Let \( A \in B(H) \) and \( \alpha \in C \) such that \( \dim \ker(A - \alpha) = 1 \). If \( X \in \{ A \}' \) and \( f \in \ker(A - \alpha) \setminus \{0\} \) then there exists \( \lambda \in C \) such that \( Xf = \lambda f \) and \( |\lambda| \leq \|X\| \).
\end{lemma}

\begin{proof}
Notice that \( AXf = XAf = \alpha Xf \), so \( Xf \in \ker(A - \alpha) \). Since \( \dim \ker(A - \alpha) = 1 \), there exists \( \lambda \in C \) such that \( Xf = \lambda f \). Finally, \( \|Xf\| = |\lambda| \cdot \|f\| \), and since \( f \neq 0 \) it follows that \( |\lambda| \leq \|X\| \). \qed
\end{proof}

2. Composition operators induced by parabolic and hyperbolic automorphisms
Theorem 2.1. If a subalgebra $A \subseteq B(H)$ admits a collection of finite-dimensional subspaces invariant under $A$, whose union is a total subset of $H$, then the algebra $A$ is strongly compact.

Shapiro [31, p. 854] observed that if a subalgebra $A \subseteq B(H)$ meets the hypotheses of Theorem 2.1 then so does its closure in the weak operator topology, and therefore the closure $\overline{A}^\sigma$ is strongly compact. He also pointed out the following immediate consequence of this observation.

Theorem 2.2. If an operator $A \in B(H)$ has a total set of eigenvectors then $\overline{\text{alg}(A)}^\sigma$ is strongly compact.

This result applies to most composition operators induced by linear fractional self-maps of the unit disk, with the only exception when the symbol is a hyperbolic non-automorphism with a fixed point on $\partial D$ and the other one in $D$. On the other hand, Shapiro [31, Theorem 4.1.1] proved the following result.

Theorem 2.3. If $\varphi$ is a parabolic or hyperbolic automorphism of the unit disk, then the algebra $\{C_\varphi\}'$ fails to be strongly compact.

The main result in this section is an immediate consequence of Theorem 2.2 and Theorem 2.3.

Theorem 2.4. If $\varphi$ is either a parabolic or a hyperbolic automorphism of the unit disk then the operator $C_\varphi$ fails to have a minimal commutant.

There is a simple, direct proof of Theorem 2.4 that does not require the notion of strong compactness. Instead, it is based on the canonical models for parabolic and hyperbolic automorphisms. These canonical models are explained in detail in the paper [31] of Shapiro. Recall that a linear fractional map is called parabolic if it has precisely one fixed point, which is necessarily on the unit circle. Using conjugation by disk automorphisms, we may assume without loss of generality that the fixed point is 1. The map $\varphi$ then has the canonical form given by the expression

$$\varphi(z) = \frac{(2-a)z + a}{-az + 2 + a},$$

where $a \in \mathbb{C}$ is a constant with $\text{Re}(a) \geq 0$. Further, $\varphi$ is an automorphism if and only if $\text{Re}(a) = 0$. Also, for each $\sigma \geq 0$, the bounded analytic function

$$e_\sigma(z) = \exp\left(-\sigma \frac{1+z}{1-z}\right)$$

satisfies the equation $C_\varphi e_\sigma = e^{-\sigma a} e_\sigma$, and hence $e_\sigma \circ \varphi = e^{-\sigma a} e_\sigma$. When $\sigma = 2\pi/|a|$, we have $e^{-\sigma a} = 1$, so the desired result follows from Theorem 1.1.

Next, recall that a fractional linear map $\varphi$ is a hyperbolic automorphism if it has two fixed points and both are on $\partial \mathbb{D}$. Without loss of generality, we may assume that these points are 1 and $-1$, in which case $\varphi$ has the canonical form

$$\varphi(z) = \frac{z+r}{1+rz},$$

for some $0 < r < 1$.

For every $t \in \mathbb{R}$, the function

$$e_{it}(z) = \left(\frac{1+z}{1-z}\right)^{it}$$

belongs to $H^2(\mathbb{D})$, it is bounded and it is an eigenfunction for $C_\varphi$ corresponding to the eigenvalue

$$\lambda(it) = \left(\frac{1+r}{1-r}\right)^{it} = \exp\left(it \log \left(\frac{1+r}{1-r}\right)\right).$$

Now, choose $t \in \mathbb{R}$ such that

$$t \log \left(\frac{1+r}{1-r}\right) = 2\pi,$$

and let $b = e_{it}$. Then, $\lambda(it) = 1$, so that $b \circ \varphi = b$, and since $b$ is not constant, the desired result follows once again from Theorem 1.1.
3. The commutant of a composition operator induced by an elliptic automorphism

In this section we consider composition operators induced by elliptic automorphisms of the unit disk. Recall that a fractional linear self-map of the unit disk \( \varphi \) is said to be elliptic if it has one fixed point in \( \mathbb{D} \) and the other is outside of \( \overline{\mathbb{D}} \). Since \( \varphi \) is an automorphism, it maps the circle to itself. Without loss of generality we may assume that \( \varphi(0) = 0 \), so by reflection \( \varphi(\infty) = \infty \). Therefore, \( \varphi \) is a rotation and it has the canonical form
\[
\varphi(z) = \omega z, \quad \text{for some } \omega \in \partial \mathbb{D} \setminus \{1\}.
\]
We will show that the commutant of \( C_\varphi \) is minimal if and only if \( \omega \) is not a root of unity.

We know that the point spectrum of \( C_\varphi \) is the set \( \{\omega^n : n \in \mathbb{N}_0\} \) with the corresponding eigenfunctions the monomials \( z^n \). If \( \omega \) is a root of unity, that is, if there is an \( m \geq 1 \) such that \( \omega^m = 1 \), then there are finitely many eigenvalues for \( C_\varphi \) of infinite multiplicity, namely, the \( m \)-th roots of unity, and the corresponding eigenspaces are given by \( H_k = \text{span}\{z^n : n \equiv k \pmod{m}\} \), \( 1 \leq k \leq m \). If \( \omega \) is not a root of unity then \( \omega^n \neq \omega^m \) for all \( n \neq m \), so that there is a countable family of eigenvalues that is dense in the unit circle. Also, the eigenvalues are all simple.

Let us denote by \( A(\mathbb{D}) \) the disk algebra. It is a standard fact that \( A(\mathbb{D}) \) coincides with the uniform closure of the polynomials; see for instance the book of Hoffman [14]. We shall use the following classical extension theorem of Rudin [28] and Carleson [3].

**Theorem 3.1 (Rudin-Carleson).** Let \( E \subset \partial \mathbb{D} \) be a closed set of Lebesgue measure zero and let \( f : E \to \mathbb{C} \) be a continuous function. Then there exists \( g \in A(\mathbb{D}) \) such that \( g|_E = f \) and \( \|g\|_\mathbb{D} \leq \|f\|_E \).

The next proposition establishes a result that is slightly more general than our target in this section.

**Proposition 3.2.** Let \( H \) be complex, separable, infinite-dimensional Hilbert space and let \( \{e_n : n \in \mathbb{N}\} \) be an orthonormal basis of \( H \). Also, let \( (d_n) \) be a sequence in \( \partial \mathbb{D} \) with \( d_n \neq d_m \) for all \( n \neq m \), and let \( D \in \mathcal{B}(H) \) be the diagonal operator defined by \( De_n = d_n e_n \) for all \( n \in \mathbb{N} \). Then \( \overline{\text{span}(\{D\}')} = \{D\}' \).

**Proof.** It is a folklore result that \( \{D\}' \) is the family of all diagonal operators with respect to \( \{e_n : n \in \mathbb{N}\} \). Indeed, if \( X \in \{D\}' \), it follows from Lemma 1.6 that \( X e_n = \lambda_n e_n \) for some bounded sequence \( (\lambda_n) \) of complex scalars.

Let \( n \in \mathbb{N} \) be fixed. Consider the finite set \( E_n = \{d_1, d_1, \ldots, d_n\} \), and the continuous function \( f_n : E_n \to \mathbb{C} \), defined by the expression \( f_n(d_k) = \lambda_k \), for every \( k = 1, 2, \ldots, n \). It follows from Theorem 3.1 that there exists a function \( g_n \in A(\mathbb{D}) \) such that \( g_n(d_k) = \lambda_k \), for every \( k = 1, 2, \ldots, n \), and such that \( \|g_n\|_\mathbb{D} \leq \|\lambda_k\|_\infty \). Next, there is a polynomial \( p_n \) such that \( \|p_n - g_n\|_\mathbb{D} < 1/n \). We claim that the sequence \( (p_n(D)) \) converges in the weak operator topology to \( X \). According to the book of Dunford and Schwartz [9], it is enough to check that \( (p_n(D)) \) is bounded and it converges pointwise in norm to \( X \) on \( \{e_k : k \in \mathbb{N}\} \). To that end, for every \( k \in \mathbb{N} \) we have \( p_n(D)e_k = p_n(d_k)e_k \), so that \( p_n(D) \) is a diagonal operator with diagonal sequence \( (p_n(d_k))_{k \in \mathbb{N}} \). Thus, for every \( n \in \mathbb{N} \), we have
\[
\|p_n(D)\| = \sup_{k \in \mathbb{N}} |p_n(d_k)| \leq \|p_n\|_\mathbb{D} \leq \|g_n\|_\mathbb{D} + \|p_n - g_n\|_\mathbb{D} \leq \|\lambda_k\|_\infty + 1.
\]
This shows that the sequence \( (p_n(D)) \) is bounded. Moreover, for all \( k \in \mathbb{N} \) and for every \( n \geq k \) we get
\[
\|p_n(D)e_k - X e_k\| = |p_n(d_k) - \lambda_k| \leq |p_n(d_k) - g_n(d_k)| + |g_n(d_k) - \lambda_k| = |p_n(d_k) - g_n(d_k)| + 0 \leq \|p_n - g_n\|_\mathbb{D} \to 0, \quad \text{as } n \to \infty.
\]
Thus, we arrived at the conclusion that the sequence \( (p_n(D)) \) converges in the strong operator topology, and therefore in the weak operator topology, to the operator \( X \), as we wanted. \( \square \)

**Theorem 3.3.** Let \( \omega \in \partial \mathbb{D} \setminus \{1\} \) and let \( \varphi(z) = \omega z \), for all \( z \in \mathbb{D} \).

1. If \( \omega \) is a root of unity then \( \{C_\varphi\}' \) is not commutative, and in particular, \( \{C_\varphi\}' \neq \text{alg}(\overline{\text{alg}(C_\varphi)})' \).
2. If \( \omega \) is not a root of unity then \( \{C_\varphi\}' = \overline{\text{alg}(C_\varphi)} \).

**Proof.** First we consider the case when the multiplier is a root of unity, that is, there is an \( m \) such that \( \omega^m = 1 \). Therefore, there are finitely many eigenvalues for \( C_\varphi \) of infinite multiplicity, namely, the \( m \)-th roots of unity, and the corresponding infinite-dimensional eigenspaces are given by

\[
H_k = \text{span}\{z^n : n \equiv k \pmod{m} \}, \quad 1 \leq k \leq m.
\]

Hence, there is a decomposition of \( H^2(\mathbb{D}) \) as an orthogonal direct sum of reducing subspaces for \( C_\varphi \),

\[
H^2(\mathbb{D}) = H_1 \oplus \cdots \oplus H_m.
\]

Notice that the restriction of \( C_\varphi \) to the subspace \( \{0\} \oplus \cdots \oplus \{0\} \oplus H_k \oplus \{0\} \oplus \cdots \oplus \{0\} \) is a scalar operator. It follows easily that \( \{C_\varphi\}' = \mathcal{B}(H_1) \oplus \cdots \oplus \mathcal{B}(H_m) \), and in particular, \( \{C_\varphi\}' \) fails to be commutative.

The case when the multiplier \( \omega \) is not a root of unity follows from Proposition 3.2 applied to Hilbert space \( H = H^2(\mathbb{D}) \), the orthonormal basis \( e_n(z) = z^n \), the diagonal operator \( D = C_\varphi \) and the diagonal sequence \( (d_n) \) defined by \( d_n = \omega^n \). \( \square \)

**Remark 3.4.** Notice that, in the periodic case, the commutant is even bigger than the algebra generated by \( C_\varphi \) and the multiplications that commute with it.

4. **The Commutant of a Composition Operator Induced by a Parabolic Non-Automorphism**

In this section we will demonstrate that, if \( \varphi \) is a parabolic non-automorphism of the unit disk, then \( C_\varphi \) has a minimal commutant. Our path to this result is based on an identification between \( H^2(\mathbb{D}) \) and the Sobolev space \( W^{1,2}(0, +\infty) \).

As we mentioned in Section 2, one may assume that a parabolic fractional linear map is of the form

\[
\varphi(z) = \frac{(2 - a)z + a}{-az + (2 + a)}, \quad \text{for some } a \in \mathbb{C} \text{ with } \text{Re}(a) \geq 0. \tag{4.1}
\]

Further, \( \varphi \) is not an automorphism if and only if \( \text{Re}(a) > 0 \). The family of eigenvalues for \( C_\varphi \) is the set \( \sigma_p(C_\varphi) = \{e^{-\sigma a} : \sigma \geq 0\} \) and every eigenvalue is simple. An eigenfunction \( e_\sigma \) corresponding to an eigenvalue \( e^{-\sigma a} \) with \( \sigma \geq 0 \) is the bounded analytic function

\[
e_\sigma(z) = \exp\left(-\sigma \frac{1 + z}{1 - z}\right). \tag{4.2}
\]

Recall that the Sobolev space \( W^{1,2}(0, +\infty) \) consists of all functions \( f \in L^2[0, +\infty) \) that are absolutely continuous on each bounded subinterval of the interval \( [0, +\infty) \) and such that \( f' \in L^2[0, +\infty) \). It is easy to prove the standard fact that \( W^{1,2}(0, +\infty) \) is Hilbert space provided with the inner product

\[
\langle f, g \rangle := \int_0^{+\infty} \left[ f(\sigma)\overline{g(\sigma)} + f'(\sigma)\overline{g'(\sigma)} \right] \, d\sigma.
\]

The corresponding norm is denoted by \( \| \cdot \|_{1,2} \).

Alfonso Montes-Rodríguez, Manuel Ponce-Escudero and Stanislav Shkarin [24] established the remarkable fact that the adjoint \( C_\varphi^* \) is similar to a certain multiplication operator on the Sobolev space \( W^{1,2}(0, +\infty) \). More precisely, they proved the following theorem.
Theorem 4.1. Let \( \varphi \) be a parabolic non-automorphism of the unit disk in the canonical form \((4.1)\) and let \( \{e_\sigma: \sigma \geq 0\} \) be the family of eigenfunctions for \( C_\varphi \) given by equation \((4.2)\). The linear map \( S \) defined by the expression \((Sf)(\sigma) := \langle f, e_\sigma \rangle\) is an isomorphism from \( H^2(\mathbb{D}) \) onto \( W^{1,2}[0, +\infty) \). Moreover, \( C^* \varphi = S^{-1}M_\varphi S \), where \( M_\varphi \) denotes the operator of multiplication by \( \varphi(\sigma) := e^{-\pi \sigma} \) on \( W^{1,2}[0, +\infty) \). Finally, the function \( \varphi \) is a cyclic vector for the operator \( M_\varphi \), that is, the family \( \{\varphi^n: n \in \mathbb{N}\} \) is a total subset of \( W^{1,2}[0, +\infty) \).

Theorem 4.2. If \( f, g \in W^{1,2}[0, +\infty) \) then \( fg \in W^{1,2}[0, +\infty) \), and moreover, \( \|fg\|_{1,2} \leq \sqrt{2}\|f\|_{1,2}\|g\|_{1,2} \). Hence, the Sobolev space \( W^{1,2}[0, +\infty) \) is a commutative Banach algebra without identity with respect to pointwise multiplication and the equivalent norm \( \|f\|_{1,2} = \sqrt{2}\|f\|_{1,2} \).

We will prove Theorem 4.2 in the Appendix. The proof of a more general result can be found in [1, Theorem 5.23]. The key for the proof of Theorem 4.2 is the so-called Sobolev embedding theorem, that can be stated as follows.

Theorem 4.3. If \( f \in W^{1,2}[0, +\infty) \), we have \( f \in L^\infty[0, +\infty) \), and moreover, \( \|f\|_\infty \leq \|f\|_{1,2} \).

Once again, we will provide the proof in the Appendix, and refer the reader to [1, Corollary 5.16] for a proof of a more general result.

Our goal is to show that the operator \( M_\varphi \), acting on \( W^{1,2}[0, +\infty) \), has a minimal commutant, whenever \( \varphi \) is a cyclic vector. We will, in fact, show that this is true on a large class of Banach algebras.

Let \( \mathcal{A} \) be a Banach algebra, let \( a \in \mathcal{A} \), and consider the multiplication operator \( M_a \) defined on \( \mathcal{A} \) by the expression \( M_ab = ab \). Notice that \( M_a \) is a bounded linear operator with \( \|M_a\| \leq \|a\| \). An element \( a \in \mathcal{A} \) is said to be cyclic when it is a cyclic vector for \( M_a \), that is, when the family \( \{a^n: n \in \mathbb{N}\} \) is a total subset of \( \mathcal{A} \). Notice that if a Banach algebra \( \mathcal{A} \) has a cyclic element then \( \mathcal{A} \) is abelian and separable.

Remark 4.4. We stated in the Introduction the definition of an operator on Hilbert space with a minimal commutant, and the same definition makes sense for an operator on a general Banach space, and in particular, for a multiplication operator on a Banach algebra.

We start our program by considering the case when \( \mathcal{A} \) has an identity.

Theorem 4.5. If \( \mathcal{A} \) is a unital Banach algebra and \( a \in \mathcal{A} \) is an element such that \( \{1, a, \ldots, a^m, \ldots\} \) is a total subset of \( \mathcal{A} \), then the multiplication operator \( M_a \) has a minimal commutant.

Proof. Let \( X \in \{M_a\}' \). We will show that \( X \in \overline{\text{alg}(M_a)}^\sigma \). It follows by induction that \( M_a^nX = XM_a^n \), for all \( n \in \mathbb{N} \). Now, since the family \( \{1, a, \ldots, a^m, \ldots\} \) is a total subset of \( \mathcal{A} \), there exists a sequence of polynomials \( \{p_n\} \) such that

\[
\lim_{n \to \infty} \|p_n(a) - X1\| = 0. \tag{4.3}
\]

We claim that \( \{p_n(M_a)\} \) converges in the strong operator topology to \( X \), and therefore \( X \in \overline{\text{alg}(M_a)}^\sigma \). Now, for the proof of our claim, the sequence of vectors \( \{p_n(a)\} \) is bounded because it is convergent, so that there is a constant \( c > 0 \) such that \( \|p_n(a)\| \leq c \), for all \( n \in \mathbb{N} \). Next, the sequence of operators \( \{p_n(M_a)\} \) is also bounded, because \( \|p_n(M_a)\| = \|M_{p_n(a)}\| \leq \|p_n(a)\| \leq c \). Finally, in order to show that
\((p_n(M_a))\) converges to \(X\) in the strong operator topology, it is enough to prove the convergence on the total set \(\{1, a, \ldots, a^m, \ldots\}\). Using (4.3) we see that
\[
\|p_n(M_a)1 - X1\| = \|M_{p_n(a)}1 - X1\| = \|p_n(a) - X1\| \to 0, \text{ as } n \to \infty.
\]
Also, for every \(m \in \mathbb{N}\), we have
\[
\|p_n(M_a)a^m - Xa^m\| = \|p_n(a)a^m - Xa^m1\| = \|a^m p_n(a) - M_a^m X1\|
= \|a^m p_n(a) - a^m X1\| \leq \|a^m\| \cdot \|p_n(a) - X1\| \to 0, \text{ as } n \to \infty,
\]
and this completes the proof of our claim. \(\Box\)

**Remark 4.6.** Notice that, under the hypotheses of Theorem 4.5, we have \(\{M_a\}' = \{M_b : b \in A\}\). Indeed, it is clear that the subalgebra on the right hand side is contained in \(\{M_a\}'\). On the other hand, if \(X \in \{M_a\}'\) then we know from Theorem 4.5 that there exists a sequence of polynomials \((p_n)\) such that \(p_n(M_a) \to X\), as \(n \to \infty\), in the strong operator topology. Let \(b = X1\). We claim that \(X = M_b\). Indeed, we have \(\|p_n(a) - b\| = \|p_n(M_a)1 - X1\| \to 0, \text{ as } n \to \infty\), and it follows that \(p_n(M_a) \to M_b\) in the strong operator topology, whence \(X = M_b\). \(\Box\)

Theorem 4.5 cannot be applied to \(W^{1,2}(0, +\infty)\) because the latter is a non-unital Banach algebra. In order to extend Theorem 4.5 we start with the following lemma.

**Lemma 4.7.** If \(A\) is a non-unital Banach algebra and \(a \in A\) is a cyclic element then
\[
\text{alg}(M_a)^\sigma = \{M_b + \beta I : b \in A, \beta \in \mathbb{C}\}^\sigma.
\]

**Proof.** It is clear that the family \(\{M_b + \beta I : b \in A, \beta \in \mathbb{C}\}\) is a subalgebra of \(\mathcal{B}(A)\) that contains \(M_a\) and the identity operator, so that it also contains \(\text{alg}(M_a)\). Hence, \(\text{alg}(M_a)^\sigma \subseteq \{M_b + \beta I : b \in A, \beta \in \mathbb{C}\}^\sigma\).

Conversely, if \(b \in A\) and \(\beta \in \mathbb{C}\), then there is a sequence of polynomials \((p_n)\) such that \(p_n(0) = 0\) and \(\|b - p_n(a)\| \to 0\), as \(n \to \infty\). If we set \(q_n(z) = p_n(z) + \beta\), then \(q_n(M_a) = M_{p_n(a)} + \beta I \to M_b + \beta I\), as \(n \to \infty\), in the strong operator topology. This shows that \(\{M_b + \beta I : b \in A, \beta \in \mathbb{C}\} \subseteq \text{alg}(M_a)^\sigma\), and it finally follows that \(\{M_b + \beta I : b \in A, \beta \in \mathbb{C}\}^\sigma \subseteq \text{alg}(M_a)^\sigma\), as we wanted. \(\Box\)

Recall that an approximate identity in a Banach algebra \(A\) is any sequence \((e_n)\) such that \(\|e_n b - b\| \to 0\), as \(n \to \infty\), for every \(b \in A\). This means that \(M_{e_n} \to I\), as \(n \to \infty\), in the strong operator topology.

**Theorem 4.8.** If \(A\) is a commutative Banach algebra with an approximate identity and \(a \in A\) is a cyclic element, then the multiplication operator \(M_a\) has a minimal commutant.

**Proof.** First, let \((e_n)\) be an approximate identity. Since \(a\) is cyclic, there is a sequence of polynomials \((p_n)\) such that \(p_n(0) = 0\) and \(\|e_n - p_n(a)\| \to 0\), as \(n \to \infty\). We have, for every \(b \in A\),
\[
\|p_n(a)b - b\| \leq \|p_n(a)b - e_n b\| + \|e_n b - b\| \\
\leq \|b\| \cdot \|p_n(a) - e_n\| + \|e_n b - b\| \to 0, \text{ as } n \to \infty,
\]
so that \((p_n(a))\) is also an approximate identity. We claim that, for every \(X \in \{M_a\}'\), we have \(XM_a = M_b\), where \(b = Xa\). Indeed, for every \(n \in \mathbb{N}\), we get \(XM_a a^n = XM_a^n a = M_a^n Xa = a^n b = ba^n = M_a a^n\), and since \(a\) is cyclic, the claim follows. Next, since \(p_n(0) = 0\), we have \(p_n(z) = zq_n(z)\), for some polynomial \(q_n\). Then, let \(X \in \{M_a\}'\). We get \(Xp_n(M_a) = XM_a q_n(M_a) = M_b q_n(M_a)\) so that \(M_b q_n(M_a) \to X\), as \(n \to \infty\), in the strong operator topology. This shows that \(X \in \{M_c + \gamma I : c \in A, \gamma \in \mathbb{C}\}^\sigma\), and the desired result finally follows from Lemma 4.7. \(\Box\)

The significance of Theorem 4.8 comes from the fact that the Sobolev algebra has an approximate identity.
Lemma 4.9. The sequence of functions \((e_n)\) defined by the expression

\[
e_n(\sigma) = \begin{cases} 
1, & \text{if } 0 \leq \sigma \leq n, \\
\sigma^{-(\sigma-n)}, & \text{if } \sigma \geq n,
\end{cases}
\]

is an approximate identity for the Sobolev algebra \(W^{1,2}(0, +\infty)\).

We will prove Lemma 4.9 in the Appendix.

Now, everything is ready for the main result of this section.

Theorem 4.10. If \(C_\varphi\) is a composition operator on \(H^2(\mathbb{D})\), induced by a parabolic non-automorphism \(\varphi\) of the unit disk, then \(C_\varphi\) has a minimal commutant.

Proof. We may assume without loss of generality that the map \(\varphi\) has the canonical form as in (4.1).

By Lemma 1.5, it suffices to show that the adjoint operator \(C_\varphi^*\) has a minimal commutant. We know from Theorem 4.2 that \(C_\varphi^*\) is similar to the multiplication operator \(M_\psi\) on the Sobolev space, where \(\psi(\sigma) = e^{-\sigma}\). Next, according to Theorem 1.4, it suffices to show that \(M_\psi\) has a minimal commutant.

We know from Theorem 4.4 that the Sobolev space \(W^{1,2}(0, +\infty)\) is a commutative Banach algebra without identity with respect to pointwise multiplication and the equivalent norm \(\|f\|_{1,2} = \sqrt{2}\|f\|_{1,2}\), and that the function \(\psi\) is a cyclic element. We also know from Lemma 4.9 that the Banach algebra \(W^{1,2}(0, +\infty)\) has an approximate identity. It finally follows from Theorem 4.8 that \(M_\psi\) has a minimal commutant. \(\Box\)

5. The commutant of a composition operator induced by a hyperbolic non-automorphism

Let us consider a composition operator induced on the Hardy space by a hyperbolic non-automorphism of the unit disk. Worner [35] showed that \(C_\varphi\) fails to have a minimal commutant using the notion of an analytic self-map \(\varphi\) of the unit disk whose iterates \((z_n)\), defined as \(z_{n+1} = \varphi(z_n)\), are interpolating.

We present in this section a direct approach to this issue that is based on the ideas of Worner but that does not require the notion of an interpolating sequence.

Recall that \(\varphi\) is a hyperbolic non-automorphism if it has one fixed point on \(\partial \mathbb{D}\) and the other one does not belong to \(\partial \mathbb{D}\). There are two possibilities, depending on whether the second fixed point belongs to \(\hat{\mathbb{C}} \setminus \mathbb{D}\) or to \(\mathbb{D}\). In the former case, the fixed points can be assumed to be 1 and \(\infty\), and the symbol has the form \(\varphi(z) = rz + (1 - r)\), for some \(0 < r < 1\). Deddens [8, Theorem 3 (iv)] showed that, in this case, the point spectrum of \(C_\varphi\) is the punctured disk

\[
\sigma_p(C_\varphi) = \{ \lambda \in \mathbb{C} : 0 < |\lambda| < r^{-1/2} \}.
\]

He also noticed that for every \(s \in \mathbb{C} \) with \(\operatorname{Re}(s) > -1/2\), the function \(e_s\) defined by \(e_s(z) = (1 - z)^s\) belongs to the Hardy space \(H^2(\mathbb{D})\) and it satisfies the functional equation

\[
(C_\varphi e_s)(z) = r^s e_s(z),
\]

so that \(e_s\) is an eigenfunction of \(C_\varphi\) corresponding to the eigenvalue \(r^s\). In particular, if we choose \(t \in \mathbb{R}\) such that \(t \log r = 2\pi\), we get \(r^t = 1\), so that \(b = e_{it}\) is a non-constant, bounded analytic function such that \(b \circ \varphi = b\). Applying Theorem 1.1 we get

Theorem 5.1. If \(\varphi\) is a linear fractional self-map of the unit disk that fixes a point on \(\partial \mathbb{D}\) and the other one in \(\hat{\mathbb{C}} \setminus \mathbb{D}\), then \(\{C_\varphi\}' \neq \text{alg}(C_\varphi)^\sigma\).

The other possibility, that the second fixed point is in \(\mathbb{D}\), is a bit more delicate, because according to Remark 1.2, all the multiplication operators in the commutant of \(C_\varphi\) are scalar multiples of the identity, so that the strategy we used in the former case does not work here. In this case the fixed points can be assumed to be 1 and 0, and the symbol has the canonical form

\[
\varphi(z) = \frac{rz}{1 - (1 - r)z}, \quad \text{for some } 0 < r < 1.
\]
Theorem 5.2. If \( \varphi \) is a fractional linear self-map of the unit disk that fixes a point on \( \partial \mathbb{D} \) and the other one in \( \mathbb{D} \), then \( \{ C_\varphi \}^\prime \neq \text{alg}(C_\varphi) \).

Proof. Without loss of generality we will assume that \( \varphi \) is given by (5.1). The following construction appears in the paper of Shapiro [31, p. 864] and some ideas go back to his joint paper with Bourdon [2]. Let \( H^2_0(\mathbb{D}) := \{ f \in H^2(\mathbb{D}) : f(0) = 0 \} \) and let (1) be the subspace of \( H^2(\mathbb{D}) \) consisting of the constant functions. Thus, there is an orthogonal decomposition \( H^2(\mathbb{D}) = (1) \oplus H^2_0(\mathbb{D}) \), where both summands are invariant under \( C_\varphi \). Now, let \( \psi(z) = rz + (1 - r) \). Then, the unitary operator \( U : H^2_0(\mathbb{D}) \to H^2(\mathbb{D}) \) defined by \( (Uf)(z) = z^{-1}f(z) \) satisfies \( U^*f(z) = zf(z) \) and

\[
(C_\varphi|_{H^2_0(\mathbb{D})})^* = U^*(rC_\varphi)U,
\]

where \( C_\varphi \) is defined on \( H^2(\mathbb{D}) \). Since the restriction of \( C_\varphi \) to the subspace \( (1) \) is the identity operator \( I_1 \), it follows that \( C_\varphi^* \) is unitarily equivalent to \( I_1 \oplus rC_\varphi \), acting on \( (1) \oplus H^2(\mathbb{D}) \). We know from Theorem 5.1 that \( C_\varphi \), and therefore \( rC_\varphi \), fail to have a minimal commutant. Also, it follows from Lemma 1.3 that \( I_1 \oplus rC_\varphi \) fails to have a minimal commutant. Finally, it follows from Lemma 1.4 and Lemma 1.5 that \( C_\varphi \) does not have a minimal commutant, and the proof is complete. \( \square \)

Remark 5.3. As a byproduct of Theorem 5.1, the Euler operator does not have a minimal commutant. Recall that the Euler operator \( E_r \), where \( 0 < r < 1 \), is defined on Hilbert space \( \ell^2 \) by

\[
(E_rf)(n) = \sum_{k=0}^n \binom{n}{k} r^k (1 - r)^{n-k} f(k).
\]

Deddens [8, p. 862] proved that \( E_r \) is unitarily equivalent to the adjoint of the composition operator \( C_\varphi \) where \( \varphi(z) = rz + (1 - r) \). Hence, \( E_r \) fails to have a minimal commutant, too.

6. The commutant of a composition operator induced by a loxodromic mapping

In this section we will demonstrate that, if \( \varphi \) is a loxodromic linear fractional map, then \( C_\varphi \) has a minimal commutant. To accomplish this goal we will establish a more general result. Namely, we will assume that \( \varphi \) is a univalent, analytic self-map of the unit disk, with a fixed point in \( \mathbb{D} \), but not necessarily a linear fractional map. The additional constraints will be imposed, instead, on the range of the associated Königs function \( \sigma \). We will show that, in that case, \( C_\varphi \) has a minimal commutant (Theorem 6.1 below). In the special case when \( \varphi \) is a loxodromic map, we will verify that all the hypotheses of Theorem 6.1 are satisfied, so it will follow that \( C_\varphi \) has a minimal commutant (Corollary 6.2 below).

Let \( \varphi \) be a univalent, analytic self-map \( \varphi \) of the unit disk, with a fixed point in \( \mathbb{D} \). Without loss of generality, we will assume that \( \varphi(0) = 0 \). Let us denote \( \alpha = \varphi'(0) \). Notice that \( \alpha \neq 0 \) because \( \varphi \) is a univalent analytic function that fixes the origin, hence it has a non-vanishing derivative at the origin. We will also assume that \( \varphi \) is not an automorphism of the unit disk, so that \( 0 < |\alpha| < 1 \). Königs [15] showed that there is a univalent, analytic function \( \sigma : \mathbb{D} \to \mathbb{C} \) such that \( \sigma \circ \varphi = \alpha \sigma \) and, therefore, \( \sigma(0) = 0 \). The function \( \sigma \) is unique up to a constant factor and it is known as the Königs function of \( \varphi \). Moreover, if \( \lambda \in \mathbb{C} \) and \( f : \mathbb{D} \to \mathbb{C} \) is an analytic function such that \( f \circ \varphi = \lambda f \), then there exists \( n \in \mathbb{N}_0 \) such that \( \lambda = \alpha^n \) and \( f \) is a constant multiple of \( \sigma^n \). In the language of operators, all eigenspaces of \( C_\varphi \) are one dimensional.

Notice that \( G = \sigma(\mathbb{D}) \) is a simply connected domain such that \( \alpha G \subseteq G \). Since \( \sigma(0) = 0 \), we have \( 0 \in G \). Also, notice that \( \sigma : \mathbb{D} \to G \) is a conformal mapping with the nice property that

\[
\varphi(z) = \sigma^{-1}(\alpha \sigma(z)), \quad \text{for all } z \in \mathbb{D}.
\]

Recall that a set \( S \subseteq \mathbb{C} \) is said to be starlike with respect to the origin provided that

\[
rS \subseteq S,
\]
for all $0 \leq r \leq 1$. Notice that if $S \subseteq \mathbb{C}$ is starlike with respect to the origin, then it is connected because, in fact, it is pathwise connected. Also, recall that a set $S \subseteq \mathbb{C}$ is said to be \textit{strictly starlike} with respect to the origin if

$$rS \subseteq S,$$

for all $0 \leq r < 1$.

The main result of this section can be stated as follows.

**Theorem 6.1.** Let $C_\varphi$ be a composition operator induced by a univalent, analytic self-map $\varphi$ of the unit disk with $\varphi(0) = 0$. Let $\sigma$ be the Königs function of $\varphi$, and suppose that its range $\sigma(\mathbb{D})$ is bounded and strictly starlike with respect to the origin. Then the operator $C_\varphi$ has a minimal commutant.

An immediate consequence of Theorem 6.1 is the following corollary.

**Corollary 6.2.** If $\varphi$ is a linear fractional loxodromic self-map of the unit disk, then the corresponding composition operator $C_\varphi$ on the Hardy space $H^2(\mathbb{D})$ has a minimal commutant.

**Proof.** Recall that $\varphi$ is loxodromic if it has one fixed point in $\mathbb{D}$ and the other one outside of $\overline{\mathbb{D}}$ and if, in addition, $\varphi$ is not an automorphism. Without loss of generality we may assume that $\varphi(0) = 0$, in which case the canonical form for $\varphi$ is

$$\varphi(z) = \frac{z}{az + b},$$

where $a$ and $b$ are complex numbers such that $|b| > 1$ and $|a| < |1 - b|$. Notice that $\varphi$ is univalent. The corresponding Königs function is the linear fractional map given by the expression

$$\sigma(z) = \frac{z}{az + b - 1}.$$ 

Then, the Königs’s domain $\sigma(\mathbb{D})$ is an open disk that contains the origin, so that $\sigma(\mathbb{D})$ is bounded and convex, hence strictly starlike with respect to the origin. It follows from Theorem 6.1, that the operator $C_\varphi$ has a minimal commutant. \hfill $\square$

The proof of Theorem 6.1 relies on several lemmas that we state and prove before we proceed with the proof of the main result.

**Lemma 6.3.** Assume $(\psi_n)$ is a sequence of analytic self-maps of the unit disk that converges pointwise to an analytic self-map $\psi$ of the unit disk, and suppose that the sequence of composition operators $(C_{\psi_n})$ is uniformly bounded. Then $C_{\psi_n} \to C_\psi$, as $n \to \infty$, in the weak operator topology.

**Proof.** We need to show that $\langle C_{\psi_n}f, g \rangle \to \langle C_\psi f, g \rangle$, as $n \to \infty$, for every $f, g \in H^2(\mathbb{D})$. Since the sequence $(C_{\psi_n})$ is uniformly bounded, it suffices to verify the convergence above when $g$ belongs to a total set. It is clear that the family of reproducing kernels $\{K_z : z \in \mathbb{D}\}$ is a total set. Finally, since $f$ is continuous in $\mathbb{D}$, we get

$$\langle C_{\psi_n}f, K_z \rangle = f(\psi_n(z)) \to f(\psi(z)) = \langle C_\psi f, K_z \rangle, \quad \text{as } n \to \infty,$$

for every $z \in \mathbb{D}$, as we wanted. \hfill $\square$

Let $G$ be a domain in $\mathbb{C}$. In what follows it will be of interest to consider the set

$$\Omega := \{w \in \mathbb{C} : w \overline{G} \subset G\}, \quad (6.2)$$

**Lemma 6.4.** Let $G \subseteq \mathbb{C}$ be a bounded domain that is strictly starlike with respect to the origin, and let $\Omega \subseteq \mathbb{C}$ be defined by equation (6.2). Then, $\Omega$ is a domain in $\mathbb{C}$ that contains the interval $[0, 1]$.

**Proof.** Inclusion $[0, 1] \subseteq \Omega$ follows from the definition of a strictly starlike set. Next, we notice that $\Omega$ is starlike with respect to the origin, hence connected. Indeed, let $0 < r < 1$, $w \in \Omega$ and $z \in \overline{G}$. Then
$wz \in G$, so $rwz \in G$, whence $rw \in \Omega$. Thus, it remains to show that $\Omega$ is open. For every $w \in \mathbb{C}$, the function $f_w$ defined by $f_w(z) := wz$, $z \in \mathbb{C}$, is continuous on $\mathbb{C}$. Next, notice that the mapping

$$F: \mathbb{C} \to C(\mathbb{G})$$

$$w \mapsto F(w) = f_w$$

is Lipschitz continuous, since $\|F(w_1) - F(w_2)\|_\infty \leq M|w_1 - w_2|$, where $M = \sup\{|z|: z \in \mathbb{G}\}$. We will show that the family $\mathcal{U} := \{f \in C(\mathbb{G}) : f(\mathbb{G}) \subseteq G\}$ is an open set in $C(\mathbb{G})$, whence it will follow that $\Omega = F^{-1}(\mathcal{U})$ is an open subset of $\mathbb{C}$.

In order to show that $\mathcal{U}$ is open, we notice that $\mathbb{G}$ is a compact set. Let $f \in \mathcal{U}$. Then $f(\mathbb{G})$ is compact as well, so there exists $\varepsilon > 0$ such that $\text{dist}(f(\mathbb{G}), G^c) \geq \varepsilon$. Let $g \in \mathcal{U}$, and suppose that $\|g - f\|_\infty < \varepsilon$. If $z \in \mathbb{G}$, then $f(z) \in G$, and $|g(z) - f(z)| < \varepsilon$, so that $g(z) \in G$, and the proof is complete. □

The strategy for the proof of Theorem 6.1 is to consider, for every $w \in \Omega$, the analytic self-map $\varphi_w$ of the unit disk defined by the expression

$$\varphi_w(z) = \sigma^{-1}(w\sigma(z)). \quad (6.3)$$

We are using $G = \sigma(\mathbb{D})$, so $\varphi_w$ is well-defined. We have $\sigma \circ \varphi_w = w\sigma$, and therefore $\sigma^n \circ \varphi_w = (\sigma \circ \varphi_w)^n = w^n\sigma^n$, so that $C_{\varphi_w}\sigma^n = w^n\sigma^n$, for all $n \in \mathbb{N}$. Also, $\varphi_w(0) = 0$, and it follows from Littlewood’s theorem [30, p. 16] that $\|C_{\varphi_w}\| \leq 1$, for all $w \in \Omega$.

**Corollary 6.5.** Under the assumptions of Theorem 6.1, we have $C_{\varphi_r} \to I$, as $r \to 1^-$, in the weak operator topology.

**Proof.** Let $(r_n)$ be a sequence of scalars with $0 < r_n < 1$ and $r_n \to 1$, as $n \to \infty$. Let $\psi_n(z) = \varphi_{r_n}(z)$ and $\psi(z) = z$, for every $z \in \mathbb{D}$. Notice that $C_\psi = I$. We know that $\|C_{\psi_n}\| \leq 1$, for all $n \in \mathbb{N}$. Then, since $\sigma^{-1}$ is continuous in $\sigma(\mathbb{D})$, which is starlike with respect to the origin by hypothesis, we have $\psi_n(z) = \sigma^{-1}(r_n\sigma(z)) \to \sigma^{-1}(\sigma(z)) = \psi(z)$, as $n \to \infty$, for every $z \in \mathbb{D}$. It follows from Lemma 6.3 that $C_{\varphi_{r_n}} \to I$, as $n \to \infty$, in the weak operator topology. □

Next, we need a result of Mattner [23] about complex differentiation of parametric integrals that can be stated as follows.

**Lemma 6.6.** Let $(\Theta, \Sigma, \mu)$ be a measure space, let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f: \Omega \times \Theta \to \mathbb{C}$ be a function such that

1. $f(w, \cdot)$ is $\Sigma$-measurable for every $w \in \Omega$,
2. $f(\cdot, \theta)$ is analytic for every $\theta \in \Theta$,
3. for every $w_0 \in \Omega$ there is some $\delta > 0$ such that $\partial(w_0, \delta) \subseteq \Omega$ and such that

$$\sup_{|w - w_0| \leq \delta} \int_{\Theta} |f(w, \theta)| d\mu(\theta) < +\infty.$$  

Then, the function $F: \Omega \to \mathbb{C}$ defined by the parametric integral

$$F(w) := \int_{\Theta} f(w, \theta) d\mu(\theta)$$

is analytic on $\Omega$.

**Lemma 6.7.** Let $\Lambda: \mathcal{B}(H^2(\mathbb{D})) \to \mathbb{C}$ be a linear functional that is continuous in the weak operator topology, and let $X \in \mathcal{B}(H^2(\mathbb{D}))$. Suppose that $\varphi$ and the corresponding König’s function $\sigma$ satisfy the hypotheses of Theorem 6.1, and let $\Omega$ be associated to $G = \sigma(\mathbb{D})$ by (6.2). Then, the function $F: \Omega \to \mathbb{C}$, defined by $F(w) := \Lambda(XC_{\varphi_w})$, is analytic.
Proof. It suffices to prove the result when $\Lambda$ has the special form $\Lambda(X) = \langle Xf, g \rangle$, for some $f, g \in H^2(\mathbb{D})$. Indeed, a classical result of Dixmier [9, p. 477] is the assertion that, if $H$ is Hilbert space, a linear functional on $B(H)$ is continuous in the weak operator topology if and only if it is continuous in the strong operator topology. Moreover, every such functional can be expressed as a linear combination of finitely many linear functionals of the form $\langle Xf, g \rangle$. Then, we have

$$F(w) = \langle XC_{\varphi_w}f, g \rangle = \overline{\langle C_{\varphi_w}f, X^*g \rangle} = \frac{1}{2\pi} \int_{0}^{2\pi} f(\sigma^{-1}(\omega\sigma(e^{i\theta})))\overline{(X^*g)(e^{i\theta})} \, d\theta.$$  

We shall apply Lemma 6.6 with $\Theta = [0, 2\pi]$ and $d\mu = d\theta/(2\pi)$ the normalized Lebesgue measure. It is clear that the function $w \mapsto f(\sigma^{-1}(\omega\sigma(e^{i\theta})))\overline{(X^*g)(e^{i\theta})}$ is analytic on $\Omega$, for every $\theta \in [0, 2\pi]$. It is also clear that the function $\theta \mapsto f(\sigma^{-1}(\omega\sigma(e^{i\theta})))\overline{(X^*g)(e^{i\theta})}$ is Lebesgue measurable, for every $w \in \Omega$. Finally, it follows from the Cauchy-Schwarz inequality that, for every $\omega \in \Omega$, we have

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left| f(\sigma^{-1}(\omega\sigma(e^{i\theta})))\overline{(X^*g)(e^{i\theta})} \right| \, d\theta \leq \|C_{\varphi_w}f\|_2 \cdot \|X^*g\|_2 \leq \|X^*\| \cdot \|f\|_2 \cdot \|g\|_2,$$

as we wanted. \qed

The heart of the proof of Theorem 6.1 is enclosed in the following result.

**Lemma 6.8.** Suppose that the map $\varphi$ and the corresponding Kőnigs function $\sigma$ satisfy the hypotheses of Theorem 6.1. There is a constant $\delta > 0$ such that, for every $w \in \mathbb{C}$ with $|w| < \delta$ and for every $X \in \{C_{\varphi}\}'$, there is a sequence of polynomials $(p_n)$ such that $\|p_n(C_{\varphi}) - XC_{\varphi}w\| \to 0$, as $n \to \infty$.

**Proof.** Let $G = \sigma(\mathbb{D})$. Since $0 \in G$ and $G$ is open, there is some $\rho > 0$ such that $\rho \mathbb{D} \subseteq G$. Also, $G$ is bounded, so there exists $M > 0$ such that $|\sigma(z)| \leq M$, for all $z \in \mathbb{D}$. Let $f \in H^2(\mathbb{D})$. The function $f \circ \sigma^{-1}$ is analytic on $G$ and its Taylor series at the origin is

$$f(\sigma^{-1}(\xi)) = \sum_{n=0}^{\infty} \frac{(f \circ \sigma^{-1})(0)}{n!} \xi^n, \quad |\xi| < \rho. \tag{6.4}$$

Let $\Omega$ be defined by (6.2). By Lemma 6.4, $\Omega$ is an open set that contains the origin. Thus, there exists $0 < \delta < \rho/M$ such that $w \in \Omega$, for all $w \in \mathbb{C}$ with $|w| < \delta$. Let $|w| < \delta$. Then $|w\sigma(z)| < \delta M < \rho$, for every $z \in \mathbb{D}$, so that

$$(C_{\varphi_w}f)(z) = f(\varphi_w(z)) = f(\sigma^{-1}(w\sigma(z))) = \sum_{n=0}^{\infty} \frac{(f \circ \sigma^{-1})(0)}{n!} w^n \sigma(z)^n.$$  

We claim that the series of analytic functions on the right hand side of the identity

$$C_{\varphi_w}f = \sum_{n=0}^{\infty} \frac{(f \circ \sigma^{-1})(0)}{n!} w^n \sigma^n \tag{6.5}$$

converges in the Hardy space $H^2(\mathbb{D})$. Indeed, it suffices to show that, for every $n \in \mathbb{N}_0$,

$$\left| \frac{(f \circ \sigma^{-1})(0)}{n!} \right| \cdot |w|^n \cdot \|\sigma^n\|_2 \leq \left( \frac{\delta M}{\rho} \right)^n \|f\|_2. \tag{6.6}$$
We have \( \|\sigma^n\|_2 \leq \|\sigma^n\|_\infty \leq M^n \), so that \( |w|^n \cdot \|\sigma^n\|_2 < (\delta M)^n \) for all \( n \in \mathbb{N}_0 \). On the other hand, if \( \gamma \) is the circle \( |\xi| = \rho \) oriented counterclockwise, then \( \gamma \) is contained in \( G \), and Cauchy’s integral formula for the derivatives yields

\[
\frac{(f \circ \sigma^{-1})^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\sigma^{-1}(\xi))}{\xi^{n+1}} \, d\xi = \frac{1}{2\pi \rho^n} \int_0^{2\pi} \frac{f(\sigma^{-1}(\rho e^{i\theta}))}{\rho e^{in\theta}} \, d\theta.
\]

Let us consider the analytic self-map of the unit disk \( \psi(z) = \sigma^{-1}(\rho z) \). Since \( \psi(0) = 0 \), we have \( \|C_\psi\| \leq 1 \).

Therefore,

\[
\left| \frac{(f \circ \sigma^{-1})^{(n)}(0)}{n!} \right| \leq \frac{1}{2\pi \rho^n} \int_0^{2\pi} |f(\sigma^{-1}(\rho e^{i\theta}))| \, d\theta \leq \frac{\rho^n}{2\pi} \left( \frac{1}{\rho} \int_0^{2\pi} |f(\sigma^{-1}(\rho e^{i\theta}))|^2 \, d\theta \right)^{1/2}.
\]

Hence, (6.6) holds for all \( n \in \mathbb{N}_0 \), and since \( \delta M/\rho < 1 \), the series in (6.5) converges in the norm of \( H^2(\mathbb{D}) \).

Next, let \( X \in \{C_\varphi\}' \). The lemma is obviously true when \( X = 0 \), so we assume that \( X \neq 0 \). The desired result is an immediate consequence of the following assertion: for every \( \varepsilon > 0 \) there is a polynomial \( p \) and an operator \( Y \in \mathcal{B}(H^2(\mathbb{D})) \) such that both of the following inequalities hold:

\[
\|p(C_\varphi) - Y\| < \frac{\varepsilon}{2}, \quad \|Y - X C_{\varphi,w}\| < \frac{\varepsilon}{2}.
\]

We start by establishing the second inequality in (6.8). Let \( n \in \mathbb{N}_0 \). It follows from Lemma 1.6, applied to the operator \( C_\varphi \), and \( f = \sigma^n \), that there is a scalar \( \lambda_n \in \mathbb{C} \) such that \( X \sigma^n = \lambda_n \sigma^n \) and \( |\lambda_n| \leq \|X\| \).

Since \( n \in \mathbb{N}_0 \) is arbitrary, (6.5) implies that, for every \( f \in H^2(\mathbb{D}) \),

\[
X C_{\varphi,w} f = \sum_{n=0}^{\infty} \lambda_n w^n \frac{(f \circ \sigma^{-1})^{(n)}(0)}{n!} \sigma^n.
\]

Now, choose \( n_0 \in \mathbb{N} \) large enough, so that

\[
\sum_{n=n_0+1}^{\infty} \left( \frac{\delta M}{\rho} \right)^n < \frac{\varepsilon}{2\|X\|},
\]

and let \( Y \in \mathcal{B}(H^2(\mathbb{D})) \) be the finite rank operator on \( H^2(\mathbb{D}) \) defined by

\[
Y f = \sum_{n=0}^{n_0} \lambda_n w^n \frac{(f \circ \sigma^{-1})^{(n)}(0)}{n!} \sigma^n, \quad \text{for } f \in H^2(\mathbb{D}).
\]

It is clear that \( Y \) is a bounded linear operator on \( H^2(\mathbb{D}) \). Since \( |\lambda_n| \leq \|X\| \), for all \( n \in \mathbb{N}_0 \), the second inequality in (6.8) follows from (6.6) and (6.9).

Thus, it remains to establish the first inequality in (6.8). Let \( \alpha = \varphi'(0) \), and recall that \( 0 < |\alpha| < 1 \). Then, there exists \( m \in \mathbb{N} \) such that \( |\alpha|^m < \delta \). Let \( \eta > 0 \) be so small that

\[
\eta \sum_{n=0}^{\infty} \left( \frac{\delta M}{\rho} \right)^n < \frac{\varepsilon}{2}. \tag{6.10}
\]
Let $K = \{\alpha^n : n \in \mathbb{N}_0\} \cup \{0\}$. It is not hard to see that $K$ is a compact set in the complex plane that has empty interior, and that $\mathbb{C} \setminus K$ is connected. Notice that the function $h : K \to \mathbb{C}$ defined by

$$h(z) = \begin{cases} \lambda_n w^n / \alpha^{mn}, & \text{if } z = \alpha_n, 0 \leq n \leq n_0, \\ 0, & \text{if } z = \alpha_n, n > n_0, \text{ or } z = 0, \end{cases}$$

is continuous. It follows from Mergelyan’s theorem that there is a polynomial $q$ such that

$$\left| q(\alpha^n) - \frac{\lambda_n w^n}{\alpha^{mn}} \right| < \eta, \quad \text{for all } 0 \leq n \leq n_0, \text{ and}$$

$$\left| q(\alpha^n) \right| < \eta, \quad \text{for all } n > n_0.$$ 

Then, consider the polynomial $p(z) = z^m q(z)$ and notice that

$$\left| p(\alpha^n) - \lambda_n w^n \right| < \eta \delta^n, \quad \text{for all } 0 \leq n \leq n_0, \text{ and}$$

$$\left| p(\alpha^n) \right| < \eta \delta^n, \quad \text{for all } n > n_0. \quad (6.11)$$

Now, let $f \in H^2(\mathbb{D})$. Using (6.1) and (6.4) we have that, for every $k \geq m$ and every $z \in \mathbb{D},$

$$(C^k_f)\langle z \rangle = f(\sigma^{-1}(\alpha^k \sigma(z))) = \sum_{n=0}^{\infty} \frac{f(\sigma^{-1})(0)}{n!} \alpha^{nk} \sigma(z)^n.$$ 

The same argument as before shows that the last series is norm convergent in $H^2(\mathbb{D})$. Indeed, the difference between the last series and the one in (6.5) is that $w$ is replaced by $\alpha^k$. Since $|\alpha^k| \leq |\alpha|^m < \delta$, for all $k \geq m$, we get an estimate analogous to (6.6), whence the series converges in the norm of $H^2(\mathbb{D})$.

If we write $p(z) = \sum_{k=m}^{d} a_k z^k$, it follows that

$$(p(C^k_f))\langle z \rangle = \sum_{k=m}^{d} a_k \sum_{n=0}^{\infty} \frac{f(\sigma^{-1})(0)}{n!} \alpha^{nk} \sigma(z)^n = \sum_{n=0}^{\infty} \frac{f(\sigma^{-1})(0)}{n!} \left( \sum_{k=m}^{d} a_k \alpha^{nk} \right) \sigma(z)^n,$$

so that

$$p(C^k_f) = \sum_{n=0}^{\infty} \frac{f(\sigma^{-1})(0)}{n!} p(\alpha^n) \sigma^n. \quad (6.12)$$

Finally, the definition of $Y$, equations (6.12), (6.7), (6.11) and the estimate $\|\sigma^n\|_2 \leq M^n$ imply that

$$\|p(C^k_f) - Yf\|_2 \leq \sum_{n=0}^{n_0} \left( \frac{f(\sigma^{-1})(0)}{n!} \right) (\rho^n) \|p(\alpha^n) \sigma^n\|_2 + \sum_{n=n_0+1}^{\infty} \left( \frac{f(\sigma^{-1})(0)}{n!} \right) (\rho^n) \|\alpha^n \sigma^n\|_2 \leq \eta \sum_{n=0}^{\infty} \left( \frac{\delta M}{\rho^n} \right) \|f\|_2.$$ 

Using (6.10) we obtain the first inequality in (6.8), and the proof is complete. \qed

Now, everything is ready for the proof of the main result of this section.

**Proof of Theorem 6.1.** We need to show that $\text{alg}(C^k_\varphi)$ is a dense linear subspace of $\{C^k_\varphi\}'$ in the weak operator topology. To that end, let $\Lambda$ be a linear functional on $H^2(\mathbb{D})$ that is continuous in the weak operator topology and that vanishes on $\text{alg}(C^k_\varphi)$. We claim that $\Lambda(X) = 0$, for all $X \in \{C^k_\varphi\}'$.

Now, for the proof of our claim, let $X \in \{C^k_\varphi\}'$, let $\sigma$ be the Kőnigs function for $\varphi$, and let $\Omega$ be the domain associated to $G = \sigma(\mathbb{D})$ by (6.2). We will consider the function $F : \Omega \to \mathbb{C}$ defined by
Proof. We prove the non trivial part of the statement. First of all, for every polynomial $p_n(C_\varphi) \to XC_{\varphi,w}$, as $n \to \infty$, in the operator norm and therefore in the weak operator topology. Now, if $|w| < \delta$ then

$$F(w) = \Lambda(XC_{\varphi,w}) = \lim_{n \to \infty} \Lambda(p_n(C_\varphi)) = 0.$$  

Since $F$ is analytic and it vanishes in an open disc, it follows from the principle of analytic continuation that $F$ vanishes identically on the domain $\Omega$. By Lemma 6.4, $(0,1) \subset \Omega$, so $F(r) = 0$, for all $0 < r < 1$. Finally, it follows from Corollary 6.5 that

$$A(X) = \lim_{r \to 1^-} \Lambda(XC_{\varphi,r}) = \lim_{r \to 1^-} F(r) = 0,$$

as we wanted. \hfill $\square$

7. Compact composition operators with univalent symbol and non-minimal commutant

As we mentioned in the Introduction, Worner [35] described a procedure for constructing a composition operator without a minimal commutant, but her technique relied on the fact that the symbol be non-univalent. She also asked the question: whether a compact composition operator with a univalent symbol has a minimal commutant.

It is the aim of this section to provide a negative answer to this question, by constructing an example of a univalent, analytic self-map $\varphi$ of the unit disk, such that $C_\varphi$ is a compact operator whose commutant fails to be minimal.

We take advantage of Tami Worner’s original idea that, for every analytic self-map $\psi$ of the unit disk, and for $\varphi = \psi \circ \psi$, the operator $C_\psi$ commutes with $C_\varphi$, and is a good candidate to be outside the closure in the weak operator topology of the unit algebra generated by the operator $C_\varphi$.

We start with an abstract result about the commutant for the square of a Hilbert space operator, that is not necessarily a composition operator on the Hardy space.

**Theorem 7.1.** Let $A \in \mathcal{B}(H)$ such that $AH \setminus \overline{A^2H} \neq \emptyset$. Then $A \in \{A^2\}'$, but $A \notin \overline{\text{alg}(A^2)}^\sigma$, hence $A^2$ fails to have a minimal commutant.

**Proof.** We prove the non trivial part of the statement. First of all, for every polynomial $p$, we have

$$p(A^2)f \in A^2H + p(0)f, \quad \text{for all } f \in H. \quad (7.1)$$

Since $AH$ is a linear subspace that contains properly $\overline{A^2H}$, it follows that there exists $g_0 \in \overline{AH}$ such that $\|g_0\| = 1$ and such that $g_0 \perp A^2H$, that is,

$$\langle A^2f, g_0 \rangle = 0, \quad \text{for all } f \in H. \quad (7.2)$$

Then, there exists a sequence $(f_n)$ of vectors in $H$ such that $Af_n \to g_0$, as $n \to \infty$. We must show that $A \notin \overline{\text{alg}(A^2)}^\sigma$. We proceed by contradiction. Let us suppose for a little while that there exists a net of polynomials $(p_d)_{d \in D}$ such that $p_d(A^2) \to A$, as $d \in D$, in the weak operator topology. It follows from equations (7.1) and (7.2) that

$$\langle p_d(A^2)f, g_0 \rangle = p_d(0)\langle f, g_0 \rangle, \quad \text{for all } f \in H. \quad (7.3)$$

Now, setting $f = g_0$ in equation (7.3), and taking limits, first as $d \in D$, and second as $n \to \infty$, we get

$$\lim_{d \in D} p_d(0) = \lim_{d \in D} \langle p_d(A^2)g_0, g_0 \rangle = \langle Ag_0, g_0 \rangle = \lim_{n \to \infty} \langle A^2f_n, g_0 \rangle = 0.$$
Then, setting $f = f_n$ in equation (7.3), and taking iterated limits, as $d \in D$, and as $n \to \infty$, we obtain

$$0 = \lim_{n \to \infty} \lim_{d \in D} \langle p_d(0), f_n \rangle = \lim_{n \to \infty} \langle p_d(A^2)f_n, g_0 \rangle = \lim_{n \to \infty} \langle Af_n, g_0 \rangle = \langle g_0, g_0 \rangle = 1,$$

and a contradiction has arrived. \qed

Our next goal is to construct a univalent, analytic self-map $\varphi$ of the unit disk such that $C^2_{\varphi}$ is compact but fails to have a minimal commutant.

It is convenient to apply a result related to the Schwarz-Christoffel formula for a conformal mapping $\tau$ from the unit disk onto the interior of a polygon $P$. This result is stated below as Lemma 7.2. We refer to the book of Gamelin [13, p. 298] for this result in connection with the Schwarz reflection principle, and for the fact that $\tau$ extends to a homeomorphism from the closed unit disk onto $P$.

The last assertion is a particular instance of Carathéodory’s theorem, that can be stated as follows: if the boundary of the range of a conformal mapping is a Jordan curve, then the conformal mapping extends to a homeomorphism from the closed unit disk onto the closure of its range. We refer the reader to the book of Rudin [27, p. 279] for a proof of Carathéodory’s theorem.

**Lemma 7.2.** Let $\tau$ be a conformal mapping from the open unit disk onto the interior of a polygon $P$ with vertices $w_j = \tau(z_j)$, where $|z_j| = 1$, and let $\pi\alpha_j$ be the angles of the polygon $P$ at the vertices $w_j$. Then, for all $1 \leq j \leq n$ there is an open neighborhood $V_j$ of the point $z_j$ and there is a function $\xi_j$ that is analytic on $V_j$, such that $\xi_j(z_j) \neq 0$, and such that

$$\tau(z) = w_j + (z - z_j)^{\pi/\alpha_j}\xi_j(z), \quad \text{for all } z \in \mathbb{D} \cap V_j, \quad (7.4)$$

Now, let $T$ be the triangle of vertices $w_1 = 1$, $w_2 = i$, and $w_3 = e^{i\pi/4}$, and consider a conformal mapping $\tau: \mathbb{D} \to \text{int}(T)$. Using an automorphism of the unit disk allows us to prescribe the images of the three points $z_1 = 1$, $z_2 = -1$ and $z_3 = i$ on the unit circle, say $\tau(1) = 1$, $\tau(-1) = i$, and $\tau(i) = e^{i\pi/4}$. Then, the final map $\varphi$ is defined by

$$\varphi(z) := \tau(z)^4, \quad z \in \mathbb{D}. \quad (7.5)$$

We know that $\tau(\mathbb{D})$ is contained in the first open quadrant, and that the quartic $q(w) := w^4$ is univalent on that quadrant. It follows that $\varphi = q \circ \tau$ is univalent on the open unit disk. We know, as a consequence of the polygonal compactness theorem [30], that $C_{\tau}$ is compact, hence $C_{\varphi} = C_{\tau}C_q$ is compact, too.

Notice that the angles of $T$ at the vertices $1$, $i$ and $e^{i\pi/4}$ are given by $\pi/8$, $\pi/8$ and $3\pi/4$, respectively. If we apply Lemma 7.2 to our conformal mapping $\tau$ then we get the following

**Lemma 7.3.** There are three functions $\xi_1, \xi_2, \xi_3$ that are analytic at $1, -1, i$, respectively, and such that

1. $\tau(z) = 1 + (z - 1)^{1/8}\xi_1(z),$
2. $\tau(z) = i + (z + 1)^{1/8}\xi_2(z),$
3. $\tau(z) = e^{i\pi/4} + (z - i)^{3/4}\xi_3(z).$

We continue with a result that contains one of the main ideas for the proof of Theorem 7.5 below.

**Lemma 7.4.** Let $\varphi$ be the map given by equation (7.5), let $p_0(z) = (z - 1)^m$, and consider the function

$$g_0(t) := p_0(\varphi(\varphi(e^{it})))\varphi'(\varphi(e^{it}))(\varphi(e^{it})e^{-it}i). \quad (7.6)$$

1. If $m \geq 64$ then $\lim_{t \to 0} g_0(t) = 0$.
2. If $m \geq 10$ then $\lim_{t \to \pi/2} g_0(t) = 0$. 


Proof. First of all, we shall use Lemma 7.3 in order to estimate the size of $|\varphi'(\varphi(e^{it}))| \cdot |\varphi'(e^{it})|$ as $t \to 0$. It will be convenient to use the big O notation. Notice that $e^{it} \to 1$ and $\varphi(e^{it}) \to 1$ as $t \to 0$. It follows from the fact that $\tau - 1$ is a factor of $\tau^4 - 1$, and from equation (1) in Lemma 7.3, that

$$
|\varphi(e^{it}) - 1| = |\tau(e^{it})^4 - 1| = O \left( |\tau(e^{it}) - 1| \right) = O \left( |e^{it} - 1|^{1/8} \right) = O \left( |t|^{1/8} \right), \quad \text{as } t \to 0.
$$

Next, we know that, on a neighborhood of $z_1 = 1$, we have

$$
\varphi'(z) = 4\tau(z)^3 \tau'(z) = 4\tau(z)^3 \left[ \frac{1}{8} (z - 1)^{-7/8} \xi_1(z) + (z - 1)^{1/8} \xi_1'(z) \right],
$$

and from this identity, we obtain the following estimate

$$
|\varphi'(z)| = O \left( |z - 1|^{-7/8} \right), \quad \text{as } z \to 1. \quad (7.7)
$$

Thus,

$$
|\varphi'(e^{it})| = O \left( |\tau(e^{it}) - 1|^{-7/8} \right) = O(|t|^{-7/8}), \quad \text{as } t \to 0,
$$

and

$$
|\varphi'(\varphi(e^{it}))| = O \left( |\varphi(e^{it}) - 1|^{-7/8} \right) = O \left( \left( |t|^{1/8} \right)^{-7/8} \right), \quad \text{as } t \to 0, \quad (7.9)
$$

Then, if we combine estimates (7.8) and (7.9), then we get

$$
|\varphi'(\varphi(e^{it}))| \cdot |\varphi'(e^{it})| = O \left( |t|^{-63/64} \right), \quad \text{as } t \to 0. \quad (7.10)
$$

Now, we proceed to estimate the size of $|\varphi'(\varphi(e^{it}))| \cdot |\varphi'(e^{it})|$ as $t \to \pi/2$. Notice that $e^{it} \to i$ and $\varphi(e^{it}) \to -1$ as $t \to \pi/2$. It follows from the fact that $\tau - e^{i\pi/4}$ is a factor of $\tau^4 + 1$, and from equation (3) in Lemma 7.3 that

$$
|\varphi(e^{it}) + 1| = |\tau(e^{it})^4 - (e^{i\pi/4})^4| = O \left( |\tau(e^{it}) - e^{i\pi/4}| \right) = O \left( |e^{it} - i|^{3/4} \right) = O \left( |t - \pi/2|^{3/4} \right), \quad \text{as } t \to \pi/2.
$$

Next, we know that, on a neighborhood of $z_2 = -1$, we have

$$
\varphi'(z) = 4\tau(z)^3 \tau'(z) = 4\tau(z)^3 \left[ \frac{1}{8} (z + 1)^{-7/8} \xi_2(z) + (z + 1)^{1/8} \xi_2'(z) \right],
$$

and from this identity we get the following estimate

$$
|\varphi'(z)| = O \left( |z + 1|^{-7/8} \right) \quad \text{as } z \to -1. \quad (7.11)
$$

Also, we know that, on a neighborhood of $z_3 = i$, we have

$$
\varphi'(z) = 4\tau(z)^3 \tau'(z) = 4\tau(z)^3 \left[ \frac{3}{4} (z - i)^{-1/4} \xi_3(z) + (z - i)^{3/4} \xi_3'(z) \right],
$$

$$
|\varphi'(z)| = O \left( |z - i|^{-1/4} \right) \quad \text{as } z \to i.
$$
and from this identity, we get the estimate
\[ |\varphi'(z)| = O \left( |z - i|^{-1/4} \right), \quad \text{as } z \to i. \] (7.12)

This yields
\begin{align*}
|\varphi'(e^{it})| &= O \left( |e^{it} - i|^{-1/4} \right) \\
&= O \left( |t - \pi/2|^{-1/4} \right), \quad \text{as } t \to \pi/2,
\end{align*}

Moreover, it follows from the estimate (7.11) that
\begin{align*}
|\varphi'(|\varphi(e^{it})|)| &= O \left( |\varphi(e^{it}) + 1|^{-7/8} \right) \\
&= O \left( \left( |t - \pi/2|^{3/4} \right)^{-7/8} \right) \\
&= O \left( |t - \pi/2|^{-21/32} \right), \quad \text{as } t \to \pi/2.
\end{align*}

Thus, we arrived at the conclusion that
\begin{align*}
|\varphi'(|\varphi(e^{it})|)| \cdot |\varphi'(e^{it})| &= O \left( |t - \pi/2|^{-21/32} \cdot |t - \pi/2|^{-1/4} \right) \\
&= O \left( |t - \pi/2|^{-29/32} \right), \quad \text{as } t \to \pi/2.
\end{align*}

Now, we consider the polynomial \( p_0(z) = (z - 1)^m \), for some \( m \in \mathbb{N} \) to be chosen later on, and we estimate the size of \( |p_0(\varphi(e^{it}))| \) as \( t \to 0 \) and as \( t \to \pi/2 \). On one hand, we have
\begin{align*}
|p_0(\varphi(e^{it}))| &= |\varphi(e^{it}) - 1|^m \\
&= O \left( |\varphi(e^{it}) - 1|^{1/8} \right)^m \\
&= O \left( |e^{it} - 1|^{3/8} \right)^m \\
&= O \left( |t|^{m/64} \right), \quad \text{as } t \to 0,
\end{align*}

and it follows from the previous estimates that
\begin{align*}
|g_0(t)| &= |p_0(\varphi(e^{it}))| \cdot |\varphi'(e^{it})| \cdot |\varphi'(e^{it})| = O \left( |t|^{(m-63)/64} \right), \quad \text{as } t \to 0,
\end{align*}

hence it suffices to take \( m \geq 64 \) to make sure that \( g_0(t) \to 0 \) as \( t \to 0 \). On the other hand, we have
\begin{align*}
|p_0(\varphi(e^{it}))| &= |\varphi(e^{it}) - 1|^m \\
&= O \left( |\tau(\varphi(e^{it}))^4 - i|^m \right) \\
&= O \left( |\tau(\varphi(e^{it})) - i|^m \right) \\
&= O \left( |\varphi(e^{it}) + 1|^{m/8} \right) \\
&= O \left( |\tau(e^{it})^4 - (e^{4\pi/4})^4|^{m/8} \right) \\
&= O \left( |\tau(e^{it}) - e^{4\pi/4}|^{m/8} \right) \\
&= O \left( |e^{it} - i|^{3m/32} \right) \\
&= O \left( |t - \pi/2|^{3m/32} \right), \quad \text{as } t \to \pi/2,
\end{align*}
The range of the Jordan curve
Remark 7.7. but it fails to have a minimal commutant.

Then, \( \psi \)

Corollary 7.6. so that the inequality (7.14) is also satisfied, as we wanted.

Finally, we get

and it follows from the previous estimates that

\[
|g_0(t)| = |p_0(\varphi(e^{it}))| \cdot |\varphi'(e^{it})| \cdot |\varphi'(e^{it})| = O \left( |t|^{\frac{3m-29}{32}} \right), \quad \text{as } t \to 0,
\]

hence it suffices to take \( m \geq 10 \) to make sure that \( g_0(t) \to 0 \) as \( t \to \pi/2 \).

\( \square \)

Theorem 7.5. If \( \varphi \) is the map defined by equation (7.5) then \( C_2^\varphi H^2(\mathbb{D}) \) fails to be dense in \( C_2 H^2(\mathbb{D}) \).

Proof. It suffices to show that there are two functions \( g_0 \in L^2[0, \pi/2] \) and \( h_0 \in H^2(\mathbb{D}) \) such that, for every polynomial \( p \), we have

\[
\int_0^{\pi/2} g_0(t)p(\varphi(e^{it})) \, dt = 0,
\]

(7.13)

\[
\int_0^{\pi/2} g_0(t)h_0(\varphi(e^{it})) \, dt \neq 0.
\]

(7.14)

Consider the closed simple curve \( \gamma(t) = \varphi(e^{it}), \) for \( 0 \leq t \leq \pi/2 \), and the polynomial \( p_0(z) = (z-1)^m \), for some \( m \geq 64 \). It follows from Cauchy’s integral theorem that for every polynomial \( p \),

\[
0 = \oint_{\gamma} p_0(z) \frac{dz}{z} = \int_0^{\pi/2} p_0(\varphi(e^{it})) \varphi'(e^{it}) \varphi'(e^{it}) ie^{it} \cdot p(\varphi(e^{it})) \, dt.
\]

Next, consider the function \( g_0 \) given by equation (7.6), that is, \( g_0(t) = p_0(\varphi(e^{it})) \varphi'(e^{it}) \varphi'(e^{it}) ie^{it} \).

Notice that \( g_0 \neq 0 \) because \( p_0 \neq 0 \) and \( \varphi \) is a conformal map. It is clear that \( g_0 \) satisfies equation (7.13).

It follows from Lemma 7.4 that \( g_0 \in L^2[0, \pi/2] \) because, in fact, \( g_0 \) is continuous on \( [0, \pi/2] \). The rest of the proof is easy. Indeed, since \( g_0 \neq 0 \), there is some \( \varepsilon > 0 \) such that

\[
\int_0^{\pi/2} |g_0(t)|^2 \, dt - \varepsilon \int_0^{\pi/2} |g_0(t)| \, dt > 0.
\]

Then, consider the compact arc \( K := \{ \varphi(e^{it}) : 0 \leq t \leq \pi/2 \} \), and notice that there is a homeomorphism \( \beta : [0, \pi/2] \to K \) given by \( \beta(t) = \varphi(e^{it}) \). Since \( K \) has empty interior, and its complement in the complex plane is connected, it follows from Mergelyan’s theorem that there is a polynomial \( h_0 \) such that

\[
|h_0(z) - \overline{g_0(\beta^{-1}(z))}| < \varepsilon, \quad \text{for all } z \in K.
\]

(7.15)

Finally, we get

\[
\left| \int_0^{\pi/2} g_0(t)h_0(\varphi(e^{it})) \, dt \right| \geq \int_0^{\pi/2} |g_0(t)|^2 \, dt - \int_0^{\pi/2} |g_0(t)| \cdot |h_0(e^{it}) - \overline{g_0(t)}| \, dt
\]

\[
\geq \int_0^{\pi/2} |g_0(t)|^2 \, dt - \varepsilon \int_0^{\pi/2} |g_0(t)| \, dt > 0,
\]

so that the inequality (7.14) is also satisfied, as we wanted.

\( \square \)

Our next result is a consequence of Theorem 7.1 and Theorem 7.5.

Corollary 7.6. Let \( \varphi \) be the analytic self-map of the unit disk defined by equation (7.5) and let \( \psi = \varphi \circ \varphi \).

Then, \( \psi \) is a univalent, analytic self-map of the unit disk such that the composition operator \( C_\psi \) is compact but it fails to have a minimal commutant.

Remark 7.7. The range of the Jordan curve \( \gamma \), defined in the proof of Theorem 7.5, is the compact set

\[
C := \{ \varphi(\varphi(t)) : 0 \leq t \leq \pi/2 \}.
\]
which is homeomorphic to the unit circle. Since the complement of $C$ in the complex plane is not connected, Mergelyan’s theorem cannot be applied. This provides a heuristic obstruction for the approximation of $C_\varphi$ by polynomials in $C_\varphi^2$. This argument is one of the main ideas in the construction of our counter-example.

We end this section with an alternative procedure to construct a composition operator $C_\varphi$ induced by a univalent, analytic self-map $\varphi$ of the unit disk so that $C_\varphi$ fails to have a minimal commutant. Now, the idea is to work with a symbol that fixes the origin, such that the Königs’s domain is bounded and starlike with respect to the origin, and such that the range of $C_\varphi$ fails to be dense. An example of such a symbol is provided by Corollary 7.10 below.

**Theorem 7.8.** Let $\varphi$ be a univalent, analytic self-map of the unit disk such that $\varphi(0) = 0$, and let $\sigma$ denote the corresponding Königs’s function. If the domain $G := \sigma(\mathbb{D})$ is bounded and starlike with respect to the origin and if the range of $C_\varphi$ fails to be dense, then $C_\varphi$ fails to have a minimal commutant.

**Proof.** Let $\alpha = \varphi'(0)$ and recall from Königs’s theorem that $\varphi(z) = \sigma^{-1}(\alpha \sigma(z))$. Since $\varphi$ is univalent and it fixes the origin, and in addition, it fails to be a disk automorphism, we have $0 < |\alpha| < 1$.

Since $G$ is starlike with respect to the origin, for every $0 \leq r \leq 1$, there exists an analytic self-map $\varphi_r$ of the unit disk defined by the expression $\varphi_r(z) := \sigma^{-1}(r \sigma(z))$. Notice that $\varphi_r \circ \varphi_s = \varphi_{rs}$, hence $C_{\varphi_r} C_{\varphi_s} = C_{\varphi_{rs}}$ for all $0 \leq r, s \leq 1$.

Observe that $\varphi_r \in \{C_\varphi\}$. Since $\varphi_r(0) = 0$, we have $\|C_\varphi\| = 1$, and it follows from Lemma 6.3 that the map $r \mapsto C_{\varphi_r}$ is continuous from $[0, 1]$ into $\mathcal{B}(H^2(\mathbb{D}))$ endowed with the weak operator topology, and in particular, $C_{\varphi_r} \to I$, as $r \to 1^-$, in the weak operator topology.

Since the range of $C_{\varphi_r}$ fails to be dense, there exists a function $f_0 \in H^2(\mathbb{D}) \setminus \{0\}$ such that $C_{\varphi_r} f_0 = 0$. Let $\epsilon > 0$ and notice that $f_0(0) = \langle f_0, e_0 \rangle = \langle f_0, C_\varphi e_0 \rangle = \langle C_{\varphi_r} f_0, e_0 \rangle = 0$. Then, consider the closed set defined by the expression

$$F := \{r \in [0, 1] : C_{\varphi_r} f_0 = 0\}.$$

Observe that $0 \in F$ because $C_{\varphi_0} f_0 = f_0(0)e_0 = 0$. Also, notice that $1 \notin F$, because $C_{\varphi_1} f_0 = f_0 \neq 0$. Moreover, $F$ is an interval. Indeed, if $0 < s < r < 1$ then $C_{\varphi_s} f_0 = C_{\varphi_s/r} C_{\varphi_r} f_0 = 0$, hence $s \in F$. Next, let $r_0 = \sup F$. It follows from the continuity of the function $r \mapsto C_{\varphi_r} f_0$, $f_0$ that there is some $\epsilon > 0$ such that $\langle C_{\varphi_r} f_0, f_0 \rangle \neq 0$, and in particular $C_{\varphi_r} f_0 \neq 0$, whenever $1 - \epsilon < r < 1$. This shows that $r_0 < 1$.

We claim that $r_0 > 0$. Indeed, since $G$ is open and $0 \in G$, there is some $\delta > 0$ such that $\delta \mathbb{D} \subseteq G$. Further, since $G$ is bounded, there is some $M > 0$ such that $\overline{G} \subseteq M \mathbb{D}$. This allows us to consider, for every $w \in \mathbb{C}$ with $|w| \leq \delta/M$, the univalent, analytic self-map $\varphi_w$ of the unit disk defined by $\varphi_w(z) := \sigma^{-1}(w \sigma(z))$.

Let $r := \delta/M$. It suffices to show that $s := r|\alpha| \in F$. Indeed, let $\theta \in \mathbb{R}$ with $\alpha = |\alpha|e^{i\theta}$, and let $w = re^{-i\theta}$. Observe that $s = \alpha w$, so that $C_{\varphi_s} = C_{\varphi} C_{\varphi_w}$, so that $C_{\varphi_s} f_0 = C_{\varphi_w} C_{\varphi} f_0 = 0$, hence, $s \in F$, and this completes the proof of our claim.

Now, the rest of the proof is easy. We know that $F = [0, r_0]$ for some $0 < r_0 < 1$. Then, there is some $0 < r < 1$ such that $r^2 < r_0 < r$, so that $r^2 \in F$, but $r \notin F$.

We finally show that $C_{\varphi_*} \notin \text{alg}(C_{\varphi})^\sigma$, and the result follows at once. We proceed by contradiction. Let us suppose for a moment that there exists a net of polynomials $(p_d)_{d \in D}$ such that $p_d(C_{\varphi}) \to C_{\varphi_*}$, as $d \in D$, in the weak operator topology. Then, for every $f \in H^2(\mathbb{D})$,

$$\langle C_{\varphi_*} f, f_0 \rangle = \lim_{d \in D} \langle p_d(C_{\varphi}) f, f_0 \rangle = \lim_{d \in D} p_d(0) \langle f, f_0 \rangle. \quad (7.16)$$

On one hand, if we set $f = C_{\varphi_*} f_0$ in equation (7.16), we get

$$\|C_{\varphi_*} f_0\|^2 = \lim_{d \in D} p_d(0) \langle C_{\varphi_*} f_0, f_0 \rangle.$$
so that $\langle C_{\varphi}f_0, f_0 \rangle \neq 0$ and $\lim_{d \to 0} p_d(0) \neq 0$, as $d \in D$, and on the other hand, if we set $f = C_{\varphi}f_0$ in equation (7.16), then we get
\[
\lim_{d \to 0} p_d(0) \langle C_{\varphi}, f_0, f_0 \rangle = 0,
\]
and it follows that $\lim_{d \to 0} p_d(0) = 0$, as $d \in D$. We arrived at a contradiction. \hfill \Box

In order to apply Theorem 7.8, we need a condition for an analytic self-map $\varphi$ of the unit disk that ensures that the range of $C_{\varphi}$ fails to be dense. Recall that an analytic self-map $\varphi$ of the unit disk is said to be univalent almost everywhere on $\partial \mathbb{D}$ provided that there is a Lebesgue measurable subset $E \subseteq \partial \mathbb{D}$ such that $|E| = 0$ and the restriction of $\varphi$ to $\partial \mathbb{D} \setminus E$ is univalent. Here $| \cdot |$ denotes the normalized arc length measure on $\partial \mathbb{D}$. We refer to the paper of Bourdon and Shapiro [2, proof of Theorem 1.7] for the following result.

**Theorem 7.9.** Let $\varphi$ be an analytic self-map of the unit disk and suppose that $C_{\varphi}$ has dense range. Then $\varphi$ is univalent on $\mathbb{D}$ and univalent almost everywhere on $\partial \mathbb{D}$.

**Corollary 7.10.** Let $0 < r_0 < 1$, and consider the slit disk $G = \mathbb{D} \setminus [r_0, 1]$. Then, consider a conformal mapping $\sigma : \mathbb{D} \to G$ such that $\sigma(0) = 0$ and $\sigma(1) = r_0$. Let $0 < r < r_0$, and let $\varphi$ be the univalent, analytic self-map of the unit disk given by the expression $\varphi(z) := \sigma^{-1}(r \sigma(z))$. Then, the composition operator $C_{\varphi}$ is compact but it fails to have a minimal commutant.

**Proof.** It is clear that $G$ is bounded and starlike with respect to the origin, so the map $\varphi$ is well defined. Also, it is clear that the composition operator $C_{\varphi}$ is compact, because $r\overline{\mathbb{D}} \subseteq G$, hence $\varphi(\overline{\mathbb{D}}) = \sigma^{-1}(r\overline{\mathbb{D}})$ is a compact subset of the unit disk. The desired result would follow as a consequence of Theorem 7.8, if we only could show that the range of $C_{\varphi}$ fails to be dense. The last assertion follows from Theorem 7.9, because $\varphi$ fails to be univalent almost everywhere on $\partial \mathbb{D}$.

Indeed, it is easy to see that there is some $\varepsilon > 0$ such that $\{e^{i\theta} : |\theta| < \varepsilon\} \subseteq \varphi^{-1}([r_0, 1])$, and such that $\varphi(e^{-i\theta}) = \varphi(e^{i\theta})$ whenever $|\theta| < \varepsilon$. This is an obstruction for $\varphi$ to be univalent almost everywhere on $\partial \mathbb{D}$. \hfill \Box

**Remark 7.11.** It follows from Corollary 7.10 that the hypothesis in Theorem 6.1 that a bounded K" onigs’s domain $G$ be strictly starlike with respect to the origin cannot be replaced by the weaker assumption that $G$ be starlike with respect to the origin.

**Remark 7.12.** The proof of Theorem 7.8 also applies if the assumption that $G$ be bounded is replaced in the statement by the condition $\varphi'(0) > 0$. Further, the proof also applies when $G$ is assumed to be unbounded and strictly starlike with respect to the origin, because according to a result of Joel Shapiro, Wayne Smith and David Stegenga [32, Lemma 4.1, p. 49], there is some $n \in \mathbb{N}$ such that $\varphi'(0)^n > 0$.

**Appendix: The Sobolev space as a Banach algebra with an approximate identity**

The aim of this section is to provide the proofs of Theorem 4.2, Theorem 4.3 and Lemma 4.9, just for the sake of completeness. For convenience, we will repeat the statements of these results.

**Theorem 4.3.** If $f \in W^{1, 2}[0, +\infty)$, we have $f \in L^{\infty}[0, +\infty)$, and moreover, $\|f\|_{\infty} \leq \|f\|_{1, 2}$.

**Proof.** Let $\sigma_0 \geq 0$ and $\sigma > \sigma_0$. By definition, $f$ is absolutely continuous on the bounded interval $[\sigma_0, \sigma]$, so the same is true of $f^2$. Hence, for every $s \in [\sigma_0, \sigma]$, we have
\[
f(\sigma_0)^2 - f(s)^2 = -\int_{\sigma_0}^{s} (f^2)'(t) \, dt \]
\[
= -2 \int_{\sigma_0}^{s} f(t)f'(t) \, dt.
\]
It follows from the Cauchy-Schwarz inequality that
\[
|f(\sigma_0)^2 - f(s)^2| = 2 \left| \int_{\sigma_0}^{s} f(t) f'(t) \, dt \right|
\]
\[
\leq 2 \left( \int_{\sigma_0}^{s} |f(t)|^2 \, dt \right)^{1/2} \left( \int_{\sigma_0}^{s} |f'(t)|^2 \, dt \right)^{1/2}
\]
\[
\leq 2 \|f\|_2 \|f'\|_2,
\]
so that \(|f(\sigma_0)|^2 \leq |f(\sigma_0)^2 - f(s)^2| + |f(s)|^2 \leq 2\|f\|_2 \|f'\|_2 + |f(s)|^2\). Taking the integral average on both sides of this inequality over the interval \([\sigma_0, \sigma]\) leads to
\[
|f(\sigma_0)|^2 \leq 2\|f\|_2 \|f'\|_2 + \frac{1}{\sigma - \sigma_0} \int_{\sigma_0}^{\sigma} |f(s)|^2 \, ds
\]
\[
\leq 2\|f\|_2 \|f'\|_2 + \frac{\|f\|_2^2}{\sigma - \sigma_0},
\]
and taking limits as \(\sigma \to +\infty\) yields \(|f(\sigma_0)|^2 \leq 2\|f\|_2 \|f'\|_2\). Since \(\sigma_0 \geq 0\) is arbitrary, we finally get
\[
\|f\|_\infty \leq 2\|f\|_2 \|f'\|_2 \leq \|f\|_2^2 + \|f'\|_2^2 = \|f\|_{1,2}^2,
\]
as we wanted. \(\square\)

Notice that the inequality in the embedding Theorem 4.3 is sharp. Indeed, the function \(f(\sigma) = e^{-\sigma}\) belongs to the Sobolev space \(W^{1,2}[0, +\infty)\), and it is easy to check that \(\|f\|_\infty = \|f\|_{1,2} = 1\).

**Theorem 4.2.** If \(f, g \in W^{1,2}[0, +\infty)\) then \(fg \in W^{1,2}[0, +\infty)\), and moreover, \(\|fg\|_{1,2} \leq \sqrt{2}\|f\|_{1,2}\|g\|_{1,2}\).

Hence, the Sobolev space \(W^{1,2}[0, +\infty)\) is a commutative Banach algebra without identity with respect to pointwise multiplication and the equivalent norm \(\|f\|_{1,2} = \sqrt{2}\|f\|_{1,2}\).

**Proof.** First, the function \(fg\) is absolutely continuous on the bounded subintervals of \([0, +\infty)\) because it is a product of two such functions. By Theorem 4.3, both \(f\) and \(g\) belong to \(L^\infty[0, +\infty)\). On the other hand, both \(f'\) and \(g'\) belong to \(L^2[0, +\infty)\) by definition. It follows that both \(fg\) and \((fg)' = f'g + fg'\) belong to \(L^2[0, +\infty)\), because the product of a function in \(L^2(0, +\infty)\) by a bounded measurable function remains in \(L^2[0, +\infty)\). This shows that \(fg \in W^{1,2}[0, +\infty)\).

Next, we show that \(\|fg\|_{1,2} \leq \sqrt{2}\|f\|_{1,2}\|g\|_{1,2}\). We start with the particular case when \(f = g\). We have
\[
\|f^2\|_{1,2}^2 = \|f^2\|_2^2 + \|(f^2)'\|_2^2
\]
\[
= \|f \cdot f\|_2^2 + 2\|f \cdot f'\|_2^2
\]
\[
\leq 2\|f \cdot f\|_2^2 + 2\|f \cdot f'\|_2^2
\]
\[
\leq 2\|f\|_2^2(\|f\|_2^2 + \|f'\|_2^2)
\]
\[
\leq 2\|f\|_{1,2}^4,
\]
so that
\[
\|f^2\|_{1,2} \leq \sqrt{2}\|f\|_{1,2}^2,
\]
as we wanted. In the general case, we may assume without loss of generality that both \(\|f\|_{1,2} \leq 1\) and \(\|g\|_{1,2} \leq 1\). The polarization identity
\[
f g = \frac{1}{4} [(f + g)^2 - (f - g)^2],
\]
implies
\[
\|fg\|_{1,2} \leq \frac{1}{4} [\|(f + g)^2\|_{1,2} + \|(f - g)^2\|_{1,2}].
\]
Now, (7.17) yields
\[ \|fg\|_{1,2} \leq \frac{\sqrt{2}}{4}(\|f + g\|^2_{1,2} + \|f - g\|^2_{1,2}). \]
Finally, using the parallelogram identity in $W^{1,2}(0, +\infty)$, we get
\[ \|fg\|_{1,2} \leq \frac{\sqrt{2}}{2}(\|f\|^2_{1,2} + \|g\|^2_{1,2}) \leq \sqrt{2}, \]
as we wanted.

Now, we have $\|fg\|^2_{1,2} = \sqrt{2}\|fg\|_{1,2} \leq 2\|f\|_{1,2}\|g\|_{1,2} = \|f\|^2_{1,2}\|g\|^2_{1,2}$, so that $W^{1,2}(0, +\infty)$ becomes a commutative Banach algebra without identity with respect to pointwise multiplication and the equivalent norm $\| \cdot \|_{1,2}$.

Lemma 4.9. The sequence of functions $(e_n)$ defined by the expression
\[ e_n(\sigma) = \begin{cases} 1, & \text{if } 0 \leq \sigma \leq n, \\ e^{-n(\sigma-n)}, & \text{if } \sigma \geq n, \end{cases} \]
is an approximate identity for the Sobolev algebra $W^{1,2}(0, +\infty)$.

Proof. It is not hard to see that, for all $n \in \mathbb{N}_0$, $e_n \in W^{1,2}[0, +\infty)$ and $\|e_n\|_{\infty} \leq 1$. Further, $e_n(\sigma) \to 1$, as $n \to \infty$, pointwise on the interval $[0, +\infty)$. Let $f \in W^{1,2}[0, +\infty)$. We must show that $\|e_n \cdot f - f\|_{1,2} \to 0$, as $n \to \infty$. We have
\[ \|e_n \cdot f - f\|^2_{1,2} = \|e_n \cdot f - f\|^2_{2} + \|(e_n \cdot f - f)\|^2_{1,2}. \]
Notice that $|e_n(\sigma)f(\sigma) - f(\sigma)|^2 \leq 4|f(\sigma)|^2$, and therefore it follows from the bounded convergence theorem that $\|e_n \cdot f - f\|_{2} \to 0$, as $n \to \infty$. Moreover, since $f' \in L^2([0, +\infty))$, the same argument shows that $\|e_n \cdot f' - f'\|^2_{2} \to 0$, as $n \to \infty$. Since $(e_n \cdot f - f)' = e'_n \cdot f + e_n \cdot f' - f'$. It remains to prove that $e'_n \cdot f \to 0$, as $n \to \infty$. Notice that, for every $\sigma \in [0, +\infty)$, $e'_n(\sigma) \to 0$, as $n \to \infty$. Finally, $\|e'_n\|_{\infty} \leq 1$ and $f$ is bounded, so another application of the bounded convergence theorem completes the proof. \qed

References


