DEDDENS ALGEBRAS AND SHIFT

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The unilateral shift S acting on Hilbert space $\mathcal{H}$ is one of the most studied operators. Much is known about its structure, and this success is in large part due to the identification of S with the operator of multiplication by z on H^2 — the Hardy space on the unit disk. From the function theory point of view, the commutant $\{S\}'$ is then simply H^∞, the Hardy space of essentially bounded functions on the disk. Thus, learning more about algebras that contain $\{S\}'$ promises to be significant for both operator theory as well as function theory.

A class of algebras with this property were studied by Deddens in [3]. We will refer to them as *Deddens algebras* and we will use the notation $\mathcal{D}_A$ when it is associated to the operator A. An operator $T \in \mathcal{D}_A$ if and only if there exists $M > 0$ such that, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$, $\|A^n Tx\| \leq M\|A^n x\|$. It is immediate that $\mathcal{D}_A$ is a unital algebra, that $\{A\}' \subset \mathcal{D}_A$ and that $\mathcal{D}_A = \mathcal{L}(\mathcal{H})$ (the algebra of all bounded linear operators) whenever A is an isometry. (E.g., when $A = S$.)

The main result of this paper is that, for $\lambda \neq 0$, $\mathcal{D}_{S-\lambda I}$ is weakly dense in $\mathcal{L}(\mathcal{H})$, but it does not contain any rank one operators. This should be compared to the situation with the *spectral radius algebra* $\mathcal{B}_{S-\lambda I}$ which, as it was shown in [8], contains a dense set of rank one operators. Recall (cf., [7, Proposition 2.3, Corollary 2.4]) that $T \in \mathcal{B}_A$ if and only if there exists $M > 0$ such that, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$, $\sum_{n\geq 0} d_m^{2n}\|A^n Tx\|^2 \leq M \sum_{n\geq 0} d_m^{2n}\|A^n x\|^2$. (Here,

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$d_m = m/(1+rm)$ and r is the spectral radius of A.) It is easy to see that $\mathcal{D}_A \subset \mathcal{B}_A$ and our result illustrates the fact that these two algebras can be very different.

Another interesting question concerns the relationship between the algebras $\mathcal{D}_{S-\lambda I}$ for different values of λ. We will show (Corollary 6) that, when λ_1 and λ_2 have different arguments, the corresponding algebras are different. In the case when they have the same argument, we establish an inclusion depending on their absolute values (Theorem 8 below).

Since $\{S-\lambda I\}' \subset \mathcal{D}_{S-\lambda I}$ (and $\{S-\lambda I\}' = \{S\}'$), the algebra $\mathcal{D}_{S-\lambda I}$ contains all multiplication operators. The first step in our analysis is to point out some other operators in this algebra. As usual, when $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function, the composition operator C_φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$. We recommend [2] for more details about these operators on various function spaces.

Theorem 1. *Let λ be a non-zero complex number, let $0 < r < 1/|\lambda|$, let $\theta \in [0, 2\pi]$, and let* $a(z) = \dfrac{z - re^{i\theta}\lambda}{1 - zre^{-i\theta}\overline{\lambda}}$. *The algebra $\mathcal{D}_{S-\lambda I}$ contains a composition operator C_a if and only if $\theta = 0$.*

Proof. It is not hard to see that $(S-\lambda I)C_a = C_a(a^{-1}(S) - \lambda I)$ so, for $f \in H^2$,

$$\|(S-\lambda I)^n C_a f\|^2 \le \|C_a\|^2 \|(a^{-1})(S) - \lambda I) f\|^2 = \frac{\|C_a\|^2}{2\pi} \int_0^{2\pi} |a^{-1}(e^{it}) - \lambda|^{2n} |f(e^{it})|^2 \, dt.$$

When $\theta = 0$, $a^{-1}(z) = \dfrac{z + r\lambda}{1 + r\overline{\lambda} z}$ and a straightforward calculation shows that, for all $z \in \mathbb{T}$,

$$|a^{-1}(z) - \lambda| \le |z - \lambda|. \tag{0.1}$$

Consequently, $\|(S-\lambda I)^n C_a f\| \le \dfrac{\|C_a\|}{\sqrt{2\pi}} \|(S-\lambda I)^n f\|$ and $C_a \in \mathcal{D}_{S-\lambda I}$.

In the other direction, we will show that, for each $\theta \neq 0$, the inequality (0.1) fails to be true on a subset of $\mathbb{T}$ of positive measure. Once we accomplish this, it will follow that there exists a set $A \subset \mathbb{T}$ of positive measure, and a number $b > 1$, such that $\left|a^{-1}(z) - \lambda\right| \ge b|z - \lambda|$ for $z \in A$. From this, we deduce that $C_a \notin \mathcal{D}_{S-\lambda I}$. Indeed, if $C_a \in \mathcal{D}_{S-\lambda I}$ then there is $M > 0$ such that, for all n and f, $\|(S-\lambda I)^n C_a f\| \le M \|(S-\lambda I)^n f\|$. A calculation shows that

$\int_{\mathbb{T}} |z-\lambda|^{2n}|f(a(z))|^2 = \int_{\mathbb{T}} |a^{-1}(z)-\lambda|^{2n}|f(z)|^2 Q(z)$, where $Q(z) \geq q \equiv \dfrac{1-r^2|\lambda|^2}{(1+r|\lambda|)^2}$. We choose n large enough so that $b^{2n}q > M^2$, and $\epsilon > 0$ that satisfies

$$\epsilon^2\left(M^2\int_{A^c}|z-\lambda|^{2n} - \int_{A_c}|a^{-1}(z)-\lambda|^{2n}Q\right) < \left(b^{2n}q - M^2\right)\int_A |z-\lambda|^{2n}. \tag{0.2}$$

Let $f \in H^2$ such that $|f(z)| = \epsilon$ when $z \in A^c$ and $|f(z)| = 1$ when $z \in A$. Such a function certainly exists because outer functions are determined by their absolute values on $\mathbb{T}$, so long as $\log|f| \in L^1(\mathbb{T})$ (cf., [9, Theorem 17.17], or [8, Theorem 5]). For such f,

$$\begin{aligned} M^2\epsilon^2\int_{A^c}|z-\lambda|^{2n} + M^2\int_A|z-\lambda|^{2n} &= M^2\|(S-\lambda I)^n f\|^2 \geq \|(S-\lambda I)^n C_a f\|^2 \\ = \int_{\mathbb{T}}|a^{-1}(z)-\lambda|^{2n}|f(z)|^2Q(z) &\geq \epsilon^2\int_{A^c}|a^{-1}(z)-\lambda|^{2n}Q(z) + \int_A|z-\lambda|^{2n}b^{2n}q. \end{aligned}$$

It follows that $\epsilon^2\left(M^2\int_{A^c}|z-\lambda|^{2n} - \int_{A_c}|a^{-1}(z)-\lambda|^{2n}Q\right) \geq \left(b^{2n}q-M^2\right)\int_A|z-\lambda|^{2n}$ contradicting (0.2).

Thus, it remains to prove that $|a^{-1}(z)-\lambda| > |z-\lambda|$ on a set of positive measure in $\mathbb{T}$. Notice that $a^{-1}(z) = \dfrac{z+re^{i\theta}\lambda}{1+zre^{-i\theta}\overline{\lambda}}$ so that both z and $a^{-1}(z)$ are of absolute value 1 and, hence, (0.1) is equivalent to

$$\operatorname{Re}(z\overline{\lambda}) \leq \operatorname{Re}\left(\overline{\lambda}\,\frac{z+re^{i\theta}\lambda}{1+zre^{-i\theta}\overline{\lambda}}\right). \tag{0.3}$$

Write $z = e^{it}$ and let $t \to \arg\lambda$. Then $z \to \lambda/|\lambda|$, and passing to the limit in (0.3) yields

$$|\lambda| \leq \frac{|\lambda|(1+2r|\lambda|\cos\theta + r^2|\lambda|^2\cos 2\theta)}{1+2r|\lambda|\cos\theta+r^2|\lambda|^2}. \tag{0.4}$$

Notice that, when $\cos 2\theta \neq 1$, the right hand side of (0.4) is strictly less than $|\lambda|$ which means that (0.3), and hence (0.1), fails on a set of positive measure. Since $\theta \neq 0$, the exceptional case, $\cos 2\theta = 1$, occurs when $\theta = \pi$. However, if $\theta = \pi$, then by letting $z \to i\lambda/|\lambda|$ in (0.3) we obtain that $0 \leq -2r|\lambda|^2/(1+r^2|\lambda|^2)$ so, once again, (0.1) must fail on a set of positive measure. This completes the proof. □

Since the shift cannot commute with a composition operator we obtain an immediate corollary.

Corollary 2. *The algebra* $\mathcal{D}_{S-\lambda I}$ *properly contains the commutant* $\{S-\lambda I\}'$.

Another consequence of Theorem 1 concerns the density of the Deddens algebra.

Theorem 3. *The algebra* $\mathcal{D}_{S-\lambda I}$ *is weakly dense in* $\mathcal{L}(\mathcal{H})$.

Proof. Since the algebra $\mathcal{D}_{S-\lambda I}$ contains the unilateral shift S, Arveson's theorem [1] implies that it is either weakly dense or has a nontrivial invariant subspace. We will demonstrate that the latter cannot happen. Indeed, if it did, this subspace would be invariant for S and, by Beurling's theorem, of the form θH^2 for some inner function θ. Our proof will consist of establishing that such an inner function θ must be constant (so that the invariant subspace is the whole H^2).

First we will establish that θ cannot have any zeros in $\mathbb{D}$. Suppose, to the contrary, that there exists $\alpha \in \mathbb{D}$ such that $\theta(\alpha)=0$. By Theorem 1, the algebra $\mathcal{D}_{S-\lambda I}$ contains composition operators C_a, where a is given by $a(z)=\dfrac{z-r\lambda}{1-r\bar{\lambda}z}$. This implies that the subspace θH^2 must be invariant for all these operators C_a, which means that, for all $f \in H^2$, $C_a(f\theta) \in \theta H^2$. In particular, $C_a(\theta) \in \theta H^2$, so $\theta(a(z))=\theta(z)h(z)$ for some $h \in H^2$. The assumption that $\theta(\alpha)=0$ now leads to $\theta(a(\alpha))=0$. Clearly, the set $\{a(\alpha) : 0<r<1/|\lambda|\}$ contains an accumulation point in $\mathbb{D}$ which is possible only if θ is the zero function.

It remains to show that θ cannot be a singular inner function. One knows (cf., [5, Theorem II.6.2]) that there exists at least one point $\alpha \in \mathbb{T}$ such that θ^* (the extension of θ to the unit circle $\mathbb{T}$) vanishes at α. Reasoning as above we see that $\theta^*(a(\alpha))=0$ for each function a that induces a composition operator C_a in $\mathcal{D}_{S-\lambda I}$. We notice that for such a and α, $a(\alpha)$ (considered as a function of r) continuously maps the interval $(0,1)$ to a subset of $\mathbb{T}$. The continuity implies that the subset in question must be an arc of the unit circle, which means that θ^* vanishes on a set of positive Lebesgue measure, again leading to the conclusion that θ is the zero function. $\square$

We will now demonstrate that, in spite of the fact that $\mathcal{D}_{S-\lambda I}$ is weakly dense, it does not contain any rank one operators. First we need a description of rank one operators in $\mathcal{B}_{S-\lambda I}$. The following result is a combination of Theorems 6–8 in [8] applied to the case $h(z) = z - \lambda$.

Theorem 4. *A rank one operator* $u \otimes v$ *belongs to* $\mathcal{B}_{S-\lambda I}$ *if and only if* $u = gp$, *where* g *is an outer function in* H^∞ *that satisfies* $|g|^2 = 1 - \dfrac{|z-\lambda|^2}{(1+|\lambda|)^2}$, *and* p *is an arbitrary function in* H^2.

Theorem 4 will allow us to show the dramatic difference between the Deddens and the spectral radius algebras associated to $S - \lambda I$.

Theorem 5. *The algebra* $\mathcal{D}_{S-\lambda I}$ *does not contain any rank one operators. Consequently,* $\mathcal{D}_{S-\lambda I} \neq \mathcal{B}_{S-\lambda I}$.

Proof. Suppose to the contrary that there exists a rank one operator $u \otimes v \in \mathcal{D}_{S-\lambda I}$. By definition of the Deddens algebra, there exists $M > 0$ such that, for all m, p, f, v,

$$\|(S-\lambda I)^m u\| |\langle f, v\rangle| \leq M \|(S-\lambda I)^m f\|.$$

In particular, it will hold for $v = f$, $\|f\| = 1$, which, together with Theorem 4, yields

$$\|(S-\lambda I)^m g(S) p\| \leq M \|(S-\lambda I)^m f\|. \tag{0.5}$$

Let us write $\lambda = |\lambda| e^{i(\pi+\alpha)}$ and let $\{\alpha_m\}$ be a sequence of real numbers such that, for each m, $\sin\left(\dfrac{|\alpha_m - \alpha|}{2}\right) = \dfrac{1}{\sqrt{m}}$. Then $|e^{i\alpha_m} - \lambda|^2 = 1 + |\lambda|^2 - \dfrac{4|\lambda|}{m}$, and $1 - \dfrac{|e^{i\alpha_m} - \lambda|^2}{(1+|\lambda|)^2} = \dfrac{4|\lambda|}{m(1+|\lambda|)^2}$.

Now we return to (0.5). For each m there exists a function p_m, $\|p_m\| = 1$, such that

$$\|(S-\lambda I)^m g(S) p_m\| \geq \|(S-\lambda I)^m g(S)\| - \frac{1}{m}.$$

Further,

$$\|(S-\lambda I)^m g(S)\| = \sup_{0 \leq t \leq 2\pi} |e^{it} - \lambda|^{2m} \left(1 - \frac{|e^{it}-\lambda|^2}{(1+|\lambda|)^2}\right) \geq |e^{i\alpha_m} - \lambda|^{2m} \left(1 - \frac{|e^{i\alpha_m}-\lambda|^2}{(1+|\lambda|)^2}\right)$$
$$= \left((1+|\lambda|)^2 - \frac{4|\lambda|}{m}\right)^m \frac{4|\lambda|}{m(1+|\lambda|)^2},$$

so we obtain that

$$\|(S-\lambda I)^m g(S)p_m\| \geq \left((1+|\lambda|)^2 - \frac{4|\lambda|}{m}\right)^m \frac{4|\lambda|}{m(1+|\lambda|)^2} - \frac{1}{m}. \tag{0.6}$$

When it comes to the right hand side of (0.5), we notice that, for each m, there exists a function f_m, $\|f_m\| = 1$, such that

$$\|(S-\lambda I)^m f_m\| - \frac{1}{m} < \inf\{\|(S-\lambda I)^m f\| : f \in H^2,\ \|f\| = 1\}.$$

Clearly, the right hand side of the last inequality is dominated by $\sup_k \inf\{\|(S-\lambda I)^{mk} f\|^{1/k} : f \in H^2,\ \|f\| = 1\}$, since it represents the value for $k = 1$. By [10, Proposition 12] the last supremum is, actually, the limit. Moreover, by [10, Proposition 13], this limit is dominated by $\inf\{|\mu| : \mu \in \sigma_{app}(S-\lambda I)^m\}$, where σ_{app} denotes the approximate point spectrum. One knows (see, e.g., [6, Problem 74]) that $\sigma_{app}(S-\lambda I)^m = [\sigma_{app}(S) - \lambda]^m$, and it is not hard to see that $\inf\{|\mu| : \mu \in \sigma_{app}(S-\lambda I)^m\} = |1-|\lambda||^m$. It follows that the right hand side of (0.5) is dominated by $M^2\left[|1-|\lambda||^m + \frac{1}{m}\right]^2$. Combining with (0.6) we obtain that

$$\left((1+|\lambda|)^2 - \frac{4|\lambda|}{m}\right)^m \frac{4|\lambda|}{m(1+|\lambda|)^2} - \frac{1}{m} \leq M^2\left[|1-|\lambda||^m + \frac{1}{m}\right]^2.$$

It is easy to see that this leads to a contradiction, because the left side behaves asymptotically as $(1+|\lambda|)^{2m}/m$ while the right side behaves as $|1-|\lambda||^{2m}$, so their quotient cannot be bounded. □

In the remainder of the paper we will compare Deddens algebras $\mathcal{D}_{S-\lambda I}$ for different values of λ. As a consequence of Theorem 1 we obtain a sufficient condition for two Deddens algebras to be distinct.

Corollary 6. *If $\lambda_1 = r_1 e^{i\varphi_1}$ and $\lambda_2 = r_2 e^{i\varphi_2}$ are complex numbers and $\varphi_1 \neq \varphi_2$ then $\mathcal{D}_{S-\lambda_1 I} \neq \mathcal{D}_{S-\lambda_2 I}$.*

When $\varphi_1 = \varphi_2$ the relationship between these algebras is less clear. We will establish an inclusion and for that we need a lemma. Since its proof requires only induction we leave the details to the reader.

Lemma 7. *Let* λ *be a complex number, let* $r > 0$*, and define two sequences of operators:* $R_n = (S - \lambda I)^{*n}(S - \lambda I)^n$ *and* $L_n = (S - r\lambda I)^{*n}(S - r\lambda I)^n$*. If* $c = (1 - r)(1 - r|\lambda|^2)$*, then* $L_n = \sum_{k=0}^{n} \binom{n}{k} c^k r^{n-k} R_{n-k}$ *and* $R_n = r^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k c^k L_{n-k}$*.*

Now we can prove the desired implication.

Theorem 8. *If* $\lambda_1 = r_1 e^{i\varphi}$ *and* $\lambda_2 = r_2 e^{i\varphi}$ *are complex numbers and either* $0 \le r_1 \le r_2 \le 1$ *or* $1 \le r_2 \le r_1$*, then* $\mathcal{D}_{S-\lambda_2 I} \subset \mathcal{D}_{S-\lambda_1 I}$*.*

Proof. Suppose that $r_1 < r_2 \le 1$ and let $r = r_1/r_2$. If X is an operator in $\mathcal{D}_{S-\lambda_2 I}$ then, for any $f \in H^2$, $\|(S - \lambda_2 I)^n X f\| \le M\|(S - \lambda_2 I)^n f\|$. By dividing by M we may assume that X satisfies $\|(S - \lambda_2 I)^n X f\| \le \|(S - \lambda_2 I)^n f\|$. With the notation of Lemma 7, where $\lambda = \lambda_2$, we obtain that $R_n - X^* R_n X$ is a positive operator for each n. Now $L_n - X^* L_n X = \sum_{k=0}^{n} \binom{n}{k} c^k r^{n-k} [R_{n-k} - X^* R_{n-k} X]$ and $c > 0$ so $L_n - X^* L_n X \ge 0$. In other words, $X \in \mathcal{D}_{S-r\lambda_2 I} = \mathcal{D}_{S-\lambda_1 I}$. The case $1 \le r_2 \le r_1$ is different only in the fact that now c is a product of two negative, instead of two positive numbers. $\square$

Unfortunately, we were unable to determine whether the inclusion $\mathcal{D}_{S-\lambda_2 I} \subset \mathcal{D}_{S-\lambda_1 I}$ in Theorem 8 is an equality. We leave this is an open question.

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