Abstract

We consider spectral radius algebras associated to C_0 contractions. When the operator A is algebraic, we describe all invariant subspaces that are common for operators in its spectral radius algebra $\mathcal{B}_A$. In all other situations $\mathcal{B}_A$ is weakly dense and we characterize a set of rank one operators in $\mathcal{B}_A$ that is weakly dense in $\mathcal{L}(\mathcal{H})$.

Keywords and phrases: spectral radius algebras, C_0 contraction, Jordan block.

SPECTRAL RADIUS ALGEBRAS AND C_0 CONTRACTIONS, II

SRDJAN PETROVIC

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Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Given an operator $A \in \mathcal{L}(\mathcal{H})$ with spectral radius r we define a sequence of positive numbers $d_m = m/(1 + rm)$ and operators $R_m = \left(\sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n\right)^{1/2}$. The *spectral radius algebra* $\mathcal{B}_A$ consists of all operators $T \in \mathcal{L}(\mathcal{H})$ such that $\sup_{m\in\mathbb{N}} \|R_m T R_m^{-1}\| < \infty$. The study of these algebras started in [6] where it was shown that, when A is compact, the algebra $\mathcal{B}_A$ has a nontrivial invariant subspace. A similar result followed for some normal operators [3]. A major role in these results was played by the ideal $\mathcal{Q}_A = \{T : \|R_m T R_m^{-1}\| \to 0\}$. We state the facts that will be used in this paper and we direct the reader to the articles [2] – [9] for more information.

Proposition 0.1. *Let A be an operator in $\mathcal{L}(\mathcal{H})$. When $AT = \lambda TA$, for some $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, then $T \in \mathcal{B}_A$. In particular, the commutant $\{A\}' \subset \mathcal{B}_A$. If there exists a non-zero compact operator in $\mathcal{Q}_A$ then $\mathcal{B}_A$ has n. i. s. Finally, $B_A = \mathcal{L}(\mathcal{H})$ if and only if the operator A is similar to a constant multiple of an isometry.*

A contraction A is *completely nonunitary* if there is no invariant subspace $\mathcal{M}$ for A such that $A|\mathcal{M}$ is a unitary operator. A completely nonunitary contraction A is said to be of *class* C_0 if there exists a nonzero function $h \in H^\infty$ such that $h(A) = 0$. The inner function v such that $vH^\infty = \{u \in H^\infty : u(A) = 0\}$ is the *minimal function* of A and is denoted by m_A. The operator A is *algebraic* if there is a polynomial p such that $p(A) = 0$.

One of the most studied concrete Hilbert spaces is the Hardy space H^2, and one of the best understood operators is the unilateral shift. Throughout the paper we will use S to denote the forward unilateral shift of multiplicity 1, and $\{e_n\}_{n=0}^{\infty}$ the orthonormal basis such that $Se_n = e_{n+1}$, $n \geq 0$. One knows that S can be viewed as multiplication by z on H^2. The classical result of Beurling states that every invariant subspace of S is of the form θH^2 for some inner function θ. The compression of S to $H^2 \ominus \theta H^2$ is called a Jordan block. This subspace will be denoted by $\mathcal{H}(\theta)$ and the compression in question by $S(\theta)$.

At this stage it is useful to point out that the term *Jordan block* has a different meaning in Linear Algebra. For example, if $\theta(z) = \mu_\alpha(z)^2\mu_\beta(z)^3$, where μ is the Möbius transformation $\mu_\lambda(z) = (z-\lambda)/(1-\overline{\lambda}z)$, then $S(\theta)$ acts on the space of dimension 5 and it is a direct sum of *two* Jordan blocks. To avoid confusion, we will say that, in this example, $S(\theta)$ is a direct sum of two *simple* Jordan blocks.

This paper can be viewed as a sequel to [9]. We continue the study of spectral radius algebras associated to C_0 contractions. However, in the previous paper, the emphasis was on establishing that the inclusion $\{A\}' \subset \mathcal{B}_A$ is proper. Here, our focus is on the structure of the algebra $\mathcal{B}_A$. In particular, we will show that the situation is significantly different depending on whether m_A is a finite Blaschke product or not. In the latter case, $\mathcal{B}_A$ is always weakly dense in $\mathcal{L}(\mathcal{H})$. (Throughout the paper all references to density will be about the *weak* density.) We will establish this fact by characterizing the set $\mathcal{N}$ of rank one operators in $\mathcal{B}_A$ and by showing that it is dense in $\mathcal{L}(\mathcal{H})$. This set will be more transparent in the case when $A = S(\theta)$ (Theorem 5) and to a lesser extent for a general contraction of class C_0 (Theorem 18). The case when A is algebraic will be studied using mostly the finite dimenisonal tools. For such an operator, the quasi-similarity model $S(\Theta)$ is a direct sum (possibly infinite) $\bigoplus_k S(\theta_k)$ but each operator $S(\theta_k)$ acts on a finite dimensional space. Therefore, $S(\Theta)$ is similar to a direct sum of simple Jordan blocks and, moreover, $S(\Theta) = S(\Theta_1) \oplus S(\Theta_2)$ where $S(\Theta_2)$ contains all the blocks with maximal eigenvalues (i.e., of absolute value equal to the spectral radius of A). Our main result for algebraic C_0 conractions (Theorem 21) is that if, relative to this decomposition, $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{B}_{S(\Theta)}$ then $T_3 = 0$ and T_4 consists of upper triangular blocks (relative to the representation of $S(\Theta_2)$ as a direct sum of simple Jordan blocks).

Before we address the general case we will investigate the basic C_0 contraction $S(\theta)$. First we assume that θ is not a finite Blaschke product so that $\mathcal{H}(\theta)$ is infinite dimensional. We will make use of two operators acting on H^2. For $f = \sum_{k\geq 0} f_k e_k \in H^2$ we define $Df = \sum_{k\geq 1} \sqrt{k} f_k e_{k-1}$ and $Jf = \sum_{k\geq 0}(f_k/\sqrt{k+1})e_{k+1}$. We start with the following result.

Proposition 0.2. *The set* $\mathcal{N} = \{u \in \mathcal{H}(\theta) : Du \in H^2\}$ *is dense in* $\mathcal{H}(\theta)$.

Proof. Suppose, to the contrary, that there exists $h \in \mathcal{H}(\theta)$ such that $h \perp \mathcal{N}$. Notice that, if g is any function satisfying $g \perp J(\theta H^2)$ then $J^* g \in \mathcal{N}$. Therefore, $h \perp J^* g$ and $Jh \perp g$ which implies that Jh belongs to the closure of $J(\theta H^2)$. In other words, there exists a sequence of polynomials $\{p_n\}$ such that $J(\theta p_n) \to Jh$ in the norm of H^2. All the more, $J(\theta p_n - h) \to 0$ weakly. Let $f \in H^2$. Then $\langle \theta p_n - h, J^* f\rangle = \langle J(\theta p_n - h), f\rangle \to 0$. Since the range of J^* is dense, it follows that $\theta p_n - h \to 0$ weakly. In particular, for any $v \in H^2$, $\langle \theta p_n - h, \theta v\rangle \to 0$. But $\langle h, \theta v\rangle = 0$ so $\langle \theta p_n, \theta v\rangle \to 0$. Taking into account that multiplication by θ is an isometry we see that $\langle p_n, v\rangle \to 0$, i.e., the sequence p_n weakly converges to 0. Consequently, the same is true of $J(\theta p_n)$. However, $J(\theta p_n) \to Jh$, and it follows that $Jh = 0$ and, hence, that $h = 0$. We conclude that $\mathcal{N}$ is dense in $\mathcal{H}(\theta)$. $\square$

Next we demonstrate the significance of $\mathcal{N}$.

Theorem 0.3. *Let θ be an inner function that is not a finite Blaschke product. A rank one operator $u \otimes v \in \mathcal{B}_{S(\theta)^*}$ if and only if $u \in \mathcal{N}$.*

Proof. The assumption on θ guarantees that the spectral radius $r(S(\theta))$ equals 1, so $d_m(S(\theta)) = d_m(S^*) = m/(m+1)$. Relative to the decomposition $H^2 = \theta H^2 \oplus \mathcal{H}(\theta)$ we have

$$R_m^2(S^*) = \begin{pmatrix} \star & \star \\ \star & R_m^2(S(\theta)^*) \end{pmatrix}.$$

Let $u \in \mathcal{N}$. Then, relative to the same decomposition, u can be identified with $w = 0 \oplus u$. Clearly, $\langle R_m^2(S^*)w, w\rangle = \langle R_m^2(S(\theta)^*)u, u\rangle$ so $\|R_m(S^*)w\| = \|R_m(S(\theta)^*)u\|$. Notice that $Dw = Du \in H^2$ which means that $\sum |\sqrt{k} w_k|^2 < \infty$. A straightforward calculation shows that $R_m(S^*)$ can be represented in the basis $\{e_k\}$ as a diagonal matrix diag $(\alpha_{m,0}, \alpha_{m,1}, \dots)$ where $\alpha_{m,k} = 1+d_m^2+d_m^4+\cdots+d_m^{2k}$. Now $R_m(S^*)w = R_m(S^*)\sum w_k e_k = \sum w_k R_m(S^*)e_k = \sum w_k \alpha_{m,k} e_k$, and $\|R_m(S^*)w\|^2 = \sum |w_k|^2 |\alpha_{m,k}|^2 \leq \sum |w_k|^2(k+1)$ so it follows that $\sup_m \|R_m(S^*)w\| < \infty$. Since R_m^{-1} is always a contraction, we see that, for any $v \in \mathcal{H}(\theta)$, $u \otimes v \in \mathcal{B}_{S(\theta)^*}$. This shows that the condition $u \in \mathcal{N}$ is sufficient and, in addition, that the algebra $\mathcal{B}_{S(\theta)^*}$ contains a dense set $\mathcal{N}$ so it is dense in $\mathcal{L}(\mathcal{H}(\theta))$.

Suppose now that $u \otimes v \in \mathcal{B}_{S(\theta)^*}$. Then $\|R_m(S(\theta)^*)u\| \|R_m^{-1}(S(\theta)^*)v\|$ is a bounded sequence, so $\sup_m \|R_m(S(\theta)^*)u\| < \infty$ or $\lim_m \|R_m^{-1}(S(\theta)^*)v\| = 0$. However, the latter is impossible. Indeed, if there exists such a non-zero vector v, then $\|R_m(S(\theta)^*)u\| \|R_m^{-1}(S(\theta)^*)v\| \to 0$ for any $u \in \mathcal{N}$. In other words, $u \otimes v \in \mathcal{Q}_{S(\theta)^*}$ and it would follow from Proposition 0.1 that the algebra $\mathcal{B}_{S(\theta)^*}$ has a nontrivial invariant subspace, contradicting the fact that it is dense. Thus, $\|R_m(S(\theta)^*)u\|$ must be bounded and, just like above, if $w = 0 \oplus u$, then $\sup_m \|R_m(S^*)w\| < \infty$. Consequently, there exists $M > 0$ such that $\sum |w_k|^2 |\alpha_{m,k}|^2 \leq M$, for all $m \in \mathbb{N}$. Since the last series converges uniformly in m, we can pass to the limit as $m \to \infty$. We obtain that $\sum |w_k|^2(k+1) \leq M$ which implies that $Dw \in H^2$ and $u \in \mathcal{N}$. □

As a consequence of Proposition 0.2 and Theorem 0.3 we obtain the following characterization.

THEOREM 0.4. *Let θ be an inner function that is not a finite Blaschke product. Then the algebra $\mathcal{B}_{S(\theta)^*}$ is dense in $\mathcal{L}(\mathcal{H}(\theta))$. Moreover, it contains a dense set of rank one operators, and $u \otimes v \in \mathcal{B}_{S(\theta)^*}$ if and only if $u \in \mathcal{N}$, with $\mathcal{N}$ as in Proposition 0.2.*

In order to describe $\mathcal{B}_{S(\theta)}$ we employ a connection between the Jordan block $S(\theta)$ and the operator $S(\tilde{\theta})^*$, where $\tilde{\theta}(z) = \overline{\theta(\overline{z})}$. We recall (cf., [1, Corollary 3.1.7]) that there exists a unitary operator $U : \mathcal{H}(\theta) \to \mathcal{H}(\tilde{\theta})$ such that $S(\tilde{\theta})^*U = US(\theta)$. Further, [3, Theorem 2.4] implies that there exists an isomorphism $\mathcal{U} : \mathcal{B}_{S(\theta)} \to \mathcal{B}_{S(\theta)^*}$ defined by $\mathcal{U}(X) = UXU^*$. Using Theorem 0.4 we obtain that $\mathcal{B}_{S(\theta)}$ is dense. We omit the proof since it is straightforward.

THEOREM 0.5. *Let θ be an inner function that is not a finite Blaschke product. Then the algebra $\mathcal{B}_{S(\theta)}$ is weakly dense in $\mathcal{L}(\mathcal{H}(\theta))$. Moreover, it contains a dense set of rank one operators and $u \otimes v \in \mathcal{B}_{S(\theta)}$ if and only if $u \in \mathcal{N}' \equiv \{u \in \mathcal{H}(\theta) : DUu \in H^2\}$, where U is the unitary operator such that $S(\tilde{\theta})^*U = US(\theta)$.*

REMARK. In [1, Exercise 5, p. 42] the operator U is given explicitly. Using this formula, a short calculation shows that the condition $DUu \in H^2$ can be written as

$$\sum_{m\geq 1} m \left| \sum_{j\geq 0} \overline{\theta}_{m+j+1} u_j \right|^2 < \infty$$

where θ_k and u_k are Taylor coefficients of θ and u, resp.

Now we turn our attention to the case when θ is a finite Blaschke product so $S(\theta)$ acts on a finite dimensional space. In this situation, $S(\theta)$ can be represented as a direct sum of simple Jordan blocks. More precisely, $S(\theta) = \bigoplus_{i=1}^{n} J_{\alpha_i}$ where

$$J_{\alpha_i} = \begin{pmatrix} \alpha_i & 1 & & & \\ & \alpha_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha_i & 1 \end{pmatrix}.$$

We start with the case when $n = 1$. The following result is a combination of [4, Theorem 4.7] and a fact that can be found in its proof. Following its lead, we will use λ_m to denote $1/(1 - |\alpha|^2 d_m^2)$, and the symbol $\sim$ to denote the same rate of growth (as $m \to \infty$).

THEOREM 0.6. *Let α be a complex number and let J_α be the simple $N \times N$ Jordan block with eigenvalue $\alpha \neq 0$. If $R_m = R_m(J_\alpha)$ then $\det(R_m) \sim \lambda_m^{N^2}$. Also, $(R_m^2)_{i,j}$ — the (i,j) entry of R_m^2 — satisfies $(R_m^2)_{i,j} = (i+j)!/(i!j!)\, \lambda_m^{i+j+1} + q_{i+j}(\lambda_m)$, where q_{i+j} is a polynomial of degree up to $i+j$. Therefore, $(R_m^2)_{i,i} \sim \lambda_m^{2i+1}$.*

Now we can describe the algebra $\mathcal{B}_{J_\alpha}$.

THEOREM 0.7. *The spectral radius algebra associated to a simple Jordan block J_α is the algebra of all upper triangular matrices.*

PROOF. We consider separately the cases $\alpha \neq 0$ and $\alpha = 0$. Let $\alpha \neq 0$, let J_α be of the size $N \times N$, and let $e_0, e_1, e_2, \ldots, e_{N-1}$ be the appropriate basis for $\mathbb{C}^N$. We will show that a rank one operator $e_i \otimes e_j$ belongs to $\mathcal{B}_{J_\alpha}$ if and only if $i \leq j$. Notice that $\|R_m e_i\|^2 = \langle R_m^2 e_i, e_i \rangle$, the (i,i) entry of R_m^2. By Theorem 0.6, $\|R_m e_i\|^2 \sim \lambda_m^{2i+1}$.

Similarly, $\|R_m^{-1}e_j\|^2$ equals the (j,j) entry of R_m^{-2}. One knows that this entry can be calculated by dividing the cofactor A_{jj} corresponding to the (j,j) entry of R_m^2 by the determinant of R_m^2. By Theorem 0.6, the latter determinant behaves asymptotically as $\lambda_m^{N^2}$, so we turn our attention to A_{jj}. Let L_1 and L_2 stand for diagonal matrices $\operatorname{diag}(\lambda_m^k)_{k=0}^{N-1}$ and $\operatorname{diag}(\lambda_m^{k+1})_{k=0}^{N-1}$, let B be the matrix with (m,n) entry $\binom{m+n}{m}$, and let L_1', L_2', and B' denote L_1, L_2, and B with the jth rows and columns deleted. The matrix M_{jj} (obtained by deleting the jth row and column from R_m^2) is asymptotically equal to $L_1'B'L_2'$. Therefore, $A_{jj} \sim \det(L_1')\det(B')\det(L_2')$. Now, $\det(L_1') = \lambda_m^{1+2+\cdots+(N-1)-j} = \lambda_m^{(N-1)N/2-j}$ and, similarly, $\det(L_2') = \lambda_m^{N(N+1)/2-(j+1)}$. Of course, $\det(B)$ is independent of m, so $A_{jj} \sim \lambda_m^{N^2-(2j+1)}$. It follows that $\|R_m^{-1}e_j\|^2 \sim \lambda_m^{-2j-1}$, and $\|R_m e_i\|^2\|R_m^{-1}e_j\|^2 \sim \lambda_m^{2(i-j)}$. Since $\lambda_m \to \infty$ the rank one operator $e_i \otimes e_j \in \mathcal{B}_{J_\alpha}$ if and only if $i \le j$. This shows that every upper triangular matrix belongs to $\mathcal{B}_{J_\alpha}$, and the converse follows from the observation that, for any matrix $A = (a_{mn})$, $e_m \otimes e_m A e_n \otimes e_n = a_{mn} e_m \otimes e_n$.

The case when $\alpha = 0$ leads to a different form for R_m. Let J_0 be the simple Jordan block of size $N \times N$ corresponding to the eigenvalue $\alpha = 0$. A calculation shows that $R_m = \operatorname{diag}(1, \alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,N-1})$ where $\alpha_{m,k} = (1 + d_m^2 + \cdots + d_m^{2k})^{1/2}$. If $T = (t_{ij})$ then $R_m T R_m^{-1} = (\alpha_{m,i} t_{ij} \alpha_{m,j}^{-1})$. Since the spectral radius of J_0 is 0 we have that $d_m = m$ and $\alpha_{m,k} \to \infty$ as $m \to \infty$. Consequently, $T \in \mathcal{B}_{J_0}$ if and only if $t_{ij} = 0$ for $i > j$, i.e., if and only if T is an upper triangular matrix. □

Next we consider a slightly more complicated scenario. Namely, we will assume that θ has more than one zero, but that they are all of the same modulus. The corresponding operator $S(\theta)$ is then a direct sum of simple Jordan blocks, which need not be of the same size. Thus, in the block representation of the matrix for this operator, the off-diagonal blocks may be rectangular. We will extend the meaning of an upper triangular matrix to apply to such blocks. Namely, if $A = (a_{ij})_{1\le i\le n,\,1\le j\le m}$, then A is upper triangular if $a_{ij} = 0$ whenever $i > j$. Similarly, A is diagonal if $a_{ij} = 0$ for $i \ne j$. Now we can prove an extension of Theorem 0.7.

Proposition 0.8. *Let N, K be positive integers, let J_α, J_β be simple Jordan blocks with* *the eigenvalues* α, β *of size* $N \times N$, $K \times K$, *resp., and suppose that* $\alpha \neq \beta$, $|\alpha| = |\beta|$. *If* $J = J_\alpha \oplus J_\beta$ *and* $\{e_k\}_{k=0}^{N+K-1}$ *is the appropriate basis for* $\mathbb{C}^{N+K}$, *then* $e_i \otimes e_j \in \mathcal{B}_J$ *if and only if* i *and* j *satisfy:* $i \leq j$ *when* $0 \leq i, j \leq N-1$ *or* $N \leq i, j \leq N+K-1$; $i \leq j - N$ *when* $0 \leq i \leq N-1$ *and* $N \leq j \leq N+K-1$; $i \leq j+N$ *when* $N \leq i \leq N+K-1$ *and* $0 \leq j \leq N-1$. *In other words, a block matrix* $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ *belongs to* $\mathcal{B}_J$ *if and only if each of the 4 blocks is upper triangular.*

Proof. We notice that $R_m(J) = R_m(J_\alpha) \oplus R_m(J_\beta)$ so the estimates for $\|R_m e_i\|^2$ depend on whether $i \leq N-1$ or $i \geq N$. Using the same computations as in the proof of Theorem 0.7, together with the fact that the quantity λ_m depends only on the modulus of the eigenvalue, we see that, when $\alpha \neq 0$, $\|R_m e_i\|^2 \sim \lambda_m^{2i+1}$ if $0 \leq i \leq N-1$ or $\|R_m e_i\|^2 \sim \lambda_m^{2(i-N)+1}$ if $N \leq i \leq N+K-1$. Similarly, $\|R_m^{-1} e_j\|^2 \sim \lambda_m^{-2j-1}$ if $0 \leq j \leq N-1$ or $\|R_m^{-1} e_j\|^2 \sim \lambda_m^{-2j+2N-1}$ if $N \leq j \leq N+K-1$. The rest of the proof, inclusing the case $\alpha = 0$, is straightforward. $\square$

Remark. If the ordered basis $\{e_k\}$ is replaced by its permutation $e_0, e_N, e_1, e_{N+1}, \ldots$, $e_{K-1}, e_{N+K-1}, e_K, e_{K+1}, \ldots, e_{N-1}$ then the matrix for T becomes upper triangular. (Clearly, we have used the assumption $N > K$. A similar permutation can be written if $N < K$. If $N = K$ no permutation is necessary since T is already upper triangular.)

It is easy to see that Proposition 0.8 can be extended to the case when θ has any finite number of (distinct) zeros of the same absolute value.

Corollary 0.9. *Let* $J = J_{\alpha_1} \oplus J_{\alpha_2} \oplus \cdots \oplus J_{\alpha_n}$, *where* $|\alpha_k| = \alpha$, $1 \leq k \leq n$, *and each simple Jordan block* J_{α_k} *is of dimension* $N_k \times N_k$. *If* $N = N_1 + N_2 + \cdots + N_n$ *then an operator* $T = (T_{ij})_{i,j=1}^N \in \mathcal{B}_J$ *if and only if each block* T_{ij} *is upper triangular.*

It remains to consider the situation in which the zeros of θ can be of different absolute value. Here we will prove a more general result which is true regardless of the dimension of Hilbert space.

Proposition 0.10. *Let* $A_k \in \mathcal{L}(\mathcal{H}_k)$, $k = 1, 2$, *let* $A = A_1 \oplus A_2$, *and suppose that*

$r(A_1) < r(A_2)$. *If an operator* $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{B}_A$ *then* T_1 *and* T_2 *can be any operators and* $T_4 \in \mathcal{B}_{A_2}$. *If, in addition,* $\mathcal{B}_{A_2}$ *contains each rank one operator* $e_i \otimes e_i$, *for some orthonormal basis* $\{e_n\}$, *then* $T \in \mathcal{B}_A$ *if and only if* $T_3 = 0$ *and* $T_4 \in \mathcal{B}_{A_2}$.

Proof. Let $C_m = \sum d_m(A)^{2n} A_1^{*n} A_1^n$. Notice that $R_m^2(A) = C_m \oplus R_m^2(A_2)$ since $d_m(A) = d_m(A_2)$. The inequality $r(A_1) < r(A_2)$ implies that the sequence C_m is norm bounded. Consequently, $\|C_m T_1 C_m^{-1}\|$ and $\|C_m T_2 R_m(A_2)^{-1}\|$ are bounded for any T_1, T_2. Further, the sequence C_m^{-1} is bounded from below. Thus, if $\tilde{u} = 0 \oplus u$ and $\tilde{v} = v \oplus 0$ then $\|R_m(A)\tilde{u} \otimes \tilde{v} R_m(A)^{-1}\| \to \infty$. This means that $T_3 = 0$ because $T \in \mathcal{B}_A$ if and only if $e_m \otimes e_m T e_n \otimes e_n \in \mathcal{B}_A$ for all m, n. Finally, $\|R_m(A_2) T_4 R_m(A_2)^{-1}\|$ is bounded if and only if $T_4 \in \mathcal{B}_A$. □

Remark. The condition that $\mathcal{B}_{A_2}$ contains rank one operators in Proposition 0.10 is essential for the conclusion that $T_3 = 0$. Indeed, if $A_2 = 0 \oplus 1$ and $T_3 = 1 \oplus 0$ then $R_m(A_2) T_3 = T_3$ whence the boundedness of C_m^{-1} implies that $\sup_m \|R_m(A_2) T_3 C_m^{-1}\| < \infty$.

Now we can prove the most general result for the case when $S(\theta)$ acts on a finite dimensional space.

Theorem 0.11. *Let* $N_1, N_2, \dots, N_n$ *and* $K_1, K_2, \dots, K_m$ *be positive integers,* $N = N_1 + \cdots + N_n$, $K = K_1 + \cdots + K_m$, *and let* $\{\alpha_i\}_{i=1}^n$, $\{\beta\}_{j=1}^m$ *be two sequences of complex numbers such that* $|\alpha_1| < |\alpha_2| < \cdots < |\alpha_n| < |\beta_1| = |\beta_2| = \cdots = |\beta_m|$. *Suppose that simple Jordan blocks* $J_{\alpha_i}, J_{\beta_j}$ *are of dimensions* $N_i \times N_i$, $K_j \times K_j$, *resp., let* $J_\alpha = J_{\alpha_1} \oplus \cdots \oplus J_{\alpha_n}$, $J_\beta = J_{\beta_1} \oplus \cdots \oplus J_{\beta_m}$, *and* $J = J_\alpha \oplus J_\beta$. *Relative to the decomposition* $\mathbb{C}^{N+K} = \mathbb{C}^N \oplus \mathbb{C}^K$, *let* $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. *Then* $T \in \mathcal{B}_J$ *if and only if* $T_3 = 0$ *and* $T_4 \in \mathcal{B}_{J_\beta}$.

Proof. Clearly $r(J_\alpha) < r(J_\beta)$ and, by Corollary 0.9, $e_i \otimes e_i \in \mathcal{B}_{J_\beta}$. The result now follows from Proposition 0.10. □

In the remainder of the paper we will apply the results about Jordan blocks to describe $\mathcal{B}_A$ for any $A \in C_0$. One knows (cf., [1, Theorem 3.5.1]) that there exist inner functions $\{\theta_k\}$ such that $\theta_{k+1} | \theta_k$, and Hilbert spaces $\mathcal{H}_k$ so that A is quasisimilar to a direct sum of Jordan blocks $S(\Theta) \equiv \bigoplus_k S(\theta_k)$, acting on $\bigoplus_k \mathcal{H}_k$. Therefore, we need to establish some fact about spectral radius algebras associated to quasi-similar operators. We start with a result from [9]. Recall that an operator $Z \in \mathcal{L}(\mathcal{H})$ is a quasi-affinity if it has trivial kernel and dense range.

LEMMA 0.12. *Suppose that A and B are quasisimilar C_0 contractions and let Y, Z be quasi-affinities such that $AY = YB$ and $ZA = BZ$. If $T \in \mathcal{B}_B$ then $YTZ \in \mathcal{B}_A$.*

Now we will establish a much stronger result.

THEOREM 0.13. *Suppose that A and B are quasi-similar C_0 contractions and let Y, Z be quasi-affinities such that $AY = YB$ and $ZA = BZ$. Then $\mathcal{B}_A$ is weakly dense in $\mathcal{L}(\mathcal{H})$ if and only if the same is true of $\mathcal{B}_B$. Moreover, if one of the algebras possesses a dense set of rank one operators, so does the other one. In fact, if there is a dense set $\mathcal{N}$ such that, for any $u \in \mathcal{N}$, $\sup_m \|R_m(A)u\| < \infty$, then $\sup_m \|R_m(B)w\| < \infty$, for any w in the dense set $Z\mathcal{N}$.*

PROOF. Suppose that $\mathcal{B}_B$ is dense, let $\epsilon > 0$, and let $W \in \mathcal{L}(\mathcal{H})$. Since Y and Z^* have dense ranges, the set $\{Yu \otimes Z^*v : u, v \in \mathcal{H}\}$ is dense in $\mathcal{L}(\mathcal{H})$ so there are $u, v \in \mathcal{H}$ such that $W_1 \equiv Yu \otimes Z^*v$ satisfies $|\langle (W_1 - W)x, y\rangle| < \epsilon\|x\|\|y\|$, for all $x, y \in \mathcal{H}$. Also, $\mathcal{B}_B$ is dense, hence there exists an operator $W_2 \in \mathcal{B}_B$ such that $|\langle (W_2 - u \otimes v)x, y\rangle| < \epsilon\|x\|\|y\|$, for all $x, y \in \mathcal{H}$. By Lemma 0.12, $YW_2Z \in \mathcal{B}_A$ and $|\langle (YW_2Z - W)x, y\rangle| < \epsilon(\|Z\|\|Y\| + 1)\|x\|\|y\|$, whence $\mathcal{B}_A$ is dense in $\mathcal{L}(\mathcal{H})$. Also, if W_2 is a rank one operator then so is YW_2Z. Finally, since A and B share the same quasi-similarity model they have the same spectral radius and, thus, $d_m(A) = d_m(B)$. Since $\|B^nZ\| = \|ZA^n\| \leq \|Z\|\|A^n\|$ we obtain that $\|R_m(B)Zu\| \leq \|Z\|\|R_m(A)u\|$. $\square$

With Theorem 0.13 in hand, we proceed to analyze the operator $S(\Theta)$. It turns out that, as before, there are two very different cases, depending on the type of the minimal function of A. We present these results separately (Theorems 0.14 and 0.17 below).

THEOREM 0.14. *Let A be a C_0 contraction and let m_A be its minimal function. If m_A is not a finite Blascke product then the algebra $\mathcal{B}_A$ contains a dense set of rank one operators, so it is dense.*

Proof. Suppose first that none of the functions θ_k in the quasi-similarity model $S(\Theta)$ is a finite Blaschke product. By Theorem 0.5, for all k, there is a dense set of vectors $\mathcal{N}_k \subset \mathcal{H}_k$ such that $\sup_m \|R_m(S(\theta_k)u\| < \infty$, for all $u \in \mathcal{N}_k$. Then $\mathcal{N} \equiv \bigoplus \mathcal{N}_k$ is dense in $\bigoplus_k \mathcal{H}_k$ and, for all $u \in \mathcal{N}$, $\sup_m \|R_m(S(\Theta))u\| < \infty$. Clearly, $Z\mathcal{N}$ is dense and, by Theorem 0.13, for any $w \in Z\mathcal{N}$, $\sup_m \|R_m(A)w\| < \infty$.

Thus, we turn our attention to the case when there exists $k_0 > 0$ such that θ_k is a finite Blaschke product for $k \geq k_0$ but not for $k < k_0$. In this situation we will use the notation $S(\Theta_1) = \bigoplus_{k \geq k_0} S(\theta_k)$ and $S(\Theta_2) = \bigoplus_{k<k_0} S(\theta_k)$, so $S(\Theta) = S(\Theta_1) \oplus S(\Theta_2)$. Notice that $r(S(\Theta_1)) < r(S(\Theta_2)) = 1$. If $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{B}_{S(\Theta)}$ (relative to the same decomposition) then it follows from Proposition 0.10 that T_1 and T_2 are arbitrary, and T_4 belongs to $\mathcal{B}_{S(\Theta_2)}$ (which contains a dense set of rank one operators). When it comes to T_3, it satisfies the condition $\sup_m \|R_m(S(\Theta_2))T_3R_m(S(\Theta_1))^{-1}\| < \infty$. Again, it is not hard to see that $R_m(S(\Theta_2)) = \bigoplus_{k<k_0} R_m(S(\theta_k))$ and $\bigoplus_{k<k_0} \mathcal{N}_k$ is dense in $\bigoplus_{k<k_0} \mathcal{H}_k$. □

It remains to consider the case when m_A is a finite Blaschke product. In this situation, the same is true of θ_0, so there is a finite number of distinct inner functions in $S(\Theta)$. Let $\{\alpha_i\}_{i=1}^n$, $\{\beta\}_{j=1}^m$ be the zeros of these inner functions such that $|\alpha_1| < |\alpha_2| < \cdots < |\alpha_n| < |\beta_1| = |\beta_2| = \cdots = |\beta_m|$. We denote J'_k (resp., J''_k) a direct sum of finitely or infinitely many copies of all simple Jordan blocks with the eigenvalue α_k (resp., β_k) so that $S(\Theta)$ is similar to a direct sum $J'_k \oplus J''_k$. In order to apply Proposition 0.10 we need to understand the algebra $\mathcal{B}_{J''_k}$. Notice that J''_k is a (possibly infinite) direct sum of a finite number of distinct simple Jordan blocks. We split these blocks into two sets — those that are repeated infinitely many times and those that are repeated finitely many times. Of course, if the former set is empty, the characterization of $\mathcal{B}_{J''_k}$ was obtained in Corollary 0.9. Our first step will be to consider the case when the latter set is empty.

Theorem 0.15. *Let $J = J_{\alpha_1} \oplus J_{\alpha_2} \oplus \cdots \oplus J_{\alpha_n}$, where $|\alpha_k| = \alpha$, $1 \leq k \leq n$, and each simple Jordan block J_{α_k} is of dimension $N_k \times N_k$. Let $N = N_1 + N_2 + \cdots + N_n$ and let A be a direct sum of infinitely may copies of J. If $T = (T_{ij})_{i,j=1}^\infty$ relative to the decomposition $\mathcal{H} = \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \dots$, then $T \in \mathcal{B}_A$ if and only if each* *block T_{ij} is upper triangular.*

Proof. We will start by switching to a different permutation of the orthonormal basis. We will achieve this in several stages. First we will permute the first N vectors, just like in Remark , so that a matrix belongs to $\mathcal{B}_J$ if and only if it is upper triangular. Then we will do the same permutation with the next N vectors, etc. Finally, if this new basis is denoted by $\{e_n\}$, we will use the ordering $e_1, e_{N+1}, e_{2N+1}, \dots, e_2, e_{N+2}, e_{2N+2}, \dots, e_{N-1}, e_{2N-1}, e_{3N-1}, \dots$. Notice that the statement that each matrix T_{ij} is upper triangular in $\{e_n\}$ is equivalent to the fact that T is block upper triangular in the new ordering; more precisely $T = (C_{ij})_{i,j=1}^{\infty}$, and $C_{ij} = 0$ when $i > j$. For the rest of the proof we will work in this new basis, which we will denote $\{\tilde{e}_n\}$. The same estimates that were used in the proof of Theorem 0.7 now show that $\mathcal{B}_A$ contains a rank one operator $\tilde{e}_i \otimes \tilde{e}_j$ if and only if the non-zero entry lies in one of the upper blocks C_{mn} (with $m \leq n$). In particular, it contains every rank one operator of the form $\tilde{e}_i \otimes \tilde{e}_i$. Consequently, $\mathcal{B}_A$ cannot contain an operator that is not in the upper block triangular form, because if t_{pq} is a nonzero entry of an operator $T \in \mathcal{B}_A$ then $\tilde{e}_p \otimes \tilde{e}_p T \tilde{e}_q \otimes \tilde{e}_q = t_{pq} \tilde{e}_p \otimes \tilde{e}_q$. That shows that the upper triangularity condition is necessary.

To prove that it is sufficient, let T be a block upper triangular matrix. If, in the basis $\{e_n\}$, the matrices for $R_m(J_\alpha)$ and $R_m^{-1}(J_\alpha)$ are $(r_{ij}^{(m)})_{i,j=1}^{N}$ and $(s_{ij}^{(m)})_{i,j=1}^{N}$, resp., then in $\{\tilde{e}_n\}$ the matrices for $R_m(A)$ and $R_m^{-1}(A)$ are $(r_{ij}^{(m)} I)_{i,j=1}^{N}$ and $(s_{ij}^{(m)} I)_{i,j=1}^{N}$. Now $R_m T R_m^{-1}$ is an $N \times N$ matrix with operator entries, so it suffice to prove that each of its N^2 blocks remains bounded as $m \to \infty$. To that end, let i, j be fixed. Then the (i,j) block of $R_m T R_m^{-1}$ equals $\sum_{k,l=1}^{N} r_{ik}^{(m)} C_{kl} s_{lj}^{(m)}$ so it suffices to prove that $\sup_m \| r_{ik}^{(m)} C_{kl} s_{lj}^{(m)} \| < \infty$ for each pair $k \leq l$. Since C_{kl} is a bounded operator the last condition is equivalent to $\sup_m |r_{ik}^{(m)} s_{lj}^{(m)}| < \infty$. Notice that $|r_{ik}^{(m)}| \leq \|(r_{1k}^{(m)}, r_{2k}^{(m)}, \dots, r_{Nk}^{(m)})\| = \|R_m e_k\|$. Similarly, $|s_{lj}^{(m)}| \leq \|(s_{l1}^{(m)}, s_{l2}^{(m)}, \dots, s_{lN}^{(m)})\| = \|(R_m^{-1})^{tr} e_l\|$, where X^{tr} stands for the transpose of X. It was observed in the proof of Theorem 0.7 that the dominant part of R_m is a symmetric matrix, so $\|(R_m^{-1})^{tr} e_l\|$ behaves asymptotically as $\|(R_m^{-1}) e_l\|$. Consequently, $\sup_m |r_{ik}^{(m)} s_{lj}^{(m)}| \leq \sup_m \|R_m e_k\| \|(R_m^{-1}) e_l\| < \infty$ since $k \leq l$, and the theorem is proved. □

Next we address the situation when J_k'' is a direct sum of simple Jordan blocks in which some blocks are repeated finitely many times, and others infintely many times.

THEOREM 0.16. *Let* $J_1 = J_{\alpha_1} \oplus J_{\alpha_2} \oplus \cdots \oplus J_{\alpha_n}$, $J_2 = J_{\alpha_{n+1}} \oplus J_{\alpha_{n+2}} \oplus \cdots \oplus J_{\alpha_{n+m}}$, *where* $|\alpha_k| = \alpha$, $1 \le k \le n+m$, *and each simple Jordan block* J_{α_k} *is of dimension* $N_k \times N_k$. *Let* A *be a direct sum of infinitely may copies of* J_1 *followed by* J_2. *If* $T = (T_{ij})_{i,j=1}^{\infty}$ *relative to the decomposition* $\mathcal{H} = \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \cdots \oplus \mathbb{C}^{N_{n+1}} \oplus \mathbb{C}^{N_{n+2}} \oplus \cdots \oplus \mathbb{C}^{N_{m+n}}$, *then* $T \in \mathcal{B}_A$ *if and only if each block* T_{ij} *is upper triangular.*

PROOF. Let $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}'$, where $\mathcal{H}' = \mathbb{C}^{N_{n+1}} \oplus \mathbb{C}^{N_{n+2}} \oplus \cdots \oplus \mathbb{C}^{N_{m+n}} \oplus \mathbb{C}^{N_{n+1}} \oplus \mathbb{C}^{N_{n+2}} \oplus \cdots \oplus \mathbb{C}^{N_{m+n}} \oplus \ldots$, and let $\hat{J} = J \oplus J_{\alpha_{n+1}} \oplus J_{\alpha_{n+2}} \oplus \cdots \oplus J_{\alpha_{m+n}} \oplus J_{\alpha_{n+1}} \oplus J_{\alpha_{n+2}} \oplus \cdots \oplus J_{\alpha_{m+n}} \oplus \ldots$. We identify an operator T acting on $\mathcal{H}$ with $T \oplus 0$ acting on $\hat{\mathcal{H}}$. Then $R_m(\hat{J})\hat{T}R_m^{-1}(\hat{J}) = R_m(J)TR_m^{-1}(J)$ and the result follows from Theorem 0.15 since $\hat{J}$ satisfies its hypotheses. □

Combining Corollary 0.9, Theorem 0.15, and Theorem 0.16 we obtain our final theorem.

THEOREM 0.17. *Let* A *be a* C_0 *contraction on* $\mathcal{H}$ *and let* m_A *be its minimal function. If* m_A *is a finite Blascke product then* A *is quasi-similar to* $S(\Theta)$ *— a finite or infinite direct sum of simple Jordan blocks. Further,* $S(\Theta) = S(\Theta_1) \oplus S(\Theta_2)$ *where all blocks in* $S(\Theta_1)$ *(resp.,* $S(\Theta_2)$*) have the eigenvalue of absolute value less than (resp., equal to) the spectral radius of* A. *If* $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ *relative to this decomposition, then* $T \in \mathcal{B}_{S(\Theta)}$ *if and only if* $T_3 = 0$ *and* $T_4 \in \mathcal{B}_{S(\Theta_2)}$. *Moreover,* $S(\Theta_2) = \bigoplus J_{\alpha_k}$ *and, relative to this decomposition, an operator* $T = (T_{ij}) \in \mathcal{B}_{S(\Theta_2)}$ *if and only if each* T_{ij} *is upper triangular.*

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Department of Mathematics

Western Michigan University

Kalamazoo, MI 49008

e-mail: srdjan.petrovic@wmich.edu.