A DILATION THEORY FOR POLYNOMIALLY BOUNDED OPERATORS

Srdjan Petrović

Abstract. In this paper we construct a special sort of dilation for an arbitrary polynomially bounded operator. This enables us to show that the problem whether every polynomially bounded operator is similar to a contraction can be reduced to a subclass of it.

1. Introduction Let $\mathcal{H}$ be a separable, infinite dimensional, complex, Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* if there exists a constant $K > 0$ such that

$$\|p(T)\| \leq K \sup\{|p(\xi)| : |\xi| = 1\} \tag{1}$$

for every polynomial p. For simplicity of reference, we shall denote the class of all polynomially bounded operators in $\mathcal{L}(\mathcal{H})$ by $PB(\mathcal{H})$. Since, by virtue of von Neumann's inequality, every contraction T in $\mathcal{L}(\mathcal{H})$ satisfies (1) with $K = 1$, one may consider the class $PB(\mathcal{H})$ as a generalization of the class of contraction operators. In fact it is a very important and difficult problem, posed explicitly by Halmos in [8], whether every operator in $PB(\mathcal{H})$ is similar to a contraction. (That is, given an operator $T \in PB(\mathcal{H})$, does there always exist an invertible operator X in $\mathcal{L}(\mathcal{H})$ such that $\|XTX^{-1}\| \leq 1$.) One of the basic tools in the study of contraction operators (cf. [15]) is the old and beautiful theorem of Sz.-Nagy [14] that every contraction has a unitary dilation, i. e., if T is a contraction in $\mathcal{L}(\mathcal{H})$, then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unitary operator $U \in \mathcal{L}(\mathcal{K})$ such that $T^n = PU^n|\mathcal{H}$ for every $n = 0, 1, 2, \ldots$, where P is the orthogonal projection in $\mathcal{L}(\mathcal{K})$ whose range is $\mathcal{H}$. It is obvious that only contractions can have *unitary* dilations, but this leaves open the question of what kind of dilation theory might be available for operators in $PB(\mathcal{H})$. The purpose of this paper is to make a start on this problem by constructing a dilation theory for polynomially bounded operators. We show, in our main theorem (Theorem 1.1), that every polynomially bounded operator has a dilation $\hat{T}$ which does have *some* good properties – namely, $\hat{T}$ is also polynomially bounded, the spectrum $\sigma(\hat{T})$ is the unit circle $\mathbb{T}$ in $\mathbb{C}$, and $\hat{T}$ satisfies

$$(\hat{T}\hat{T}^*)(\hat{T}^*\hat{T}) = (\hat{T}^*\hat{T})(\hat{T}\hat{T}^*) \quad .$$

Before stating the main results of this paper we briefly mention some notation and terminology. As usual, $\mathbb{N}$ is the set of positive integers, $\mathbb{C}$ denotes the complex plane, $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, and $\mathbb{T}$ is the unit circle $\mathbb{T} = \partial\mathbb{D}$ in $\mathbb{C}$. A subspace $\mathcal{M} \subset \mathcal{H}$ is said to be invariant for an operator T in $\mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$, and in this situation we denote by $T|\mathcal{M}$ the restriction of T to $\mathcal{M}$. A subspace $\mathcal{M}$ is said to be semi-invariant for T if there exist invariant subspaces $\mathcal{N}_1 \supset \mathcal{N}_2$ for T such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$, and in this situation we denote by $T_{\mathcal{M}}$ the compression of T to $\mathcal{M}$, i. e. $T_{\mathcal{M}} = P_{\mathcal{M}}T|\mathcal{M}$, where $P_{\mathcal{M}}$ is the (orthogonal) projection in $\mathcal{L}(\mathcal{H})$ with range $\mathcal{M}$. We shall use the notation Ker T and Ran T for the kernel and the range of T , respectively. By dim $\mathcal{M}$ we denote the orthogonal dimension of a Hilbert space $\mathcal{M}$. Finally, if $\mathcal{M}$ is a subspace of $\mathcal{H}$, we denote by $\mathcal{M}^\perp$ the subspace $\mathcal{H} \ominus \mathcal{M}$.

We recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *quasinormal* if T commutes with T^*T . The structure of quasinormal operators was determined by A. Brown in [1]. Clearly a quasinormal operator T satisfies

$$(T^*T)(TT^*) = (TT^*)(T^*T) \quad . \tag{2}$$

Operators T satisfying (2), which we shall call *weakly centered* operators, have been studied in [3], [4], and [5], under the name binormal operators (cf. [2]).

THEOREM 1.1. *For every polynomially bounded operator in $\mathcal{L}(\mathcal{H})$ there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an operator $\hat{T} \in \mathcal{L}(\mathcal{K})$ such that:*

(a) *$\hat{T}$ is polynomially bounded*,

(b) *$\mathcal{H}$ is a semi-invariant subspace for $\hat{T}$* ,

(c) *$\sigma(\hat{T})$ is the unit circle*, and

(d) *$\hat{T}$ is weakly centered.*

For some time it was an open question whether every power bounded operator T in $\mathcal{L}(\mathcal{H})$ is similar to a contraction (notation $T \in \mathcal{SC}(\mathcal{H})$), but this was finally negatively settled by Foguel [6] (see also [7] for a somewhat simpler proof). Several authors have addressed the problem whether every polynomially bounded operator is similar to a contraction (cf. [9], [10], [11]), and we mention in particular, some nice progress made by Paulsen [12], but as of this writing, the question remains open. The folowing theorem shows that it suffices to establish this fact for a subclass of $PB(\mathcal{H})$.

THEOREM 1.2. *Every polynomially bounded operator in $\mathcal{L}(\mathcal{H})$ is similar to a contraction if and only if every weakly centered polynomially bounded operator in $\mathcal{L}(\mathcal{H})$ whose spectrum is the unit circle is similar to a contraction.*

2. Some preliminary lemmas. The proofs of these theorems are based on some preliminary lemmas. We will omit the proofs of the first two lemmas, since they are straightforward.

LEMMA 2.1. *Suppose $T \in \mathcal{SC}(\mathcal{H})$, and let $\mathcal{M}$ be an invariant subspace for T . Then $T|\mathcal{M}$ belongs to $\mathcal{SC}(\mathcal{M})$.*

LEMMA 2.2. *Suppose $T \in \mathcal{SC}(\mathcal{H})$. If $\mathcal{M}$ is a semi-invariant subspace for T, then the compression $T_{\mathcal{M}}$ belongs to $\mathcal{SC}(\mathcal{M})$.*

LEMMA 2.3. *Let T, D, and X be operators in $\mathcal{L}(\mathcal{H})$ such that D is a unilateral weighted shift of infinite multiplicity with weight sequence $\{d_n\}_{n=1}^{\infty}$ defined as*

$$\begin{cases} d_1 = \frac{1}{\log 2}, \\ d_n = \left(\frac{n-1}{n}\right)^2 \frac{\log n}{\log(n+1)}, \quad n \geq 2 \end{cases} \tag{3}$$

and Ran $X^* \subset$ Ker D^* . Then the operator $\tilde{T}$ in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, defined by

$$\tilde{T} = \begin{pmatrix} T & X \\ 0 & D^* \end{pmatrix},$$

belongs to the class $PB(\mathcal{H} \oplus \mathcal{H})$ if and only if $T \in PB(\mathcal{H})$.

Proof. It is clear that the restriction of a polynomially bounded operator to an invariant subspace is also a polynomially bounded operator, so we confine our attention to the other half of the proof. Thus, let $T \in PB(\mathcal{H})$. It is not hard to see that for every nonnegative integer k ,

$$\tilde{T}^k = \begin{pmatrix} T^k & G_k \\ 0 & {D^*}^k \end{pmatrix},$$

where

$$\begin{cases} G_0 = 0, \qquad \text{and} \\ G_k = T^{k-1}X + T^{k-2}XD^* + T^{k-3}X{D^*}^2 + \cdots + TX{D^*}^{k-2} + X{D^*}^{k-1} \quad , \; k \geq 1 \, . \end{cases}$$

Thus if p is the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, then

$$p(\tilde{T}) = \sum_{k=0}^{n} a_k \begin{pmatrix} T^k & G_k \\ 0 & {D^*}^k \end{pmatrix} = \begin{pmatrix} p(T) & \sum_{k=1}^{n} a_k G_k \\ 0 & p(D^*) \end{pmatrix}.$$

By [13, Proposition 2], $\|D^2\| = \sup\{|d_k d_{k+1}| \; : \; k \in \mathbb{N}\}$. Hence

$$\begin{aligned} \|D^2\| = \; & max\Big\{\sup_{k\geq 2}\Big\{\Big(\frac{k-1}{k}\Big)^2 \frac{\log k}{\log(k+1)} \cdot \Big(\frac{k}{k+1}\Big)^2 \frac{\log(k+1)}{\log(k+2)}\Big\} \; , \; d_1 d_2\Big\} \\ = \max & \Big\{\sup_{k\geq 2}\Big\{\Big(\frac{k-1}{k+1}\Big)^2 \frac{\log k}{\log(k+2)}\Big\} \; , \; \frac{1}{\log 2} \cdot \frac{1}{4} \cdot \frac{\log 2}{\log 3}\Big\} \leq 1 \; . \end{aligned}$$

Thus D^2 is a contraction. By a well–known argument (cf. [8]), D is similar to a contraction, hence in $PB(\mathcal{H})$, so the same is true of D^* . Therefore $\tilde{T}$ will be polynomially bounded if and only if there exists $K > 0$ (independent of P) such that $\| \sum_{k=1}^{n} a_k G_k\| \leq K \sup\{|p(\xi)| : |\xi| = 1\} = K\|p\|$. (Here we use the obvious fact that the norm of a 2×2 matrix with operator entries is less than or equal to the sum of the norms of its four entries.) To establish the existence of such a K , note that

$$\begin{aligned} \sum_{k=1}^{n} a_k G_k &= \sum_{k=1}^{n} a_k \Big(\sum_{i=1}^{k} T^{k-i} X {D^*}^{i-1}\Big) = \sum_{i=1}^{n} \sum_{k=i}^{n} a_k T^{k-i} X {D^*}^{i-1} = \\ &= \sum_{i=1}^{n} \sum_{m=0}^{n-i} a_{m+i} T^m X {D^*}^{i-1} \quad , \end{aligned} \tag{4}$$

where we have written $m = k - i$. If we define $p_{(0)} = p$, $p_{(1)}(z) = \frac{p(z)-p(0)}{z}$, and, by induction, $p_{(n)}(z) = \frac{p_{(n-1)}(z)-p_{(n-1)}(0)}{z}$, then it is not hard to see that the right hand side of (4) is exactly

$$\sum^{n} p_{(i)}(T) X {D^*}^{i-1} \quad , \tag{5}$$

and it is well-known (cf. [16, page 418]) that for all polynomials p and for every $n \geq 2$, $\|p_{(n)}\| \leq 6 \log n \|p\|$. Thus

$$
\begin{aligned}
\|\sum_{k=1}^{n} a_k G_k\| &= \|\sum_{i=1}^{n} p_{(i)}(T) X {D^*}^{i-1}\| \\
&\leq \sum_{i=1}^{n} \|p_{(i)}(T)\| \|X {D^*}^{i-1}\| \leq \sum_{i=1}^{n} M \|X {D^*}^{i-1}\| \|p_{(i)}\| \qquad (6) \\
&\leq 2M\|X\|\|p\| + \sum_{i=2}^{n} 6 \log i \|p\| M \|X {D^*}^{i-1}\| \ ,
\end{aligned}
$$

where M is the polynomial bound of T . By definition of D, there exists an infinite dimensional Hilbert space $\mathcal{G}$ and a decomposition $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{G}_n$, where $\mathcal{G}_n = \mathcal{G}$, $n \in \mathbb{N}$, such that, relative to this decomposition, D has the matrix

$$
\begin{pmatrix}
0 & & & \\
d_1 1_{\mathcal{G}} & 0 & & \\
 & d_2 1_{\mathcal{G}} & 0 & \\
 & & \ddots & \ddots \\
 & & & \ddots & \ddots
\end{pmatrix} . \qquad (7)
$$

Furthermore X may thus be regarded as an operator mapping $\bigoplus_{n=1}^{\infty} \mathcal{G}_n$ into $\mathcal{H}$ and hence has a matrix $X = (X_1 \ , \ X_2 \ , \ \dots \ )$, where $X_n \ : \ \mathcal{G}_n \to \mathcal{H}$, $n \in \mathbb{N}$. Moreover, since Ker $D^* =$ Ran X^* , it is clear that $X_n = 0$ for all $n \geq 2$. Thus $\|X {D^*}^{i-1}\| = \|(d_1 \dots d_{i-1}) X_1\|$, and it follows trivially from this and (6) that

$$
\begin{aligned}
\|\sum_{k=1}^{n} a_k G_k\| &\leq 2M\|X_1\|\|p\| + \sum_{i=2}^{n} 6M \log i \ (d_1 \dots d_{i-1}) \|X_1\| \|p\| \\
&\leq M\|X_1\| \left(2 + \pi^2\right) \|p\| \quad .
\end{aligned}
$$

Thus, $\tilde{T}$ is polynomially bounded and the lemma is proved. ∎

3. Two matricial constructions. Before we can turn to the proof of Theorem 1.1, we need two more preliminary results. For the sake of simplicity we shall use the notation $\mathcal{M}^{(3)}$ for $\mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M}$, where $\mathcal{M}$ is an arbitrary Hilbert space.

PROPOSITION 3.1. *Suppose $T \in PB(\mathcal{H})$. Then there exists an operator $\tilde{T}$ in $\mathcal{L}(\mathcal{H}^{(3)})$ such that*

(a) $\mathcal{H} \oplus (0) \oplus (0)$ *is invariant for* $\tilde{T}$,

(b) $T = \tilde{T}|(\mathcal{H} \oplus (0) \oplus (0))$,

(c) $\tilde{T} \in PB(\mathcal{H}^{(3)})$,

(d) Ran $(\tilde{T})$ *is closed*, dim(Ker $\tilde{T}$)$= \aleph_0$, *and* dim(Ker $\tilde{T}^*$)$= \aleph_0$.

Proof. Let M be the polynomial bound for T . Define $\tilde{T}$ to be the following 3×3 operator matrix acting on $\mathcal{H}^{(3)}$ in the usual way:

$$\tilde{T} = \begin{pmatrix} T & \sqrt{M^2 - TT^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{8}$$

It is obvious from the definition that (a) and (b) are valid, so we first show that $\tilde{T}$ is polynomially bounded. An easy computation shows that

$$p(\tilde{T}) = \begin{pmatrix} p(T) & p_{(1)}(T)\sqrt{M^2 - TT^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $p_{(1)}$ is as was defined in the proof of Lemma 2.3 . Since $\|p_{(1)}\| \leq 2\|p\|$, clearly $\tilde{T}$ is polynomially bounded. In order to show that the range of $\tilde{T}$ is closed, it suffices to prove the same fact for $\tilde{T}^*$. We will prove that $\tilde{T}^*$ is bounded below on $(\text{Ker } \tilde{T}^*)^{\perp} = \mathcal{H} \oplus (0) \oplus (0)$. So, let $x \in \mathcal{H}$. Then,

$$\|\tilde{T}^*(x \oplus 0 \oplus 0)\|^2 = \|T^* x\|^2 + \|\sqrt{M^2 - TT^*}x\|^2 = M^2\|x\|^2.$$

This shows that $\tilde{T}^*$ is bounded below on its initial space , and completes the proof. ∎

PROPOSITION 3.2. *Suppose* $T \in PB(\mathcal{H})$, $\mathcal{K} = \mathcal{H}^{(3)}$, *and let* $\tilde{T} \in \mathcal{L}(\mathcal{K})$ *be as in Proposition 3.1. Then there exists an operator* $\hat{T} \in PB(\mathcal{K}^{(3)})$ *such that*

(a) $\mathcal{M} = (0) \oplus \mathcal{K} \oplus (0)$ *is semi-invariant for* $\hat{T}$,

(b) $\hat{T}_{\mathcal{K}} = \tilde{T}$,

(c) $\hat{T} \in PB(\mathcal{K}^{(3)})$,

(d) $\sigma(\hat{T})$ *is the unit circle , and*

(e) $\hat{T}$ *is weakly centered*

Proof. Note that, by (d) of Proposition 3.1, both the kernel and cokernel of $\tilde{T}$ are infinite dimensional. Let D be a weighted unilateral shift of infinite multiplicity in $\mathcal{L}(\mathcal{K})$ with weight sequence $\{d_n\}_{n=1}^{\infty}$ as defined in (3). Then there exist a partial isometry U in $\mathcal{L}(\mathcal{K})$ with initial space Ker $\tilde{T}$ and final space Ker D^*, and a partial isometry V in $\mathcal{L}(\mathcal{K})$ with initial space Ker D^* and final space Ker $\tilde{T}^*$. Let a be a positive number, let M be the polynomial bound for T , and let $A = aU$, $C = MV$. We define $\hat{T}$ to be the following matrix, acting on $\mathcal{K}^{(3)}$ in the usual way:

$$\begin{pmatrix} D & A & 0 \\ 0 & \tilde{T} & C \\ 0 & 0 & D^* \end{pmatrix} . \tag{9}$$

It is obvious from this definition that conclusions (a) and (b) are valid. Since $\tilde{T}$ is polynomially bounded and Ran $C^* = \text{Ker}D^*$, it follows from Lemma 2.3 that the same is true for the compression $T_1 = \hat{T}_{\left((0)\oplus\mathcal{K}\oplus\mathcal{K}\right)}$ and its adjoint. Furthermore, another application of the same lemma (with ${T_1}^*$ replacing T) gives that $\hat{T}^*$ is polynomially bounded, and hence so is $\hat{T}$. We next show that $\hat{T}$ is invertible. To accomplish this it suffices to exhibit its inverse. Since Ran $\tilde{T}$ is closed and $\tilde{T}$ maps (Ker $\tilde{T})^{\perp}$ injectively onto Ran $\tilde{T}$, there exists a bounded linear operator $Q \in \mathcal{L}(\mathcal{H})$ such that $\tilde{T}Q = P_{\text{Ran }\tilde{T}}$ and $Q\tilde{T} = P_{(\text{Ker }T)^{\perp}}$. It is obvious that in the polar decomposition $D = S(D^*D)^{1/2} = S\Delta$ of D , S is an isometry with final space Ran D , and Δ is invertible. If we define $D_1 = S\Delta^{-1}$, then it is easy to verify that the operator

$$\begin{pmatrix} {D_1}^* & 0 & 0 \\ (1/a^2)A^* & Q & 0 \\ 0 & (1/M^2)C^* & D_1 \end{pmatrix},$$

is the inverse of $\hat{T}$.

Next, we prove that $\sigma(\hat{T}) \subset \mathbb{T}$. First we note that, since $\hat{T}$ is polynomially bounded, $\sigma(\hat{T}) \subset \mathbb{D}^-$. On the other hand, $0 \notin \sigma(\hat{T})$, so it clearly suffices to prove that $\partial\sigma(\hat{T}) \subset \mathbb{T}$. But, as is well known, $\partial\sigma(\hat{T}) \subset \sigma_{\text{ap}}(\hat{T})$, the approximate point spectrum of $\hat{T}$, so it suffices to show that $\sigma_{\text{ap}}(\hat{T}) \subset \mathbb{T}$. Suppose $\lambda_0 \in \sigma_{\text{ap}}(\hat{T})$, and let $\{\tilde{x}_n\}$ be a sequence of unit vectors in $\mathcal{K}^{(3)}$ such that $\|(\hat{T} - \lambda_0)\tilde{x}_n\| \to 0$. Write $\tilde{x}_n = x_n \oplus y_n \oplus z_n$. Then a calculation shows that $(\hat{T} - \lambda_0)\tilde{x}_n = ((D - \lambda_0)x_n +$

$Ay_n\big) \oplus \big((\tilde{T} - \lambda_0)y_n + Cz_n\big) \oplus (D^* - \lambda_0)z_n$. Therefore

$$\|(D - \lambda_0)x_n + Ay_n\| \to 0 \ , \tag{10}$$

$$\|(T - \lambda_0)y_n + Cz_n\| \to 0 \ , \tag{11}$$

$$\|(D^* - \lambda_0)z_n\| \to 0 \ . \tag{12}$$

Now (10) implies that $\|D^*\big((D - \lambda_0)x_n + Ay_n\big)\| \to 0$, and since $D^*A = 0$ by definition of A ,

$$\|D^*Dx_n - \lambda_0 D^*x_n\| \to 0 \ .$$

An easy matricial calculation shows that D^*D is invertible, and hence that

$$\|\big((D^*D)^{-1}D^* - 1/\lambda_0\big)x_n\| \to 0 \ .$$

Thus, if $\|x_n\| \not\to 0$, $1/\lambda_0 \in \sigma\left((D^*D)^{-1}D^*\right)$. But another easy matricial calculation shows that $(D^*D)^{-1}D^*$ is a backward weighted shift of infinite multiplicity with weight sequence $\{1/d_n\}_{n=1}^{\infty}$, and since $d_n \to 1$, one knows (cf. [13, Proposition 15]) that $\sigma\big((D^*D)^{-1}D^*\big) = \mathbb{D}^-$, and hence $|\lambda_0| \geq 1$. Thus either $\|x_n\| \to 0$ or $\lambda_0 \in \mathbb{T}$. In the former case (10) becomes $\|Ay_n\| \to 0$. In this situation write $y_n = {y_n}' \oplus {y_n}''$ relative to the decomposition $\mathcal{K} = \text{Ker } A \oplus \text{Ran } A^*$. Since ${y_n}' \in \text{Ker } A = \text{Ran } \tilde{T}^*$, there exists a sequence $\{v_n\} \in \text{Ran } \tilde{T}$ such that ${y_n}' = \tilde{T}^* v_n$ for all n . On the other hand, it is easy to see that (11) implies that

$$\|\tilde{T}y_n + Cz_n\| - |\lambda_0|\|y_n\| \to 0 \ .$$

Thus

$$\|\tilde{T}y_n + Cz_n\|^2 - |\lambda_0|^2\|y_n\|^2 \to 0 \ ,$$

and since Ran $\tilde{T}$ is orthogonal to Ran C,

$$\|\tilde{T}y_n\|^2 + \|Cz_n\|^2 - |\lambda_0|^2\|y_n\|^2 \to 0 \ .$$

In particular, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for, $n \geq n_0$,

$$\|\tilde{T}y_n\|^2 - |\lambda_0|^2\|y_n\|^2 < \epsilon \ . \tag{13}$$

Note that $\|A{y_n}''\| = \|Ay_n\| \to 0$. But ${y_n}'' \in \text{Ran } A^*$, the initial space of the partial isometry $(1/a)A$, so $\|{y_n}''\| \to 0$. So (13) becomes

$$\|\tilde{T}{y_n}' + \tilde{T}{y_n}''\|^2 - |\lambda_0|^2\|{y_n}' + {y_n}''\|^2 < \epsilon \ , \quad n \geq n_0$$

and since $\|y_n''\| \to 0$, we have that there exists $n_1 \in \mathbb{N}$ such that

$$\|\tilde{T}y_n'\|^2 - |\lambda_0|^2\|y_n'\|^2 < 2\epsilon \quad , \quad n \geq n_1 \quad .$$

Therefore

$$\|\tilde{T}\tilde{T}^* v_n\|^2 - |\lambda_0|^2\|\tilde{T}^* v_n\|^2 < 2\epsilon \quad , \quad n \geq n_1 \quad .$$

Using (8) we have that

$$\left\| \begin{pmatrix} M^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_n \right\|^2 - |\lambda_0|^2 \left\| \begin{pmatrix} T^* & 0 & 0 \\ \sqrt{M^2 - TT^*} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_n \right\|^2 < 2\epsilon \quad , \quad n \geq n_1 \quad .$$

Since $v_n \in \text{Ran } \tilde{T} \subset \mathcal{K} \oplus (0) \oplus (0)$, we can write v_n as $w_n \oplus 0 \oplus 0$. So we obtain that $\|M^2 w_n\|^2 - |\lambda_0|^2 \left\| \begin{pmatrix} T^* w_n \\ \sqrt{M^2 - TT^*} w_n \\ 0 \end{pmatrix} \right\|^2 < 2\epsilon$, i. e. ,

$$(M^4 - |\lambda_0|^2 M^2)\|w_n\|^2 < 2\epsilon \quad , \quad n \geq n_1 \quad .$$

Since ϵ was arbitrary and $M^4 - |\lambda_0|^2 M^2 > 0$, it follows that $\|w_n\| \to 0$ and the same is true of v_n and $y_n' = \tilde{T}^* v_n$. Since y_n'' also tends to 0, we conclude that $\|y_n\| \to 0$ (under the assumption, made earlier, that $\|x_n\| \to 0$).

Next, under this same assumption, write $z_n = z_n' \oplus z_n''$ relative to the decomposition $\mathcal{K} = \text{Ker } C \oplus \text{Ker } D^*$. Then, $\|C z_n''\| = \|C z_n\| \to 0$ by (11). Since $(1/M)C$ is a partial isometry with initial space Ker D^* , it follows that $\|z_n''\| \to 0$. On the other hand, $z_n' \in \text{Ker } C = \text{Ran } D$, so there exists a sequence $\{u_n\} \subset \mathcal{K}$ such that $z_n' = D u_n$. Then (12) gives that $\|D^* D u_n - \lambda_0 D u_n\| \to 0$. As was already observed, $D^* D$ is invertible, so

$$\|(1/\lambda_0) u_n - (D^* D)^{-1} D u_n\| \to 0 \quad .$$

Now an easy computation shows that $(D^* D)^{-1} D$ is a forward unilateral weighted shift of infinite multiplicity with weight sequence $\{1/d_n\}$, and it has been already observed that the spectrum of this operator is equal to $\mathbb{D}^-$. Thus either $\|u_n\| \to 0$ or $\lambda_0 \in \mathbb{T}$. If $\|u_n\| \to 0$, then $\|z_n'\| \to 0$ which, together with the previous conclusions, implies that $\|\tilde{x}_n\| \to 0$. Since $\|\tilde{x}_n\| = 1$, $n \in \mathbb{N}$, this is a contradiction.

and it follows that $\lambda_0 \in \mathbb{T}$. Thus, we have shown that $\sigma(\hat{T}) \subset \mathbb{T}$. To prove the opposite inclusion, we note that by [13, Proposition 15], $\sigma_{\mathrm{ap}}(D) = \mathbb{T}$, and from (9), $\sigma_{\mathrm{ap}}(D) \subset \sigma_{\mathrm{ap}}(\hat{T})$, so $\sigma(\hat{T}) = \mathbb{T}$.

Finally, we show that $\hat{T}$ is weakly centered. A simple calculation shows that

$$\hat{T}\hat{T}^* = \begin{pmatrix} DD^* + AA^* & 0 & 0 \\ 0 & \tilde{T}\tilde{T}^* + CC^* & 0 \\ 0 & 0 & D^*D \end{pmatrix} ,$$

and

$$\hat{T}^*\hat{T} = \begin{pmatrix} D^*D & 0 & 0 \\ 0 & A^*A + \tilde{T}^*\tilde{T} & 0 \\ 0 & 0 & C^*C + DD^* \end{pmatrix} .$$

By definition of D , there exists an infinite dimensional Hilbert space $\mathcal{G}$ and a decomposition $\mathcal{K} = \bigoplus_{n=1}^{\infty} \mathcal{G}_n$, where $\mathcal{G}_n = \mathcal{G}$, $n \in \mathbb{N}$, such that, relative to this decomposition, D has the matrix (7). We observe that relative to this decomposition $DD^* + AA^*$, D^*D , and $C^*C + DD^*$ are diagonal operators with scalar multiples of $1_{\mathcal{G}}$ as diagonal entries, so $DD^* + AA^*$ commutes with D^*D and D^*D commutes with $C^*C + DD^*$. Thus it suffices to show that $\tilde{T}\tilde{T}^* + CC^*$ commutes with $A^*A + \tilde{T}^*\tilde{T}$. Using (8), we obtain that $\tilde{T}\tilde{T}^* + CC^* = \tilde{T}\tilde{T}^* + M^2 P_{\mathrm{Ker}\ \tilde{T}^*} =$

$$= \begin{pmatrix} T & \sqrt{M^2 - TT^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T^* & 0 & 0 \\ \sqrt{M^2 - TT^*} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + M^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$= M^2 I_{\mathcal{H}^{(3)}}$, which obviously commutes with $A^*A + \tilde{T}^*\tilde{T}$. This completes the proof of the proposition. ∎

4. Proofs of the theorems

Proof of Theorem 1.1. Let T be a polynomially bounded operator in $\mathcal{L}(\mathcal{H})$. Then applying propositions 3.1 and 3.2 we obtain the operator $\hat{T}$ that satisfies (a), (c), and (d) of this theorem. Finally, by proposition 3.2 $\mathcal{K}$ is semi-invariant for $\hat{T}$, and by proposition 3.1 $\mathcal{H}$ is invariant for $\tilde{T} = \hat{T}_{\mathcal{K}}$. Therefore, $\mathcal{H}$ is semi-invariant for $\hat{T}$. ∎

Proof of Theorem 1.2. If every polynomially bounded operator in $\mathcal{L}(\mathcal{H})$ is similar to a contraction, then the same is true, in particular, for every weakly centered polynomially bounded operator whose spectrum is the unit circle. To prove the nontrivial implication, let T be a polynomially bounded operator in $\mathcal{L}(\mathcal{H})$. By Proposition 3.1 and Lemma 2.2 there exists a certain polynomially bounded operator $\tilde{T}$ in $\mathcal{L}(\mathcal{H}\oplus\mathcal{H})$ with closed range such that if $\tilde{T}$ is similar to a contraction, the same is true for T . Thus, we can apply Proposition 3.2 and conclude that there exists a weakly centered polynomially bounded operator $\hat{T}$, whose spectrum is the unit circle, such that $\tilde{T}$ is a compression of $\hat{T}$ to a semi-invariant subspace. By hypothesis, $\hat{T}$ is similar to a contraction, so Lemma 2.2 implies that the same is true for $\tilde{T}$ and, hence, for T . ∎

NOTE. This paper is part of the author's Ph. D. thesis, written at the University of Michigan under the direction of Professor Carl Pearcy. The author wishes to acknowledge partial support by the National Science Foundation and Texas A. & M. University during the period when this paper was being written.

BIBLIOGRAPHY

1. BROWN A. , On a class of operators , *Proc. Amer. Math. Soc.* **4** (1953) , 723–728.

2. BROWN A. , The unitary equivalence of binormal operators , *Amer. J. Math.* **76** (1954) , 414–434.

3. CAMPBELL S. L. , Linear operators for which T^*T and TT^* commute , *Proc. Amer. Math. Soc.* **34** (1972) , 177–180.

4. CAMPBELL S. L. , Linear operators for which T^*T and TT^* commute (II) , *Pacific J. Math.* **53** (1974) , 355–361.

5. EMBRY M. R. , Similarities involving normal operators on Hilbert space , *Pacific J. Math.* **35** (1970) , 331–336.

6. FOGUEL S. R. , A counterexample to a problem of Sz.-Nagy , *Proc. Amer. Math. Soc.* **15** (1964) , 788–790.

7. HALMOS P. R. , On Foguel's answer to Nagy's question , *Proc. Amer. Math. Soc.* **15** (1964) , 791–793.

8. HALMOS P. R. , Ten problems in Hilbert space , *Bull. Amer. Math. Soc.* **76**

(1970) , 887–933.

9. HOLBROOK J. A. R. , Spectral dilations and polynomially bounded operators , *Indiana Univ. Math. J.* **20** (1971) , 1027–1034.

10. LEBOW A. , On von Neumann's theory of spectral sets , *J. Math. Anal. and Appl.* **7** (1963) , 64–90.

11. MLAK W. , Algebraic polynomially bounded operators , *Ann. Pol. Math.* **29** (1974) , 133–139.

12. PAULSEN V. I. , Every completely polynomially bounded operator is similar to a contraction , *J. Funct. Anal.* **55** (1984) , 1–17.

13. SHIELDS A. L. , Weighted shift operators and analytic function theory , *Topics in Operator Theory* , Mathematical Surveys **13** (1974) , 49–128.

14. SZ.-NAGY B. , Sur les contractions de l'espace de Hilbert , *Acta Sci. Math.* **15** (1953) , 87–92.

15. SZ.-NAGY B. and FOIAS C. , *Harmonic analysis of operators on Hilbert space* , American Elsevier , New York , 1970

16. TITCHMARSH E. C. , *The theory of functions* , Oxford University Press, Glasgow , 1949.

Department of Mathematics University of Michigan Ann Arbor, MI 48109

Department of Mathematics Texas A. & M. University College Station, TX 77843