POLYNOMIALLY UNBOUNDED PRODUCT OF TWO POLYNOMIALLY BOUNDED OPERATORS

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In this paper we study the operator of Pisier

$$T = \begin{pmatrix} S^* & Y_\alpha \\ 0 & S \end{pmatrix}$$

where S is a unilateral forward shift of infinite multiplicity, S^* is its adjoint, and Y_α is a matrix $(\alpha_{i+j} C_{i+j})$, with $\{\alpha_k\}_{k\in\mathbb{N}}$ a sequence of complex numbers, and $\{C_k\}_{k\in\mathbb{N}}$ a family of operators satisfying the "canonical anticommutation relations". We exhibit a sequence $\{\alpha_k\}$ such that T is polynomially bounded but $T \otimes T$ is not. This shows that the product of two commuting polynomially bounded operators need not be polynomially bounded itself.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex, Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* (notation: $T \in$ (PB)$(\mathcal{H})$ = (PB)) if there exists an $M \geq 1$ such that

$$\|p(T)\| \leq M \sup\{|p(\zeta)| : |\zeta| = 1\} \tag{1}$$

for every polynomial p.

Since, by von Neumann's inequality, contractions satisfy (1) with $M = 1$ polynomially bounded operators can be regarded as a generalization of the former class. Therefore, it is natural to ask what properties of contraction operators carry over. In particular, one may ask whether the product of two polynomially bounded operators is polynomially bounded itself.

It is easy to see that, in general, $T_1, T_2 \in$ (PB) does not imply that $T_1 T_2 \in$ (PB). Indeed, it suffices to take a noncontractive polynomially bounded operator T and to notice that in this situation T^* is also polynomially bounded, but T^*T cannot belong to (PB) because polynomially bounded normal operators *are* contractions. (Normal operators have the norm equal to the spectral radius, and the spectrum of an operator in (PB) is always contained in the closed unit disk.)

Naturally, one wonders whether the addition of commutativity might force the product to be polynomially bounded. In other words, if T_1, T_2 is a pair of commuting operators in (PB), is T_1T_2 in (PB)? A related question is whether $T_1, T_2 \in$ (PB) implies $T_1 \otimes T_2 \in$ (PB). Here, $T_1 \otimes T_2$ denotes the bounded linear operator acting in a natural way on the Hilbert space completion of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is *similar to a contraction* (notation: $T \in$ (SC)) if there exists an invertible operator S in $\mathcal{L}(\mathcal{H})$ such that $\|S^{-1}TS\| \leq 1$. Clearly, if either of the operators T_1, T_2 is similar to a contraction, then $T_1 \otimes T_2$ is polynomially bounded by [4, Theorem 2.3]. It follows that both T_1 and T_2 should be polynomially bounded but not similar to contractions. Whether such operators existed or not was a long standing open problem. Recently, Pisier solved it by constructing an operator $T \in$ (PB) $\setminus$ (SC) (cf. [6]), and in view of the previous remarks, we consider $T \otimes T$.

THEOREM 1. *There exists a polynomially bounded operator T such that $T \otimes T$ is not polynomially bounded.*

By considering $T \otimes I$ and $I \otimes T$ we obtain

COROLLARY 2. *The product of two commuting polynomially bounded operators need not be polynomially bounded.*

NOTE. The author wishes to express his gratitude to professors Hari Bercovici and Michael Lacey for many helpful conversations, to the Indiana University for its support during the period when this paper was being written, as well as to professors Pisier, Davidson, and Paulsen for the preprints of their papers.

Before we prove the main results of this paper, we need to establish some notation and terminology, as well as some facts that will be used later. We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of nonnegative integers. If $n \in \mathbb{N} \cup \{\aleph_0\}$ and $\mathcal{K}$ is any complex Hilbert space, we write $\mathcal{K}^{(n)}$ for the (orthogonal) direct sum of n copies of $\mathcal{K}$. As usual, ℓ^p will denote the Banach space of all complex sequences $\{\alpha_k\}_{k\in\mathbb{N}_0}$ such that $\sum_{k\in\mathbb{N}_0} |\alpha_k|^p < \infty$. If the operators A and B are unitary equivalent we will use the notation $A \cong B$. Finally, $A(\mathbb{D})$ is the disk algebra — the class of all functions analytic in the open unit disk $\mathbb{D}$ and continuous on its boundary. It is well known that if $T \in$ (PB) then (1) holds for every p in $A(\mathbb{D})$.

We say that a sequence $\{C_k\}_{k\in\mathbb{N}_0} \subset \mathcal{L}(\mathcal{H})$ satisfies the "Canonical Anticommutation Relations" (CAR) if

$$C_iC_j + C_jC_i = 0$$
$$C_iC_j{}^* + C_j{}^*C_i = \delta_{ij}I.$$

Such operators possess some remarkable properties and we are going to state a few that are going to be of use in the remaining portion of the paper. For their proofs we direct the reader to [5].

LEMMA 3. *If* $\{\gamma_k\}_{k\in\mathbb{N}} \in \ell^2$ *then*

$$\|\sum_{k\in\mathbb{N}} \gamma_k C_k\|^2 = \sum_{k\in\mathbb{N}} |\gamma_k|^2. \tag{4}$$

LEMMA 4. *If* $\{\gamma_k\}_{k\in\mathbb{N}} \in \ell^1$ *then*

$$\frac{1}{2}\sum_{k\in\mathbb{N}} |\gamma_k| \leq \|\sum_{k\in\mathbb{N}} \gamma_k C_k \otimes C_k\| \leq \sum_{k\in\mathbb{N}} |\gamma_k|. \tag{5}$$

The following example comes from [6]. Let $\mathcal{K} = \mathcal{H}^{(\aleph_0)}$, a countably infinite direct sum of copies of $\mathcal{H}$, let S be a unilateral, forward shift of infinite multiplicity in $\mathcal{L}(\mathcal{K})$, and let S^* be its adjoint. We define $Y_\alpha \in \mathcal{L}(\mathcal{K})$ to be the operator matrix $(\, \alpha_{i+j} C_{i+j}\,)_{i,j\geq 0}$, relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H} \oplus \ldots$, where $\{\alpha_k\}_{k\in\mathbb{N}}$ is a sequence of complex numbers, and $\{C_k\}_{k\in\mathbb{N}}$ is a sequence of operators in $\mathcal{L}(\mathcal{H})$ satisfying (CAR). Now we define the operator T acting on $\mathcal{K} \oplus \mathcal{K}$ as

$$T = \begin{pmatrix} S^* & Y_\alpha \\ 0 & S \end{pmatrix}. \tag{6}$$

As we have mentioned before, Pisier has shown that if

$$\alpha_n = \begin{cases} 2^{-k} & \text{if } n = 2^k \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

then the operator (6) is polynomially bounded but not similar to a contraction. Davidson and Paulsen have improved this result by showing that the same conclusion is obtained whenever the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the conditions

$$\sup_k k^2 \sum_{i\geq k} |\alpha_i|^2 < \infty \tag{8}$$

and

$$\sum_k k^2 |\alpha_k|^2 = \infty. \tag{9}$$

Furthermore, they have shown that T is polynomially bounded if and only if (8) holds, and that (9) implies that T is not similar to a contraction.

PROOF OF THEOREM 1: Let T be the operator (6) with α_n as in (7). Since an easy computation shows that condition (8) is satisfied, it follows that T is polynomially bounded. We will show that $T \otimes T$ is not in (PB). In order to accomplish this we will consider $p(T \otimes T)$ where $p(z) = \sum a_k z^k$ is in the disk algebra. Then

$$p(T \otimes T) = \sum a_k T^k \otimes T^k = \sum a_k \begin{pmatrix} {S^*}^k & k{S^*}^{k-1}Y_\alpha \\ 0 & S^k \end{pmatrix} \otimes T^k$$

and thus

$$p(T \otimes T) \cong \begin{pmatrix} \sum a_k {S^*}^k \otimes T^k & \sum k a_k {S^*}^{k-1} Y_\alpha \otimes T^k \\ 0 & \sum a_k S^k \otimes T^k \end{pmatrix}. \tag{10}$$

Since diagonal entries in the last matrix are $p(S^* \otimes T)$ [resp., $p(S \otimes T)$] and since $T \in$(PB), it follows from [3, Theorem 2.3] that these entries are bounded. So we concentrate on the off-diagonal entry in (10), $F_1 = \sum k a_k {S^*}^{k-1} Y_\alpha \otimes T^k$, and we will show that it is not bounded. Clearly,

$$\begin{aligned} F_1 &\cong \sum k a_k Y_\alpha S^{k-1} \otimes \begin{pmatrix} {S^*}^k & k{S^*}^{k-1}Y_\alpha \\ 0 & S^k \end{pmatrix} \\ &\cong \begin{pmatrix} Y_\alpha \otimes I \sum k a_k S^{k-1} \otimes {S^*}^k & Y_\alpha \otimes I \sum k^2 a_k S^{k-1} \otimes {S^*}^{k-1} Y_\alpha \\ 0 & Y_\alpha \otimes I \sum k a_k S^{k-1} \otimes S^k \end{pmatrix}. \end{aligned}$$

We see that the diagonal entries are $Y_\alpha \otimes S^* p'(S \otimes S^*)$ [resp., $Y_\alpha \otimes S p'(S \otimes S)$]. A calculation shows that they are exactly the off-diagonal entries in $p(T \otimes S^*)$ [resp., $p(T \otimes S)$] and in view of the polynomial boundedness of T, via [4] they are both bounded. Therefore, we concentrate on $F_2 = \sum k^2 a_k {S^*}^{k-1} Y_\alpha \otimes {S^*}^{k-1} Y_\alpha$. By the definition of Y_α

$$\begin{aligned} {S^*}^{k-1} Y_\alpha \otimes {S^*}^{k-1} Y_\alpha &= (\, \alpha_{k+i+j-1} C_{k+i+j-1} \,) \otimes (\, \alpha_{k+m+n-1} C_{k+m+n-1} \,) \\ &\cong (\, (\, \alpha_{k+i+j-1} C_{k+i+j-1} \otimes \alpha_{k+m+n-1} C_{k+m+n-1} \,)_{i,j \geq 0} \,)_{m,n \geq 0} \,. \end{aligned}$$

The norm of any matrix certainly dominates the norm of its $(0,0)$ entry, so

$$\|F_2\| \geq \| \sum_k k^2 a_k \,(\, \alpha_{k+i+j-1} C_{k+i+j-1} \otimes \alpha_{k-1} C_{k-1} \,)_{i,j \geq 0} \|.$$

Applying the same estimate once again we obtain that

$$\|F_2\| \geq \| \sum_k k^2 a_k \alpha_{k-1} C_{k-1} \otimes \alpha_{k-1} C_{k-1} \|.$$

Using (5) we get

$$\|F_2\| \geq \sum_k k^2 |a_k| \, |\alpha_{k-1}|^2. \tag{11}$$

If we denote the expression on the right side of (11) as $F(p)$, it suffices to show that $\sup\{F(p) : p \in A(\mathbb{D}), \|p\|_\infty = 1\} = \infty$. A remarkable result of Kislyakov ([2]) states that if $\{b_k\}_{k\in\mathbb{N}_0}$ is in ℓ^2 then there exists a function $p \in A(\mathbb{D})$ such that $p(z) = \sum a_k z^k$, $|b_k| \le |a_k|$ for every $k \in \mathbb{N}_0$, and $\|p\|_\infty \le K\|\{b_k\}_{k\in\mathbb{N}_0}\|_{\ell^2}$ where K is an absolute constant. Clearly, the supremum above dominates the supremum taken over functions in the disk algebra obtained this way, and thus

$$\begin{aligned}\sup\{F(p) : p \in A(\mathbb{D}), \|p\|_\infty = 1\} &\ge \sup\{\sum_k k^2|\alpha_{k-1}|^2\,|b_k| : \|\{b_k\}_{k\in\mathbb{N}_0}\|\ell^2 \le \frac{1}{K}\}\\ &= \frac{1}{K}\|\{k^2|\alpha_{k-1}|^2\}_{k\in\mathbb{N}}\|^2_{\ell^2}\\ &= \frac{1}{K}\sum_k k^4|\alpha_{k-1}|^4.\end{aligned}$$

From definition of the sequence $\{\alpha_k\}$ it follows easily that the last series is divergent. This completes the proof. $\square$

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1991 Mathematics Subject Classification. Primary 47A20; Secondary 47B35.