AN EXTREMAL PROBLEM IN INTERPOLATION THEORY

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Abstract. If $z_1, z_2, \dots, z_n$ are complex numbers in the open unit disk $\mathbb{D}$ and $A_1, A_2, \dots, A_n$ are $N \times N$ matrices, let $\mathcal{F}$ denote the family of analytic functions, bounded in $\mathbb{D}$, such that for each $F \in \mathcal{F}$, $F(z_k) = A_k$, $k = 1, 2, \dots, n$. We consider $\rho = \inf_{F \in \mathcal{F}} \sup_{z \in \mathbb{D}} |F(z)|_{\mathrm{sp}}$ where $|\cdot|_{\mathrm{sp}}$ denotes the spectral radius. H. Bercovici has raised the question whether this infimum is attained. We will show that the answer is affirmative for $N \leq 3$, and we point out at the obstructions to generalize this result to the case $N > 3$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. If $f : \mathbb{D} \to \mathbb{C}$ we use the notation $\|f\| = \sup\{|f(z)| : z \in \mathbb{D}\}$, and $\mathcal{B} = \{f : \mathbb{D} \to \mathbb{C} : \|f\| \leq 1\}$. Let n be a positive integer, and let $z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_n$ be complex numbers in $\mathbb{D}$ such that $z_i \neq z_j$ if $i \neq j$. The classical Nevanlinna–Pick interpolation problem asks for conditions under which there exists a function $f \in \mathcal{B}$ such that $f(z_k) = w_k$, $k = 1, 2, \dots, n$. In a more general form, one can replace the complex numbers $w_1, w_2, \dots, w_n$ by $N \times N$ matrices, for an arbitrary positive integer N, or even operators on infinite dimensional Hilbert space (see e. g. [6]). Also, the class $\mathcal{B}$ can be replaced by various classes of functions. Some of the commonly asked questions in these circumstances are whether a solution exists and, if it does, one can further investigate whether the solution is unique. Finally, in the absence of uniqueness, one can search for a solution possessing some extremal properties.

In this paper we consider the case when $w_1, w_2, \dots, w_n$ are $N \times N$ complex matrices $A_1, A_2, \dots, A_n$ for an arbitrary positive integer N, and the class of functions that are analytic and bounded in the open unit disk $\mathbb{D}$ with values in M_N — the algebra of complex $N \times N$ matrices. Let $\mathcal{F} = \mathcal{F}(z_1, z_2, \dots, z_N; A_1, A_2, \dots, A_N)$ consist of all such functions that, in addition, satisfy $F(z_k) = A_k$, $k = 1, 2, \dots, N$. For each function $F \in \mathcal{F}$ and each $z \in \mathbb{D}$ it is of interest to evaluate the spectral

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radius of $F(z)$ (notation: $|F(z)|_{\mathrm{sp}}$) and then to define

$$\rho = \inf_{F\in\mathcal{F}} \sup_{z\in\mathbb{D}} |F(z)|_{\mathrm{sp}}.$$

In [1] Hari Bercovici has posed the following extremal problem:

Problem 1.1. Let n and N be positive integers, let $z_1, z_2, \ldots, z_n \in \mathbb{D}$, and let $A_1, A_2, \ldots, A_n$ be $N\times N$ matrices. Let $\mathcal{F}$ denote the set of functions $F : \mathbb{C} \to M_N$ that are analytic and bounded in $\mathbb{D}$, and such that for every $F \in \mathcal{F}$ and $1 \le k \le n$, $F(z_k) = A_k$. Let

$$\rho = \inf_{F\in\mathcal{F}} \sup_{z\in\mathbb{D}} |F(z)|_{\mathrm{sp}}.$$

Is there a function $F \in \mathcal{F}$ s.t. $\rho = \sup_{z\in\mathbb{D}} |F(z)|_{\mathrm{sp}}$?

It is the purpose of this note to show that the answer is affirmative in the cases $N = 2$ and $N = 3$. Thus, our main results are:

Theorem 1.2. *Problem 1.1 has a solution $F \in \mathcal{F}$ in the case $N = 2$.*

Theorem 1.3. *Problem 1.1 has a solution $F \in \mathcal{F}$ in the case $N = 3$.*

The reason that we have stated these assertions separately lies in the fact that the proofs that we will give are substantially different. In particular, we were unable to use the result of Theorem 1.2 in the proof of Theorem 1.3. It appears that the approach used in the proof of the latter is more general and that it could have been employed in the proof of the former. Yet, since either method may turn out to be fruitful in investigating the case $N > 3$ we were inclined to present both.

Before that, however, we need to introduce some notation and terminology. This will be done in Section 2, together with several reductions which will show that in order to prove Theorem 1.2 and Theorem 1.3 it suffices to prove their special cases (Theorem 2.1 and Theorem 2.2 below). It turns out that proofs of the latter results rely heavily upon some factorization properties of interpolating functions. We will explore these and establish the needed facts in Section 3. In Section 4 we will prove Theorem 2.1 and thus complete the proof of Theorem 1.2. Finally, in Section 5 we will prove Theorem 2.2 and thus complete the proof of Theorem 1.3.

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a number of helpful sugestions. In particular, the proof of Lemma 2.4 is much shorter and more elegant than in its original form.

2. Preliminaries

As usual, $\mathbb{N}$ is the set of positive integers. When $N \in \mathbb{N}$ and A is an $N \times N$ matrix we will use $tr(A)$ to denote the trace of A. It is well known that $tr(A)$ is the sum of the diagonal elements of A. Also, $\Delta(A)$ will denote the determinant of A. The characteristic polynomial of A is $p_A(\lambda) = \Delta(A - \lambda I)$, where I is the identity $N \times N$ matrix. When $N = 3$, the characteristic polynomial can be written as

$$p_A(\lambda) = \lambda^3 - tr(A)\lambda^2 + \delta(A)\lambda - \Delta(A)$$

where $\delta(A)$ is the sum of all 2×2 principal minors of A. We will say that two $N \times N$ matrices A and B are similar if there exists an invertible $N \times N$ matrix X such that $B = XAX^{-1}$. Each $N \times N$ matrix A can be viewed as a linear transformation on N-dimensional Hilbert space and, consequently, we will use $\|A\|$ to denote the norm of this linear transformation.

As we have mentioned, for any bounded function f on $\mathbb{D}$, $\|f\|$ denotes the standard norm on the Hardy space H^∞, namely $\|f\| = \sup\{|f(z)| : z \in \mathbb{D}\}$. Since we plan to use no other norm, we will omit the usual subscript (as in $\|f\|_\infty$). One knows that functions in H^∞ possess the factorization property: if $f \in H^\infty$ then $f = B_f e^g$, where B_f is a Blaschke product. When $z_1, z_2, \ldots, z_n$ are in $\mathbb{D}$ we will write

$$B(z_1, z_2, \ldots, z_n) = \prod_{k=1}^{n} \frac{z - z_k}{1 - z\overline{z_k}}$$

or simply B when it is clear which complex numbers $z_1, z_2, \ldots, z_n$ are used. Finally, as is customary, for a complex number $z = a + ib$, where a and b are real numbers, we use the notation $a = \mathrm{Re} z$, $b = \mathrm{Im} z$.

Although Theorem 1.2 and Theorem 1.3 are stated for arbitrary $N \times N$ matrices, we will show that there is no loss of generality in assuming that they possess some additional properties.

Theorem 2.1. *Let n be a positive integer, and let $z_1, z_2, \ldots, z_n \in \mathbb{D}$. Suppose that $A_1, A_2, \ldots, A_n$ are 2×2 matrices such that for each k, $1 \leq k \leq n$, the entries of A_k on the main diagonal are equal and such that if the $(1, 2)$ entry of A_k is 0 then so is the $(2, 1)$ entry. Let $\mathcal{F}$ denote the set of functions $F : \mathbb{C} \to M_2$ that are analytic and bounded in $\mathbb{D}$, and such that for every $F \in \mathcal{F}$ and $1 \leq k \leq n$,*

$F(z_k) = A_k$. Let

$$\rho = \inf_{F\in\mathcal{F}} \sup_{z\in\mathbb{D}} |F(z)|_{sp}$$

and suppose that $\rho \le 1$. Then there is a function $F \in \mathcal{F}$ s.t. $\rho = \sup_{z\in\mathbb{D}} |F(z)|_{sp}$.

In order to formulate a similar result in the case $N = 3$, we introduce the following concept.

Definition 1. An $N \times N$ matrix $A = \big(a_{ij}\big)_{i,j=1}^{N}$ is said to be in a *modified Jordan form* if $a_{ij} = 0$ when $j \ne i$ or $i + 1$, and if $a_{12} \ne 0$ whenever A is not a scalar multiple of the identity.

The significance of this form comes from the following result.

Theorem 2.2. *Let n be a positive integer, and let $z_1, z_2, \dots, z_n \in \mathbb{D}$. Suppose that $A_1, A_2, \dots, A_n$ are 3×3 matrices such that for each k, $1 \le k \le n$, A_k is in a modified Jordan form. Let $\mathcal{F}$ denote the set of functions $F : \mathbb{C} \to M_3$ that are analytic and bounded in $\mathbb{D}$, and such that for every $F \in \mathcal{F}$ and $1 \le k \le n$, $F(z_k) = A_k$. Let*

$$\rho = \inf_{F\in\mathcal{F}} \sup_{z\in\mathbb{D}} |F(z)|_{sp}$$

and suppose that $\rho \le 1$. Then there is a function $F \in \mathcal{F}$ s.t. $\rho = \sup_{z\in\mathbb{D}} |F(z)|_{sp}$.

We will now prove several easy lemmas which will show that Theorem 2.1 implies Theorem 1.2 and that Theorem 2.2 implies Theorem 1.3. First we will establish that each matrix A_k, $1 \le k \le n$, can be replaced by a matrix similar to it. Actually, we can make this reduction regardless of the size of these matrices.

Lemma 2.3. *Let Problem $\widetilde{1.1}$ denote Problem 1.1 where matrices $A_1, \dots, A_n$ are replaced by, respectively, similar matrices $\tilde{A}_1, \dots, \tilde{A}_n$. Then Problem 1.1 has a solution iff Problem $\widetilde{1.1}$ does.*

Proof. Let $\{X_k\}_{k=1}^n$ be invertible $N \times N$ matrices and denote $\tilde{A}_k = X_k A_k X_k^{-1}$, $k = 1, 2, \dots, n$. One knows, (cf. [2]) that every invertible matrix has a logarithm. In other words, for $k = 1, 2, \dots, n$, there is a matrix Y_k such that $X_k = e^{Y_k}$ (and, hence, $X_k^{-1} = e^{-Y_k}$). Let G be an analytic function that interpolates the sequence $\{Y_k\}$ i. e., such that $G(z_k) = Y_k$, $1 \le k \le n$. Furthermore, let $\tilde{F} = e^G F e^{-G}$. Then F is an analytic function satisfying $F(z_k) = A_k$, $k = 1, 2, \dots, n$, if and only if $\tilde{F}$ is an analytic function satisfying $\tilde{F}(z_k) = \tilde{A}_k$, $k = 1, 2, \dots, n$. In addition, $|F(z)|_{\mathrm{sp}} = |\tilde{F}(z)|_{\mathrm{sp}}$ for all $z \in \mathbb{D}$, so ρ is the same in Problem 1.1 and Problem

$\widetilde{1.1}$. We conclude that F is a solution of Problem 1.1 if and only if $\tilde{F}$ is a solution of Problem $\widetilde{1.1}$. $\square$

Our next result deals with the fact that matrices in Theorem 2.1 are of very special form (constant main diagonal).

Lemma 2.4. *Let A be an $N \times N$ matrix with trace t. Then A is similar to a matrix $\tilde{A}$ which has every entry on the main diagonal equal to t/N. Furthermore, when $N = 2$, $\tilde{A} = \left(\tilde{a}_{ij}\right)_{i,j=1}^{2}$ can be selected in such a way that either $\tilde{a}_{12} \neq 0$ or both $\tilde{a}_{12} = \tilde{a}_{21} = 0$.*

Proof. First we consider the case $N = 2$. The Jordan form of a 2×2 matrix comes in one of the three types:

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha \neq \beta.$$

Moreover,

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\alpha+\beta}{2} & \dfrac{\alpha-\beta}{2} \\ \dfrac{\alpha-\beta}{2} & \dfrac{\alpha+\beta}{2} \end{pmatrix}.$$

Since every matrix is similar to its Jordan form this completes the proof when $N = 2$. An inspection of possible forms verifies the last assertion of the lemma.

When $N \geq 3$ the result is an easy consequence of the Toeplitz-Hausdorff theorem (cf. [3, Problem 210]). Indeed, let $e_1, e_2, \ldots, e_N$ be an orthonormal basis of $\mathbb{C}^N$ and let $X \in M_N$ satisfy $tr(X) = 0$. Then $\sum_{k=1}^{N} (Xe_k, e_k) = 0$ and the convexity of the numerical range implies that there exists a unit vector $f_1 \in \mathbb{C}^N$ such that $(Xf_1, f_1) = 0$. Denote by $f_{12}, f_{13}, \ldots, f_{1N}$ an orthonormal basis of $\mathcal{H}_1^{\perp}$, the orthogonal complement of the one-dimensional subspace spanned by f_1. Then $\sum_{k=2}^{N} (Xf_{1k}, f_{1k}) = 0$ so the compression of A to $\mathcal{H}_1^{\perp}$ has zero trace and, consequently, there is a unit vecor $f_2 \in \mathcal{H}_1^{\perp}$ satisfying $(Xf_2, f_2) = 0$. Continuing this process we obtain an ortonormal basis $f_1, f_2, \ldots, f_N$ for $\mathbb{C}^N$ in which the matrix for X has zeros on the main diagonal. Applying this observation to $X = A - t/N$ completes the proof. $\square$

Our next reduction is directed at ρ. Without loss of generality we can assume that $\rho \leq 1$. Indeed, it is easy to see that, if M is an arbitrary positive number, $\tilde{A}_k = A_k/M$, $k = 1, 2, \ldots, n$, and $\tilde{F} = F/M$, then F is an analytic function such that $F(z_k) = A_k$ if and only if $\tilde{F}$ is an analytic function satisfying $\tilde{F}(z_k) = \tilde{A}_k$, $k = 1, 2, \dots, n$. Using the natural notation

$$\tilde{\rho} = \inf_{\tilde{F}\in\mathcal{F}} \sup_{z\in\mathbb{D}} |\tilde{F}(z)|_{\mathrm{sp}}.$$

it is clear that $\tilde{\rho} = \rho/M$ and that each of the two extremal problems has a solution if and only if the other one does. Therefore, there is no loss of generality in assuming that $\rho \leq 1$. This shows that Theorem 2.1 implies Theorem 1.2.

Now we turn our attention to the form introduced in Definition 1. Our next result justifies the word "canonical".

Lemma 2.5. *Let A be an $N \times N$ matrix. Then A is similar to a matrix $\tilde{A}$ which is in the modified Jordan form.*

Proof. Clearly, A is similar to a matrix A' which is in Jordan form. Therefore, A' is a direct sum of Jordan blocks. If at least one of these blocks has a nonzero off-diagonal entry a permutation of blocks leads to the desired form. If each block is a diagonal matrix, and if A' is not a scalar multiple of identity, then there is a permutation X such that $XA'X^{-1} = \hat{A} = \big(\hat{a}_{ij}\big)$ satisfies $\hat{a}_{11} \neq \hat{a}_{22}$. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \hat{a}_{11} & 0 \\ 0 & \hat{a}_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{a}_{11} & \hat{a}_{22} - \hat{a}_{11} \\ 0 & \hat{a}_{22} \end{pmatrix},$$

and since $\hat{A}$ can be written as a direct sum

$$\hat{A} = \begin{pmatrix} \hat{a}_{11} & 0 \\ 0 & \hat{a}_{22} \end{pmatrix} \oplus \hat{A}'$$

we can take

$$Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I.$$

Then $\tilde{A} = Y\hat{A}Y^{-1}$ is in the modified Jordan form. □

Finally, we notice that Lemma 2.5, together with Lemma 2.3 and the remark following the proof of Lemma 2.4, shows that Theorem 2.2 implies Theorem 1.3.

3. Factoring Interpolating Functions

In this section we are interested in the following problem. Let $\{z_1, z_2, \dots, z_n\}$ and $\{w_1, w_2, \dots, w_n\}$ be subsets of $\mathbb{D}$ and let G be a bounded analytic function on $\mathbb{D}$ that is interpolating for these sets i. e., let $G(z_k) = w_k$, $k = 1, 2, \dots, n$. If each member of the latter set is written as a product of two complex numbers, say $w_k = a_k b_k$, $k = 1, 2, \ldots, n$, does it follow that G can be written as a product of two functions u and v, bounded and analytic on $\mathbb{D}$, such that $u(z_k) = a_k$ and $v(z_k) = b_k$, $k = 1, 2, \ldots, n$? The following example shows that the answer is negative.

Example 3.1. Let $G(z) = z$, $n = 1$, $z_1 = 0$, $w_1 = 0$, $a_1 = 0$, and $b_1 = 0$. The conditions on u and v mean that $u(0) = 0$ and $v(0) = 0$. Thus, $u(z) = zu_0(z)$ and $v(z) = zv_0(z)$ for some bounded analytic functions u_0 and v_0. If the factorization $G(z) = u(z)v(z)$ were valid for all $z \in \mathbb{D}$ then it would follow that $z = zu_0(z) \cdot zv_0(z)$ so that $u_0(z)v_0(z) = 1/z$ contradicting the boundedness of u_0 and v_0.

It turns out that the obstruction featuring in Example 3.1 is the only one, and we will shortly demonstrate (Proposition 3.3 below) that without it the problem posed at the beginning of this section has an affirmative answer. In view of this one can ask whether, assuming that $a_k, b_k \in \mathbb{D}$, $k = 1, 2, \ldots, n$ and that $\|G\| \le 1$, it is possible to select functions u and v in such a way that $\|u\|, \|v\| \le M$, where M is a constant independent of G. Unfortunately, this is not possible in the general case as the following example shows.

Example 3.2. Let $m \in \mathbb{N}$, $G(z) = z^m$, $n = 2$, $z_1 = 0$, $z_2 = 1/2$ (so that $w_1 = 0$, $w_2 = 1/2^m$). Let $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1/2^m$. Since $v(0) = 1$ it follows that $u(z) = z^m u_0(z)$ where $u_0 \in H^\infty$. On the other hand, $u(1/2) = 1$ so $u_0(1/2) = 2^m$. Therefore, $\|u_0\| \ge 2^m$ and, hence, $\|u\| \ge 2^m$. Yet, $\|G\| = 1$.

An analysis of Example 3.2 reveals that this obstacle can be avoided if $u(z)$ can "share" the zeros of G with $v(z)$. The following result summarizes the ideas of this section.

Proposition 3.3. *Let $n \in \mathbb{N}$, and let $w_1, w_2, \ldots, w_n \in \mathbb{D}$. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers in $\mathbb{D}$ satisfying $w_k = a_k b_k$, with the property that $a_k = 0 \Rightarrow b_k = 0$, $k = 1, 2, \ldots, n$. Suppose that $z_1, z_2, \ldots, z_n \in \mathbb{D}$, and let G be a bounded analytic function on $\mathbb{D}$ such that $G(z_k) = w_k$, $k = 1, 2, \ldots, n$, and such that if for some k, $1 \le k \le n$, $a_k = b_k = 0$ then G has at z_k a zero of multiplicity no less than two. Then there exist bounded analytic functions u and v on $\mathbb{D}$ such that $G = uv$, $u(z_k) = a_k$, and $v(z_k) = b_k$, $k = 1, 2, \ldots, n$. Moreover,* *if*

$$\begin{aligned}
&\alpha = \min\{|z_i - z_j| : i, j = 1, 2, \ldots, n,\ i \neq j\}\\
&\eta = \min\{\{|a_k| : a_k \neq 0\} \cup \{1\}\}\\
&M_0 = \pi + \max\{|\ln \eta|, |\ln(2/\alpha)^n|\}\\
&M = 3e^{M_0 n (2/\alpha)^{n-1}} \max\{\|G\|, 1\}
\end{aligned} \tag{1}$$

then u, v can be chosen so that $\|u\| \leq M/3$, $\|v\| \leq M/3$. In addition, the function u can be chosen in such a way that if $u = B_u e^g$, where B_u is a Blaschke product and $g \in H^\infty$, then $\|g\| \leq M_0 n (2/\alpha)^{n-1}$.

In order to prove this assertion we need the following lemma.

Lemma 3.4. *Let $n \in \mathbb{N}$. If $z_1, z_2, \ldots, z_n \in \mathbb{D}$, and $w_1, w_2, \ldots, w_n$ are complex numbers such that $|w_k| \leq M$, $k = 1, 2, \ldots, n$, then there exists a function $f : \mathbb{D} \to \mathbb{C}$ possesing the following properties:*

- (i) *f is bounded and analytic in $\mathbb{D}$,*
- (ii) *$f(z_k) = w_k$, $k = 1, 2, \ldots, n$,*
- (iii) *$\|f\| \leq Mn(2/\alpha)^{n-1}$, where $\alpha = \min\{|z_i - z_j| : i \neq j\}$.*

Proof. Let $\hat{B}_i(z)$ be the Blaschke product that has a simple zero at all of $\{z_k\}_{k=1}^n$ except at z_i i. e., $\hat{B}_i(z) = \prod_{j \neq i} \frac{z - z_j}{1 - z\overline{z_j}}$, $i = 1, 2, \ldots, n$. Let $f_i(z) = \hat{B}_i(z) \cdot \frac{w_i}{\hat{B}_i(z_i)}$, and define $f(z) = \sum_{i=1}^n f_i(z)$. Conclusions (i) and (ii) follow immediately. To show (iii) we notice that

$$\|f\| \leq \sum_{i=1}^n \|f_i\| \leq \sum_{i=1}^n \|\hat{B}_i\| \cdot \frac{|w_i|}{|\hat{B}_i(z_i)|} \leq M \sum_{i=1}^n \frac{1}{|\hat{B}_i(z_i)|}.$$

Since, for $i \neq j$,

$$\frac{|z_i - z_j|}{|1 - z_i\overline{z_j}|} \geq \frac{\alpha}{1 + |z_i\overline{z_j}|} \geq \frac{\alpha}{2}$$

it follows that

$$|\hat{B}_i(z_i)| \geq \left(\frac{\alpha}{2}\right)^{n-1}, \tag{2}$$

and

$$\|f\| \leq \sum_{i=1}^n \frac{M}{\left(\frac{\alpha}{2}\right)^{n-1}} = Mn\left(\frac{2}{\alpha}\right)^{n-1}.$$

□

Now we can prove Proposition 3.3.

Proof. Let $G = B_G \cdot G_0 = \prod_{k=1}^{s} B_k \cdot G_0$ be the usual factorization of G, where B_k are Blaschke factors and G_0 does not vanish in $\mathbb{D}$. The elements of the set $\{z_k\}_{k=1}^{N}$ can be numbered in such a way that $a_1 = a_2 = \cdots = a_j = 0$ and $a_k \neq 0$ for $k > j$. We will show that there are functions u_0, v_0 analytic and bounded in $\mathbb{D}$ such that $G_0 = u_0 v_0$ and such that if

$$u(z) = B_1(z) \dots B_j(z)\, u_0(z), \tag{3}$$

$$v(z) = B_{j+1}(z) \dots B_s(z)\, v_0(z), \tag{4}$$

then u, v have desired properties.

If $a_k = 0$, $k = 1, 2, \dots, n$, then we can take $u_0(z) = 1$. In this situation $\|u\| = 1$ and $\|v\| = \|G\|$ and we are done. So, let us assume that $a_k \neq 0$ for some k, $1 \leq k \leq n$. By (3), the prescribed (non-zero) values of u_0 are of the form

$$u_0(z_k) = \frac{u(z_k)}{B_1(z_k) \dots B_j(z_k)} = \frac{a_k}{B_1(z_k) \dots B_j(z_k)},$$

and we denote these complex numbers by c_k. Since

$$|B_m(z_k)| = \left| \frac{z_k - z_m}{1 - z_k \overline{z_m}} \right| \geq \frac{\alpha}{1 + |z_k||\overline{z_m}|} \geq \frac{\alpha}{2}, \quad 1 \leq m \leq j,$$

we conclude that

$$|c_k| \leq \frac{1}{\left(\frac{\alpha}{2}\right)^j} \leq \frac{1}{\left(\frac{\alpha}{2}\right)^n}.$$

On the other hand, by definition of η,

$$|c_k| \geq \frac{\eta}{|B_1(z_k)| \dots |B_j(z_k)|} \geq \eta.$$

Thus, complex numbers $\{c_k\}_{k=1}^{n}$, lie in the annulus $\{\eta \leq |z| \leq (2/\alpha)^n\}$. This implies that $\ln \eta \leq \ln |c_k| \leq \ln(2/\alpha)^n$, $k = 1, 2, \dots, n$, and consequently, $\{\ln c_k\}_{k=1}^{n}$ can be chosen in the rectangle $\{\ln \eta \leq Re(z) \leq \ln(2/\alpha)^n,\ -\pi \leq Im(z) \leq \pi\}$. In particular, with M_0 as in (1), all complex numbers $\{\ln c_k\}_{k=1}^{n}$ lie in the circle with center the origin and of radius M_0. By Lemma 3.4 there exists an analytic function g such that $g(z_k) = \ln c_k$ and $\|g\| \leq M_0 n (2/\alpha)^{n-1}$. We take $u_0 = e^g$. Then, clearly, $u_0(z_k) = c_k$ as prescribed, and

$$\|u_0\| = \sup_{z \in \mathbb{D}} |e^{g(z)}| \leq \sup_{z \in \mathbb{D}} e^{|g(z)|} \leq e^{M_0 n (2/\alpha)^{n-1}} \leq \frac{M}{3}.$$

Naturally, we take $v_0 = G_0 e^{-g}$. Again, it is obvious that v_0 attains all of its prescribed values, so it remains to establish the estimate on v_0. Since $\|G_0\| = \|G\|$ we have

$$\|v_0\| = \|G_0 e^{-g}\| \leq \|G_0\| \cdot \|e^{-g}\| \leq \|G\| e^{\|g\|} \leq \|G\| e^{M_0 n(2/\alpha)^{n-1}} \leq \frac{M}{3}.$$

□

4. PROOF OF THEOREM 2.1

We notice that if the class $\mathcal{F}$, as defined in Theorem 2.1, were sequentially compact in the topology of uniform convergence on compact subsets of $\mathbb{D}$ the theorem would follow immediately. Motivated by this observation we define, for $M > 0$, a subclass $\tilde{\mathcal{F}}$ of $\mathcal{F}$ as the set of all $f \in \mathcal{F}$ satisfying $\|\tilde{F}\| \leq M$. We will show that if $\mathcal{F}$ is replaced by $\tilde{\mathcal{F}}$ that does not affect ρ.

Let $\mathcal{F}_1 = \{F \in \mathcal{F} : \sup_{z \in \mathbb{D}} |F(z)|_{\mathrm{sp}} \leq 2\}$. Using the notation

$$F(z) = \begin{pmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{pmatrix},$$

as well as the fact that the eigenvalues $\lambda_1(z)$ and $\lambda_2(z)$ of $F(z)$ are the solutions of the equation $\lambda^2 - tr(F)\lambda + \Delta(F) = 0$ we obtain that, for $F \in \mathcal{F}_1$,

$$\|f_1 + f_4\| \leq 4, \qquad \|f_1 f_4 - f_2 f_3\| \leq 4. \tag{5}$$

Now, we proceed with a more dramatic reduction of the class $\mathcal{F}$. We start by defining

$$\begin{aligned} \alpha &= \min\{\{|z_i - z_j| : i, j = 1, 2, \ldots, n,\ i \neq j\} \\ \eta &= \min\{\{|f_2(z_k)| : f_2(z_k) \neq 0\} \cup \{1\}\} \\ M_0 &= \pi + \max\{|\ln \eta|, |\ln(2/\alpha)^n|\} \\ M &= 24 e^{M_0 n (2/\alpha)^{n-1}} \end{aligned}$$

similarly to definitions in Proposition 12. Next we consider the analytic function

$$G = \frac{(f_1 + f_4)^2}{4} - (f_1 f_4 - f_2 f_3).$$

Our goal is to factor G into a product of two analytic functions u and v so that

$$u(z_k) = f_2(z_k), \quad v(z_k) = f_3(z_k), \quad k = 1, 2, \ldots, n. \tag{6}$$

Such a factorization is provided by Proposition 1. However in order to apply that result we need to address the following issue: If, for some k, u and v are both required to vanish at z_k, then the function G must have a zero of multiplicity no less than 2 at z_k. So assume that $u(z_k) = v(z_k) = 0$. Then, in view of (6), $f_2(z_k) = f_3(z_k) = 0$. Consequently, $f_2(z) = (z - z_k)\tilde{f}_2(z)$, and $f_3(z) = (z - z_k)\tilde{f}_3(z)$, for some bounded analytic functions $\tilde{f}_2$, $\tilde{f}_3$. Furthermore, by assumption $F(z_k)$ has all entries on the main diagonal equal, so $f_1(z_k) = f_4(z_k)$. Therefore, $f_1(z) - f_4(z) = (z - z_k)\tilde{f}(z)$, where $\tilde{f}$ is a bounded analytic function. We conclude that

$$\begin{aligned} G(z) \qquad &= \frac{(f_1(z) - f_4(z))^2}{4} + f_2(z)f_3(z) \\ &= \frac{(z - z_k)^2\tilde{f}(z)^2}{4} + (z - z_k)^2\tilde{f}_2(z)\tilde{f}_3(z). \end{aligned}$$

This shows that, in this situation, z_k is a zero of multiplicity at least 2. Also, using (5), we see that

$$\|G\| \leq \frac{\|f_1 + f_4\|^2}{4} + \|f_1f_4 - f_2f_3\| \leq 8.$$

Therefore, all hypotheses of Proposition 1 are satisfied and we obtain the desired factorization $G = uv$ with u, v satisfying (6), as well as

$$\|u\| \leq \frac{M}{3}$$

and

$$\|v\| \leq \frac{M}{3}.$$

Next we define

$$\tilde{F}(z) = \begin{pmatrix} \dfrac{f_1(z) + f_4(z)}{2} & u(z) \\ v(z) & \dfrac{f_1(z) + f_4(z)}{2} \end{pmatrix}.$$

Clearly, $\|\tilde{F}\| \leq \|f_1 + f_4\| + \|u\| + \|v\| \leq 4 + M/3 + M/3 \leq M$. Furthermore, $tr(\tilde{F}) = f_1 + f_4 = tr(F)$ and $\Delta(\tilde{F}) = (f_1 + f_4)^2/4 - uv = f_1f_4 - f_2f_3 = \Delta(F)$. This implies that the characteristic polynomials of $\tilde{F}$ and F are the same and, therefore,

$$|\tilde{F}(z)|_{\mathrm{sp}} = |F(z)|_{\mathrm{sp}}, \quad z \in \mathbb{D}.$$

Finally, (6) and the equalities $f_1(z_k) = f_4(z_k)$, $k = 1, 2, \ldots, n$, show that $\tilde{F}(z_k) = F(z_k)$.

We conclude that $\tilde{F} \in \tilde{\mathcal{F}}$. Since functions in the class $\tilde{\mathcal{F}}$ are uniformly bounded by M, Montel's Theorem ([4, Theorem 17.17]) shows that $\tilde{\mathcal{F}}$ is conditionally compact. This means that if $\{F_n\} \subset \tilde{\mathcal{F}}$ is a sequence satisfying $\sup\limits_{z\in\mathbb{D}} |F_n(z)|_{\mathrm{sp}} \leq \rho + 1/n$, $n \in \mathbb{N}$, then there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that F_{n_k}

converges uniformly on compact sets in $\mathbb{D}$ (as $k \to \infty$) to a function F. Clearly, F is analytic in $\mathbb{D}$ and satisfies $\|F\| \le M$ as well as $F(z_k) = A_k$, $k = 1, 2, \ldots, n$. Finally, it is easy to see that $\sup_{z\in\mathbb{D}} |F(z)|_{\mathrm{sp}} \le \rho$. This completes the proof of the theorem.

5. PROOF OF THEOREM 2.2

The proof of Theorem 2.2 goes along the same lines as the proof of Theorem 2.1. We have decided to omit some of the computations that are straightforward albeit tedious. Again, we assume that, for any $F \in \mathcal{F}$, $\sup_{z\in\mathbb{D}} |F(z)|_{\mathrm{sp}} \le 2$, and we write

$$F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{pmatrix}.$$

We start by defining

$$\alpha = \min\{\{|z_i - z_j| : i, j = 1, 2, \ldots, n,\ i \ne j\} \tag{7}$$

$$\beta = \max\{\|A_k\| : k = 1, 2, \ldots, n\} \tag{8}$$

$$\phi = \min\{1 - |z_k| : k = 1, 2, \ldots, n\} \tag{9}$$

$$\begin{aligned} \psi = \min\{\, & |f_9(z_k) - f_1(z_k)||f_9(z_k) - f_5(z_k)| : \\ & |f_9(z_k) - f_1(z_k)||f_9(z_k) - f_5(z_k)| \ne 0 \,\} \end{aligned} \tag{10}$$

$$\eta = \min\{\{|f_2(z_k)| : f_2(z_k) \ne 0\} \cup \{1\}\} \tag{11}$$

$$M_0 = \pi + \max\{|\ln \eta|, |\ln(2/\alpha)^n|\} \tag{12}$$

$$M_2 = \beta n \left(\frac{2}{\alpha}\right)^{n-1} \tag{13}$$

$$M_3 = \frac{1}{\phi}\left(\frac{2}{\alpha}\right)^{n-1}\left[M_2 + \frac{12M_2}{\psi} + \frac{16M_2^2}{\psi} + \frac{8}{\psi}\right] \tag{14}$$

$$\tau = \max\{6, M_2, M_3 n \left(\frac{2}{\alpha}\right)^{n-1}\} \tag{15}$$

$$M = 360 e^{2M_0 n (2/\alpha)^{n-1}} \tau^3. \tag{16}$$

We remark here that definitions (7), (11), and (12) are the same as in Proposition 3.3 and Lemma 3.4.

We claim that there exists a function

$$\tilde{F} = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix}$$

such that

$$\tilde{F}(z_k) = F(z_k), \qquad k = 1, 2, \ldots, n \tag{17}$$

$$\|\tilde{F}\| \le M \tag{18}$$

$$|\tilde{F}(z)|_{\mathrm{sp}} = |F(z)|_{\mathrm{sp}}, \qquad z \in \mathbb{D}. \tag{19}$$

In order to ensure (19) we will require that, for any $z \in \mathbb{D}$, the characteristic polynomials of $F(z)$ and $\tilde{F}(z)$ coincide. By comparing the coefficients of those polynomials we obtain the equations

$$\begin{gathered} u_1 + u_5 + u_9 = tr(F) \\ u_1u_5 + u_1u_9 + u_5u_9 - u_2u_4 - u_3u_7 - u_6u_8 = \delta(F) \\ u_1u_5u_9 + u_2u_6u_7 + u_3u_4u_8 - u_1u_5u_9 - u_2u_4u_9 - u_3u_5u_7 = \Delta(F) \end{gathered} \tag{20}$$

while (17) translates into

$$u_j(z_k) = f_j(z_k), \qquad j = 1, 2, \ldots, 9, \qquad k = 1, 2, \ldots, n.$$

Finally, we will obtain functions u_j so that

$$\|u_j\| \le M/9, \quad j = 1, 2, \ldots, 9.$$

Since each A_k is in the modified Jordan form we may take

$$u_3 \equiv 0 \quad \text{and} \quad u_8 \equiv 0.$$

Next, we choose functions u_1 and u such that $u_1(z_k) = f_1(z_k)$ and $u(z_k) = f_9(z_k)$. Since $|f_1(z_k)|, |f_9(z_k)| \le \beta$, $k = 1, 2, \ldots, n$, Lemma 3.4 shows that this choice can be made in such a manner that $\|u_1\| \le \beta n \, (2/\alpha)^{n-1}$ and $\|u\| \le \beta n \, (2/\alpha)^{n-1}$. That way, using (13)

$$\|u_1\| \le M_2, \qquad \|u\| \le M_2. \tag{21}$$

Of course, (21), (15), and (16) imply that

$$\|u_1\| \le \frac{M}{9}.$$

Let $B = B(z_1, z_2, \ldots, z_n)$ and let $\theta_1, \theta_2, \ldots, \theta_n$ be complex numbers defined as

$$\theta_k = -\frac{u'(z_k)}{B'(z_k)} + \frac{u(z_k)\delta(F)'(z_k) - tr(F)'(z_k)u^2(z_k) - \Delta(F)'(z_k)}{B'(z_k)[3u^2(z_k) - 2tr(F)(z_k)u(z_k) + \delta(F)(z_k)]} \tag{22}$$

for all k such that $3u^2(z_k)-2tr(F)(z_k)u(z_k)+\delta(F)(z_k)\neq 0$, and $\theta_k=0$ otherwise. Then, using Lemma 3.4 once again, there exists a function v such that

$$v(z_k)=\theta_k, \quad k=1,2,\dots,n. \tag{23}$$

In order to use full strength of that lemma we need to estimate the size of $|\theta_k|$, $k=1,2,\dots,n$. One knows (cf. [5, page 272]), that if $f\in H^\infty$, $\|f\|\leq 1$, $z\in\mathbb{D}$, and $f(z)=w$ then

$$|f'(z)|\leq\frac{1-|w|^2}{1-|z|^2}.$$

An easy calculation yields the estimates

$$\begin{aligned}|u'(z_k)|&\leq\frac{M_2}{\phi}\\ |\delta(F)'(z_k)|&\leq\frac{12}{\phi}\\ |tr(F)'(z_k)|&\leq\frac{6}{\phi}\\ |\Delta(F)'(z_k)|&\leq\frac{8}{\phi}.\end{aligned}\tag{24}$$

Furthermore, it can be shown using (10) that, whenever $\theta_k\neq 0$, we have the inequality

$$|3u^2(z_k)-2tr(F)(z_k)u(z_k)+\delta(F)(z_k)|\geq\psi. \tag{25}$$

Finally, using the notation $B_j(z)=\prod_{m\neq j}\frac{z-z_m}{1-z\overline{z_m}}$, it is an exercise in elementary calculus to show, using (2), that

$$|B'(z_k)|\geq\left(\frac{\alpha}{2}\right)^{n-1}. \tag{26}$$

We conclude from (22), (14), (24), (25), and (26) that

$$|\theta_k|\leq\frac{M_2}{\phi}\left(\frac{2}{\alpha}\right)^{n-1}+\frac{2^{n-1}\left[M_2(12/\phi)+(6/\phi)M_2^2+(8/\phi)\right]}{\psi\alpha^{n-1}}=M_3.$$

Now Lemma 3.4 gives the estimate $\|v\|\leq M_3n(2/\alpha)^{n-1}$. We define

$$u_9(z)=u(z)+B(z)v(z), \tag{27}$$

and we notice that

$$\|u_9\|\leq M_2+M_3n(2/\alpha)^{n-1}. \tag{28}$$

In view of (15) and (16) it follows that $\|u_9\| \leq 2\tau \leq M/9$. Finally, we take

$$u_5 = tr(F) - u_1 - u_9. \tag{29}$$

By (21) and (28), $\|u_5\| \leq 6 + M_2 + M_2 + M_3 n(2/\alpha)^{n-1}$. Once again, using (15) we obtain that

$$\|u_5\| \leq 4\tau \tag{30}$$

and thus, via (16), that $\|u_5\| \leq M/9$.

Now we turn our attention to the remaining functions to be determined: u_2, u_4, u_6, and u_7. The system (20) can be written as

$$u_2 u_4 = u_1 u_5 + u_1 u_9 + u_5 u_9 - \delta(F) \tag{31}$$

$$u_2 u_6 u_7 - u_2 u_4 u_9 = \Delta(F) - u_1 u_5 u_9 \tag{32}$$

$$u_j(z_k) = f_j(z_k), \qquad j = 1, 2, \ldots, 9, \qquad k = 1, 2, \ldots, n. \tag{33}$$

First we consider the equation (31). In order to apply Proposition 3.3 we need to show that if, for some k, $u_2(z_k) = u_4(z_k) = 0$ then $H = u_1 u_5 + u_1 u_9 + u_5 u_9 - \delta(F)$ has at z_k a zero of multiplicity at least 2. Clearly, $H(z_k) = 0$ for all k. We notice that if $u_2(z_k) = 0$ then A_k is a scalar multiple of the identity. A straightforward calculation shows that, in this situation, $H'(z_k) = 0$. So, z_k is a double zero of H. Therefore, there exist analytic functions u_2 and u_4 satisfying (31) and (33). Furthermore, since (21), (30), (28), and (15) imply that

$$\|u_1 u_5 + u_1 u_9 + u_5 u_9 - \delta(F)\| \leq M_2(4\tau) + M_2(2\tau) + 4\tau(2\tau) + 12 \leq 15\tau^2, \tag{34}$$

Proposition 3.3 shows that $\|u_2\|, \|u_4\| \leq e^{M_0 n(2/\alpha)^{n-1}} \max\{15\tau^2, 1\}$ and, by (16), that $\|u_2\|, \|u_4\| \leq M/9$. In addition, if $u_2 = B_{u_2} e^g$ then

$$\|g\| \leq M_0 n \left(\frac{2}{\alpha}\right)^{n-1}. \tag{35}$$

Now we turn our attention to equation (32) which, in view of (31), becomes

$$u_2 u_6 u_7 = [u_1 u_5 + u_1 u_9 + u_5 u_9 - \delta(F)] u_9 + \Delta(F) - u_1 u_5 u_9. \tag{36}$$

In order to find functions u_6 and u_7 that satisfy (36) we will apply Proposition 3.3. However, we need to make sure that $R(z_k)$, the right hand side of (36) evaluated at any z_k, $k = 1, 2, \ldots, n$, has a zero of sufficiently large multiplicity.

The fact that $R(z_k) = 0$ follows from direct calculation and the observation that A_k is in the modified Jordan form. Next, suppose that at some z_k the expression $u_2 u_6 u_7$ has a zero of multiplicity 2. Since $u_7(z_k) = 0$, $k = 1, 2, \ldots, n$, and $u_2(z_k) = 0$ implies that $u_6(z_k) = 0$ we must have (at least) $u_6(z_k) = 0$. In order to show that R has a double zero at such z_k we consider its derivative R'. Using (27), which implies that $u_9' = u' + B'v + Bv'$, and the fact that $B(z_k) = 0$,

$$\begin{aligned} -R'(z_k) &= [v(z_k)B'(z_k) + u'(z_k)][3u^2(z_k) - 2tr(F)(z_k)u(z_k) + \delta(F)(z_k)] \\ &\quad + [u(z_k)\delta(F)'(z_k) - tr(F)'(z_k)u^2(z_k) - \Delta(F)'(z_k)]. \end{aligned} \tag{37}$$

If, for some k, $3u^2(z_k) - 2tr(F)(z_k)u(z_k) + \delta(F)(z_k) \neq 0$ then (22) and (23) show that the right hand side of (37) vanishes. Here, we use the fact that $B'(z_k) \neq 0$. On the other hand, if $3u^2(z_k) - 2tr(F)(z_k)u(z_k) + \delta(F)(z_k) = 0$ then $u(z_k)$ (and hence $u_9(z_k)$) is a zero of the derivative of the characteristic polynomial. It can be shown that $u_9(z_k)$ is a zero of the characteristic polynomial and, thus, $u_9(z_k)$ is a double eigenvalue of A_k. In particular, either $f_1(z_k) = f_9(z_k)$ or $f_5(z_k) = f_9(z_k)$. We omit the computations which show that, in this situation, the last expression in brackets in (37) must also vanish at z_k.

Finally, if the multiplicity of some z_k is 3 then $u_2(z_k) = 0$ which, in its turn, implies that $A_k = \lambda I$. Since $u_6(z_k) = 0$ we already know that z_k is at least a double zero of R. In order to show that it is a triple zero, it suffices to show that $R''(z_k) = 0$. Again, we leave the details to the reader.

We conclude that, in this sitaution, Proposition 3.3 applies. Therefore, the analytic function

$$G = \frac{[u_1u_5 + u_1u_9 + u_5u_9 - \delta(F)]u_9 + \Delta(F) - u_1u_5u_9}{u_2}$$

can be factored as $G = u_6u_7$. Since, by (34), (28), (21), (30), and (35),

$$\|G\| \leq \left(15\tau^2 \cdot 2\tau + 8 + \tau \cdot 4\tau \cdot 2\tau\right) e^{M_0 n (2/\alpha)^{n-1}} \leq 40\tau^3 e^{M_0 n (2/\alpha)^{n-1}} \tag{38}$$

we obtain from Proposition 3.3 that $\|u_6\|, \|u_7\| \leq e^{M_0 n(2/\alpha)^{n-1}}\, 40\tau^3 e^{M_0 n (2/\alpha)^{n-1}}$. The observation that, by (16), $\|u_6\|, \|u_7\| \leq M/9$ completes the proof.

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