INVARIANT SUBSPACES AND LIMITS OF SIMILARITIES

Alan Lambert and Srdjan Petrovic

Abstract. Let $\{D_n\}$ be a sequence of bounded invertible operators on Hilbert space $\mathcal{H}$. It is shown that the collection of operators T for which the norm-limit $\lim D_n T D_n^{-1}$ exists is an algebra. Furthermore, some sufficient conditions on this sequence are established for the corresponding algebra to have a nontrivial invariant subspace. By considering specific sequences of operators several invariant subspace results are obtained.

1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is said to be a *nontrivial invariant subspace* for T if $(0) \subsetneq \mathcal{M} \subsetneq \mathcal{H}$ and $T\mathcal{M} \subset \mathcal{M}$. If, in addition, $R\mathcal{M} \subset \mathcal{M}$ for every operator R that commutes with T, then $\mathcal{M}$ is said to be a *nontrivial hyperinvariant subspace* for T. In what follows we will use the abbreviations n. i. s. for a nontrivial invariant subspace, and n. h. s. for a nontrivial hyperinvariant subspace.

The question whether every operator $T \in \mathcal{L}(\mathcal{H})$ has a n. i. s., also known as the *invariant subspace problem*, is still open and represents one of the most difficult and most studied problems in operator theory. Although it is, in general, a more difficult task, it is often useful to look for a n. i. s. of a system of operators — most often of an operator algebra. In this paper we study a specific algebra, namely the set of all operators T in $\mathcal{L}(\mathcal{H})$ such that for a fixed choice of invertible operators D_n $(n = 1, 2, \dots)$ in $\mathcal{L}(\mathcal{H})$ the sequence $D_n T D_n^{-1}$ is convergent in some sense. We will be mostly interested in the convergence in the norm of $\mathcal{L}(\mathcal{H})$ although our results remain valid when one considers, instead, the weak operator topology (see definition below). It turns out that under some additional conditions on the sequence D_n the algebra in question has a n. i. s. As a consequence, we will get some new sufficient conditions for $T \in \mathcal{L}(\mathcal{H})$ to have a n. i. s. or a n. h. s. Finally, we study the relationship between an operator T and the limit of $D_n T D_n^{-1}$ (when it exists).

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Before we state our main result we briefly mention some notation and terminology. As usual, $\mathbb{N}$ is the set of positive integers. When $m \in \mathbb{N}$ we use $M_m(\mathcal{S})$ to denote the set of $m \times m$ matrices with entries from the set $\mathcal{S}$. In particular, when $\mathcal{S}$ is the complex plane $\mathbb{C}$ we will write M_m for $M_m(\mathbb{C})$. Two operators A and B are similar if there exists an invertible operator X in $\mathcal{L}(\mathcal{H})$ such that $B = XAX^{-1}$. In this situation X is called a similarity.

As it is customary I stands for the identity operator, although we will occasionally abbreviate expressions of the form λI, for $\lambda \in \mathbb{C}$, to just λ when there is no possibility of confusion. The kernel of an operator $T \in \mathcal{L}(\mathcal{H})$ will be denoted by Ker$T$. Also, $\sigma(T)$ stands for the spectrum of T and $\rho(T)$ for the resolvent set of T i. e. the complement of $\sigma(T)$ in $\mathbb{C}$. We will use $\|T\|_{\text{sp}}$ to denote the spectral radius of T.

Finally, as is well known, three different topologies (and, hence, three different types of convergence) are often used in $\mathcal{L}(\mathcal{H})$. We say that T_n converges to T in the norm, and we write $\lim\limits_{n\to\infty} T_n = T$, if $\|T_n - T\| \to 0$ as $n \to \infty$. It converges in the strong operator topology (or, simply, converges strongly) if $T_n x \to Tx$, as $n \to \infty$, for all $x \in \mathcal{H}$. It converges in the weak operator topology (or, simply, converges weakly) if $< T_n x, y > \to < Tx, y >$, as $n \to \infty$, for all $x, y \in \mathcal{H}$. Of course, $< \cdot, \cdot >$ denotes the inner product on $\mathcal{H}$. We write $SOT\text{-}\lim\limits_{n\to\infty} T_n = T$ [resp., $WOT\text{-}\lim\limits_{n\to\infty} T_n = T$] to denote the strong [resp., weak] convergence.

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2. Some preliminary results

Our starting point is the paper of Bercovici, Foias, and Tannenbaum [1] where they have proved the following result:

Theorem 1. *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$, and $\{D_j : j = 1, 2, \dots\} \subset \mathcal{L}(\mathcal{H})$ with each D_j invertible, so that*

(1) $\lim\limits_{n\to\infty} D_n T D_n^{-1} = T_0$, *and*

(2) *the set $\{D_n, D_n^{-1} : n = 1, 2, \dots\}$ is contained in a finite dimensional subspace.*

Then $\|T_0\|_{\text{sp}} = \|T\|_{\text{sp}}$.

We briefly mention that without the assumption (2) the theorem is false. This follows, for example, from Theorem 5.8 of [5], with N a positive operator with spectrum $[2, 3]$ and A a positive operator with spectrum $[1, 4]$. The strength of the mentioned assumption was further demonstrated in the following result from [7].

Theorem 2. *Under the same assumptions as in Theorem 1, the unbounded connected components of $\rho(T)$ and $\rho(T_0)$ are the same.*

Finally, it was noticed in [BL] that the relationship between $\rho(T)$ and $\rho(T_0)$ is not restricted to unbounded components only. Roughly, when passing from $\sigma(T)$ to $\sigma(T_0)$ one takes a hole in the spectrum of T and either leaves it alone, or fills it up completely. More precisely, we have:

Theorem 3. *Under the same assumptions as in Theorem 1, if $\lambda \in \rho(T_0)$ then the bounded component Ω_λ of $\rho(T)$ that contains λ satisfies $\Omega_\lambda \subset \rho(T_0)$.*

In this paper we turn our attention to the relationship between the structure of T and T_0. We will show that, even under weaker hypotheses, T and T_0 are either similar or they both have n. i. s.

We start with the observation that, due to (2), $D_n/\|D_n\|$ must have a convergent subsequence and the same is true of $D_n^{-1}/\|D_n^{-1}\|$. We pass to a subsequence for which both limits exist and, with a slight abuse of notation, we still call it D_n. Let

$$D_+ = \lim_{n\to\infty} \frac{D_n}{\|D_n\|}, \quad D_- = \lim_{n\to\infty} \frac{D_n^{-1}}{\|D_n^{-1}\|}. \tag{3}$$

In what follows we will consider not only sequences $\{D_n\}$ that satisfy (2) but also those for which the limits in (3) exist. Of course, if one is allowed to replace a sequence $\{D_n\}$ by one of its subsequences (as will be the case in most of our results) it suffices to assume the existence of D_+ and D_-.

Now we turn to the main object of our study — the set $\mathcal{M} = \{T : \lim_{n\to\infty} D_n T D_n^{-1}$ exists$\}$. It is easy to see that $\mathcal{M}$ is a unital subalgebra of $\mathcal{L}(\mathcal{H})$. In [1] it was shown that, under condition (2), $\mathcal{M}$ is a closed algebra. The following example shows that under a weaker assumption (3) it need not be closed in the norm topology (and, hence, in any topology).

Example 4. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. For every $n \in \mathbb{N}$ let P_n denote the projection in $\mathcal{L}(\mathcal{H})$ with range the one-dimensional subspace spanned by e_n, and let $D_n = 1 + n^2 P_n$. Finally, for every $k \in \mathbb{N}$, let T_k be an operator in $\mathcal{L}(\mathcal{H})$ such that, relative to the basis $\{e_j\}_{j\in\mathbb{N}}$,

$$T_k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{k} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

An easy computation shows that $D_n^{-1} = 1 - \left(1 - \frac{1}{n^2+1}\right) P_n$. Therefore, for any $n \in \mathbb{N}$,

$$D_n T_k D_n^{-1} = (1 + n^2 P_n) T_k \left[1 - \left(1 - \frac{1}{n^2+1}\right) P_n\right]$$

$$= T_k + n^2 P_n T_k - \left(1 - \frac{1}{n^2+1}\right) T_k P_n - n^2 \left(1 - \frac{1}{n^2+1}\right) P_n T_k P_n$$

Clearly, $T_k e_n = 0$ for $n > 1$ and $k \in \mathbb{N}$, so $T_k P_n = 0$. Furthermore, for $n > k$, $P_n T_k = 0$. Thus, for $n > k$, $D_n T_k D_n^{-1} = T_k$ and it follows that $\lim_{n\to\infty} D_n T_k D_n^{-1} = T_k$ for all $k \in \mathbb{N}$. We conclude that $T_k \in \mathcal{M}$ for $k \in \mathbb{N}$.

Next, let

$$T = \begin{pmatrix} 1 & 0 & 0 & \dots \\ \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $T = \lim_{k\to\infty} T_k$, but T does not belong to $\mathcal{M}$. Indeed, for $n > 1$, $TP_n = 0$ so

$$D_n T D_n^{-1} = T + n^2 P_n T. \tag{4}$$

However, the matrix of $P_n T$, relative to $\{e_j\}_{j\in\mathbb{N}}$, has the only nonzero entry equal to $1/n$ and thus $\|n^2 P_n T\| = n$. Obviously, $D_n T D_n^{-1}$ does not converge, as $n \to \infty$, so $T \notin \mathcal{M}$.

Now we turn our attention to the mapping $\phi : \mathcal{M} \to \mathcal{L}(\mathcal{H})$ defined as $\phi(T) = \lim_{n\to\infty} D_n T D_n^{-1}$. It is not hard to see that ϕ is an algebra homomorphism. However, it need not be injective. To verify this assertion it suffices to write $\mathcal{H}$ as $\mathcal{H}_1 \oplus \mathcal{H}_2$, the orthogonal direct sum of two infinite-dimensional closed subspaces of $\mathcal{H}$ and to consider, relative to this decomposition,

$$D_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $D_n T D_n^{-1} \to 0$ as $n \to \infty$.

3. Main result

In this section we state and prove our main result that may be viewed as a generalization of Theorem 1.

Theorem 5. *Suppose that* $\{D_n\}$ *is a sequence of invertible operators on* $\mathcal{H}$*, and* $\mathcal{M} = \{T : \lim_{n\to\infty} D_n T D_n^{-1}$ *exists*$\}$*. For* $T \in \mathcal{M}$ *let* $\phi(T) = \lim_{n\to\infty} D_n T D_n^{-1}$*. Let* $\mathcal{N}$ *be a norm closed, unital subalgebra of* $\mathcal{M}$*, and let* $\psi = \phi|\mathcal{N}$*. Then:*
(i) The map ψ *is a norm continuous algebra homomorphism on* $\mathcal{N}$ *which is completely spectrally isometric i. e., for each positive integer* m *and each* $(T_{ij})_{i,j=1}^m \in M_m(\mathcal{N})$*,* $\|(T_{ij})_{i,j=1}^m\|_{\mathrm{sp}} = \|(\psi(T_{ij}))_{i,j=1}^m\|_{\mathrm{sp}}$*.*
(ii) Assuming (3), either ϕ *is given by a similarity or* $\mathcal{M}$ *and* $\phi(\mathcal{M})$ *have n. i. s.*

Proof. (i) For each positive integer n define ψ_n on $\mathcal{N}$ by $\psi_n(T) = D_n T D_n^{-1}$. Then $\{\psi_n\}_{n\in\mathbb{N}}$ is a pointwise bounded sequence of linear transformations on $\mathcal{N}$, so by the uniform boundedness principle [Dunford, Theorem II. 3. 6.] there is a constant C such that, for every $T \in \mathcal{N}$, $\|\psi(T)\| \le C\|T\|$.

Now, let $T \in \mathcal{M}$, and let $\lambda \in \sigma(T)$. Then

$$\phi(T) - \lambda = \lim_{n\to\infty} D_n T D_n^{-1} - \lambda = \lim_{n\to\infty} D_n(T-\lambda)D_n^{-1}.$$

In view of the norm convergence of $\{D_n(T-\lambda)D_n^{-1}\}$ and the fact that the set of invertible operators in $\mathcal{L}(\mathcal{H})$ is open it follows that $\lambda \in \sigma(\phi(T))$ and so $\sigma(T) \subset \sigma(\phi(T))$. In particular, $\|T\|_{\mathrm{sp}} \le \|\phi(T)\|_{\mathrm{sp}}$. In order to prove the opposite inequality we consider $T \in \mathcal{N}$ and an arbitrary positive integer k. Then

$$\|(\psi(T))^k\|^{1/k} = \|(\psi(T^k))\|^{1/k} \le \left(\|\psi\|\|T^k\|\right)^{1/k} = \|\psi\|^{1/k}\|T^k\|^{1/k}.$$

Passing to the limit as $k \to \infty$ we obtain that $\|\psi(T)\|_{\mathrm{sp}} \le \|T\|_{\mathrm{sp}}$. Thus $\|\psi(T)\|_{\mathrm{sp}} = \|T\|_{\mathrm{sp}}$.

As for the assertion regarding the complete spectral isometry, let m be a positive integer. For each $n \in \mathbb{N}$, define $D_n^{(m)}$ to be the $m \times m$ matrix

$$\begin{pmatrix} D_n & 0 & \dots & 0 \\ 0 & D_n & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & D_n \end{pmatrix}.$$

Then, for $(T_{ij})_{i,j=1}^m \in M_m(\mathcal{N})$,

$$D_n^{(m)}\,(\,T_{ij}\,)_{i,j=1}^m\,(D_n^{(m)})^{-1} = (\,D_n T_{ij} D_n^{-1}\,)_{i,j=1}^m \to (\,\psi(T_{ij})\,)_{i,j=1}^m\,.$$

Since our analysis of the sequence $\{D_n\}$ and the algebra $\mathcal{N}$ applies just as well to the sequence $\{D_n^{(m)}\}$ and the algebra $M_m(\mathcal{N})$ we may deduce the corresponding spectral radii equality.

(ii) First we consider the case when $\liminf_{n\to\infty}\|D_n\|\|D_n^{-1}\| < +\infty$. (Of course, $\|D_n\|\|D_n^{-1}\| \geq 1$.) Then there is a positive number M and a subsequence $\{D_{n_i}\}_{i\in\mathbb{N}}$ of $\{D_n\}_{n\in\mathbb{N}}$ such that $\|D_{n_i}\|\|D_{n_i}^{-1}\| \to M$ as $i\to\infty$. Now

$$D_+D_- = \lim_{i\to\infty}\frac{D_{n_i}}{\|D_{n_i}\|}\lim_{i\to\infty}\frac{D_{n_i}^{-1}}{\|D_{n_i}^{-1}\|} = \lim_{i\to\infty}\frac{D_{n_i}D_{n_i}^{-1}}{\|D_{n_i}\|\|D_{n_i}^{-1}\|} = \frac{1}{M}.$$

Similarly, $D_-D_+ = \dfrac{1}{M}$. Thus D_+ and D_- are both invertible and $D_- = \dfrac{1}{M}D_+^{-1}$. Also, for $T\in\mathcal{M}$,

$$D_+T = \left(\lim_{n\to\infty}\frac{D_n}{\|D_n\|}\right)T = \lim_{n\to\infty}(D_nTD_n^{-1})\frac{D_n}{\|D_n\|} = T_0D_+ \tag{5}$$

which shows that $T_0 = D_+TD_+^{-1}$. Since this holds for every pair T and $T_0 = \phi(T)$ we see that ϕ is given by a similarity.

On the other hand, if the sequence $\{\|D_n\|\|D_n^{-1}\|\}_{n\in\mathbb{N}}$ does not possess a bounded subsequence, i. e., if $\lim_{n\to\infty}\|D_n\|\|D_n^{-1}\| = +\infty$ then $D_+D_- = 0$ and $D_-D_+ = 0$. Also, as before, $D_+T = T_0D_+$ and, analogously, $TD_- = D_-T_0$. Obviously, the kernel of D_+ is invariant for T while $\mathrm{Ker}D_-$ is invariant for T_0. Furthermore, it follows from (3) that $\|D_+\| = \|D_-\| = 1$. In particular, $D_+\neq 0$ and $D_-\neq 0$ so $\mathrm{Ker}D_+\neq\mathcal{H}$ and $\mathrm{Ker}D_-\neq\mathcal{H}$. On the other hand, if $\mathrm{Ker}D_+ = (0)$ then $D_+x\neq 0$ for any nonzero $x\in\mathcal{H}$ or, equivalently, for any $x\in\mathcal{H}$ we have that $D_+x = 0$ implies $x = 0$. Since $D_+(D_-x) = 0$ for any $x\in\mathcal{H}$ it follows that $D_-x = 0$ for any $x\in\mathcal{H}$ so that $D_- = 0$ which is impossible. Therefore, $\mathrm{Ker}D_+\neq(0)$ and a similar argument shows that $\mathrm{Ker}D_-\neq(0)$. Consequently, $\mathrm{Ker}D_+$ is n. i. s. for T while $\mathrm{Ker}D_-$ is n. i. s. for T_0. In view of the arbitrary choice of $T\in\mathcal{M}$ and $T_0 = \phi(T)$, we conclude that both $\mathcal{M}$ and $\phi(\mathcal{M})$ have n. i. s. $\square$

Remark 6. If, instead of $\mathcal{M}$, we consider the set $\tilde{\mathcal{M}} = \{T : WOT\text{-}\lim_{n\to\infty} D_nTD_n^{-1}$ exists$\}$ then $\tilde{\mathcal{M}}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $\mathcal{M}$ and the natural extension $\tilde{\phi}$ of ϕ to $\tilde{\mathcal{M}}$ is an algebra homomorphism. We notice that $\tilde{\mathcal{M}}$ need not be norm-closed, and thus not closed in any topology. Indeed, take D_n, T_k, and T as in Example 4. Let x be a vector in $\mathcal{H}$ given by $x = \sum_{n=1}^{\infty} n^{-3/4}e_n$. Then, in view of (4),

$$< D_nTD_n^{-1}e_1, x > = < Te_1, x > + n^2 < P_nTe_1, x > .$$

Since $P_nTe_1 = (1/n)e_n$ it follows that $n^2 < P_nTe_1, x >= n < e_n, x >= n \cdot n^{-3/4} = n^{1/4}$. That way, $D_nTD_n^{-1}$ does not converge weakly. This shows that $T \notin \tilde{\mathcal{M}}$ and, therefore, $\tilde{\mathcal{M}}$ is not closed.

Furthermore, part (ii) of Theorem 5 remains valid when ϕ and $\mathcal{M}$ are replaced by $\tilde{\phi}$ and $\tilde{\mathcal{M}}$. Indeed, the proof is a word by word repetition of the proof above. The only place that may need additional attention is the formula (5). Since, by assumption, both $D_n/\|D_n\|$ and $D_n^{-1}/\|D_n^{-1}\|$ converge in the norm topology they also converge in the weak operator topology. Consequently,

$$D_+T = \left(WOT\text{-}\lim_{n\to\infty} \frac{D_n}{\|D_n\|} \right) T = WOT\text{-}\lim_{n\to\infty} (D_nTD_n^{-1}) \frac{D_n}{\|D_n\|}.$$

One knows that if $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ so that $WOT\text{-}\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$ then $WOT\text{-}\lim_{n\to\infty} A_nB_n = AB$. Therefore,

$$D_+T = WOT\text{-}\lim_{n\to\infty} (D_nTD_n^{-1}) \lim_{n\to\infty} \frac{D_n}{\|D_n\|} = T_0D_+.$$

Although the following assertion represents a part of what was proved in Theorem 5, we state it here separately, for reference.

Corollary 7. *Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of invertible operators in $\mathcal{L}(\mathcal{H})$ and let $T \in \mathcal{L}(\mathcal{H})$ so that (1) and (3) hold, and such that $\|D_n\|\|D_n^{-1}\| \to \infty$ as $n \to \infty$. Then T has a n. i. s. If, in addition, A is an operator commuting with each D_n, $n \in \mathbb{N}$, then A has a n. i. s.*

4. Applications

As our first application of Theorem 5 we consider the situation when there is an invertible operator $D \in \mathcal{L}(\mathcal{H})$ such that $D_n = D^n$. Now the existence of limits in (3) can be written as

$$\lim_{n\to\infty} \frac{D^n}{\|D^n\|} = D_+, \quad \lim_{n\to\infty} \frac{D^{-n}}{\|D^{-n}\|} = D_- \tag{6}$$

and we have the following corollary.

Corollary 8. *Let $D \in \mathcal{L}(\mathcal{H})$ satisfy (6). Then D has a n. i. s.*

Proof. If $\lim_{n\to\infty} \|D^n\|\|D^{-n}\| = \infty$ the assertion follows from Corollary 7. If, on the other hand, $\|D^n\|\|D^{-n}\| \le M$, for all $n \in \mathbb{N}$, then it was shown in [3, Remark after Question 2] that D is similar to a scalar multiple of a unitary operator. Either way, D has a n. i. s. $\square$

We point out briefly some operators satisfying (6). If the span of D^n and the span of D^{-n} are both finite dimensional then the formula in question is true at least for some subsequence. Such is the case, for example, when D acts on a finite dimensional Hilbert space. Also, (6) holds if $D = I + N$, where N is a nilpotent operator. The last observation leads to the following corollary.

Corollary 9. *Let X, W, and Z be operators in $\mathcal{L}(\mathcal{H})$. Suppose that A is a nonzero operator in $\mathcal{L}(\mathcal{H})$ such that $AZ = ZA = 0$ and $XA = AW$. Then the operator*

$$T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

acting on the orthogonal direct sum of two copies of $\mathcal{H}$, has a n. i. s.

Proof. The conditions $AZ = ZA = 0$ and $XA = AW$ imply that T commutes with

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}.$$

□

As our next application we consider the case when Q is a quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and we define $D_n = I + nQ$. Before we study this case in detail we make the following observation.

Proposition 10. *Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of invertible operators in $\mathcal{L}(\mathcal{H})$ such that both $\{D_n/\|D_n\|\}_{n\in\mathbb{N}}$ and $\{D_n^{-1}/\|D_n^{-1}\|\}_{n\in\mathbb{N}}$ converge. If $D_n/\|D_n\| \to D_+$ and $D_n^{-1}/\|D_n^{-1}\| \to D_-$ $(n \to \infty)$ then $\|D_n\|\|D_n^{-1}\| \to \infty$ $(n \to \infty)$ if and only if D_+ is non-invertible.*

Proof. Suppose first that D_+ is invertible. We will show that in this case the sequence $\{\|D_n\|\|D_n^{-1}\|\}_{n=1}^{\infty}$ has a bounded subsequence. Assume to the contrary that $\|D_n\|\|D_n^{-1}\| \to \infty$. Then

$$D_+D_- = \lim_{n\to\infty} \frac{D_n}{\|D_n\|} \lim_{n\to\infty} \frac{D_n^{-1}}{\|D_n^{-1}\|} = \lim_{n\to\infty} \frac{I}{\|D_n\|\|D_n^{-1}\|} = 0$$

Since D_+ is invertible, $D_+D_- = 0 \Rightarrow D_- = 0$ which is impossible. So, the sequence $\{\|D_n\|\|D_n^{-1}\|\}_{n=1}^{\infty}$ does not diverge to infnity.

On the other hand, if D_+ is non-invertible, then [6, Lemma 10.7] shows that

$$\left\| \left(\frac{D_n}{\|D_n\|} \right)^{-1} \right\| \to \infty$$

as $n \to \infty$. But $\|(D_n/\|D_n\|)^{-1}\| = \|D_n\|\|D_n^{-1}\|$ and the proposition is proved. □

Our central result concerning quasinilpotent operators is the following theorem.

Theorem 11. *If Q is a non-zero quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and if the sequence $(I+nQ)^{-1}/\|(I+nQ)^{-1}\|$ converges as $n \to \infty$ then Q has a n. h. s.*

Proof. Let

$$\lim_{n\to\infty} \frac{(I+nQ)^{-1}}{\|(I+nQ)^{-1}\|} = Q_-.$$

In order to apply Corollary 7 we need, beside the hypothesis of this theorem, that $(I+nQ)/\|I+nQ\| \to Q_+$ and that $\|I+nQ\|\|(I+nQ)^{-1}\| \to \infty$. In view of Proposition 10 the last condition is equivalent to Q_+ being non-invertible. Therefore, it suffices to show that

$$\lim_{n\to\infty} \frac{I+nQ}{\|I+nQ\|} = \frac{Q}{\|Q\|}.$$

Clearly, $n\|Q\| - 1 \le \|I+nQ\| \le 1 + n\|Q\|$. Thus

$$\|Q\| - \frac{1}{n} \le \frac{\|I+nQ\|}{n} \le \frac{1}{n} + \|Q\|,$$

so $\lim_{n\to\infty} \|I+nQ\|/n = \|Q\|$. On the other hand, $\lim_{n\to\infty}(I+nQ)/n = Q$. Therefore

$$\lim_{n\to\infty} \frac{I+nQ}{\|I+nQ\|} = \lim_{n\to\infty} \frac{(I+nQ)/n}{\|I+nQ\|/n} = \frac{Q}{\|Q\|}$$

and the theorem is proved. □

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Alan Lambert, Department of Mathematics, Univeristy of North Carolina at Charlotte, Charlotte, NC 28223

E-mail address: `allamber@email.uncc.edu`

Srdjan Petrovic, Department of Mathematics and Statistics, Western Michigan University, Kalamazoo, MI 49008-5152

E-mail address: `petrovic@wmich.edu`