BEYOND HYPERINVARIANCE FOR COMPACT OPERATORS

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Abstract. We introduce a new class of operator algebras on Hilbert space. To each bounded linear operator a spectral algebra is associated. These algebras are quite substantial, each containing the commutant of the associated operator, frequently as a proper subalgebra. We establish several sufficient conditions for a spectral algebra to have a nontrivial invariant subspace. When the associated operator is compact this leads to a generalization of V. Lomonosov’s theorem.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space with inner product $\langle , \rangle$, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be invariant for an operator $T$ if $T\mathcal{M} \subseteq \mathcal{M}$. It is nontrivial if different from $(0)$ and $\mathcal{H}$. We will frequently use the abbreviation n. i. s. for a nontrivial invariant subspace. The collection of all invariant subspaces of any operator $T$ is a lattice and it is denoted $\text{Lat}(T)$. The invariant subspace problem asks for a description of $\text{Lat}(T)$ for an arbitrary $T \in \mathcal{L}(\mathcal{H})$. Since this problem is open, one may consider a weaker (open) question — whether there exists an operator whose lattice is precisely $\{(0), \mathcal{H}\}$.

In 1973 V. Lomonosov proved in [vL73] the following remarkable result.

**Theorem 1.1.** If $K$ is a nonzero compact linear operator on a complex Banach space then there exists a nontrivial subspace that is invariant under every operator commuting with $K$.

In other words, if $K$ is compact, its commutant $\{K\}'$ has a n. i. s. Such a subspace is called a hyperinvariant subspace for $K$. Recall that a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ is said to be transitive if the only subspaces invariant for $\mathcal{A}$ (meaning for every operator in $\mathcal{A}$) are $(0)$ and $\mathcal{H}$. In the language of algebras Theorem 1.1 can be extended as in [RR73, Theorem 8.23].

**Theorem 1.2.** A weakly closed transitive algebra which contains a nonzero compact operator must be $\mathcal{L}(\mathcal{H})$.

The new technique of Lomonosov had a great impact on research in operator theory in the 1970’s. The paper [PS74] is an excellent source of information about [vL73] as well as the progress made within a short time period after the appearance

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of this result. Also, one should consult [cP78] for further accomplishments along the same lines. The paper [PS74] did an excellent job of identifying important results that follow from but were not explicitly stated in [vL73]. In particular, it was noticed there that Lomonosov’s new technique yields the following result.

**Theorem 1.3.** If $\mathcal{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{H})$ and if $K$ is a nonzero compact operator in $\mathcal{L}(\mathcal{H})$ then there exists an operator $A \in \mathcal{A}$ such that $AK$ has $1$ as an eigenvalue (i.e., there exists a nonzero vector $x$ in $\mathcal{H}$ such that $AKx = x$).

In this work we will use the following reformulation of Theorem 1.3. Recall that an operator is *quasinilpotent* if its spectrum consists of $0$ alone. We will denote the class of quasinilpotent operators as $\mathcal{Q}$.

**Proposition 1.4.** Let $\mathcal{A}$ be a unital subalgebra of $\mathcal{L}(\mathcal{H})$. The following are equivalent:

(a) $\mathcal{A}$ has a n. i. s.

(b) There exists a nonzero compact operator $K$ such that $AK \subset \mathcal{Q}$.

(c) There exists a rank one operator $F$ such that $\mathcal{A}F \subset \mathcal{Q}$.

(d) There exists a rank one operator $F$ such that the square of every operator in $\mathcal{A}F$ is $0$.

**Proof.** Implications (d)$\Rightarrow$(c) and (c)$\Rightarrow$(b) are obvious, while (b)$\Rightarrow$(a) is a consequence of Theorem 1.3. Indeed, if (a) were false, then there would exist an operator $A \in \mathcal{A}$ such that $AK$ has eigenvalue $1$, contradicting the assumption that it must be quasinilpotent. Thus, it remains to prove that (a)$\Rightarrow$(d). So suppose that $\mathcal{A}$ has a n. i. s. Then there exist nonzero vectors $u$ and $v$ such that $\mathcal{A}u$ is orthogonal to $v$. It follows that, for any operator $T$ in $\mathcal{A}$, $(T (u \otimes v))^2 = (Tu \otimes v)^2 = (Tu, v)(Tu \otimes v) = 0$, and the proposition is proved.

Theorem 1.1 shows that, when $K$ is a compact operator, the algebra $\{K\}'$ possesses a n. i. s. On the other hand, judging by Proposition 1.4, one might hope that a relationship with a specific compact operator might lead to the existence of a n. i. s. for other algebras. The main result of the present paper is that there is such a class of algebras, which we call spectral algebras and denote $B_K$, so that for a compact operator $K$, the algebra $B_K$ has a n. i. s. In addition, we will show that whenever the spectral radius of $K$ is positive, $B_K$ properly contains the commutant of $K$.

In order to motivate the definition of a spectral algebra we present a modification, based on the work of Gilfeather [fG78], of a result of Rota [gR60]. Let $A \in \mathcal{L}(\mathcal{H})$. For $m \geq 1$, let

$$d_m = d_m(A) = \frac{1}{m + r(A)} = \frac{m}{1 + mr(A)},$$

where $r(A)$ is the spectral radius of $A$. Define

$$R_m = R_m(A) = \left( \sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n \right)^{1/2}.$$
Proposition 1.5. \( \| R_m A R_m^{-1} \| \to r(A) \), as \( m \to \infty \).

Note that the series above converges in the norm topology, and each \( R_m \) is invertible with \( \| R_m^{-1} \| \leq 1 \).

Proof. Let \( m \) be an arbitrary positive integer. Then \( r(d_m A) < 1 \). A calculation shows that \( \| R_m(A)(d_m A)R_m^{-1}(A) \| < 1 \). Thus, \( \| R_m A R_m^{-1} \| < 1/d_m \), for all \( m \), and
\[
\limsup_{m \to \infty} \| R_m A R_m^{-1} \| \leq r(A).
\]
Of course, \( r(A) = r(R_m A R_m^{-1}) \leq \| R_m A R_m^{-1} \| \) for all \( m \), hence
\[
\liminf_{m \to \infty} \| R_m A R_m^{-1} \| \geq r(A)
\]
and the proposition is proved.

Throughout the sequel, the symbols \( d_m (= d_m(A)) \) and \( R_m (= R_m(A)) \) will have the meanings ascribed to them in Proposition 1.5. We will only use the longer forms when forced to do so to avoid ambiguity.

We now introduce the focal object of this investigation. If \( A \) is an operator in \( \mathcal{L}(\mathcal{H}) \) and \( R_m \) is as in Proposition 1.5, we associate with \( A \) the collection
\[
\mathcal{B}_A = \left\{ T \in \mathcal{L}(\mathcal{H}) : \sup_{m} \| R_m A R_m^{-1} \| < \infty \right\}.
\]
It is easy to see that \( \mathcal{B}_A \) is indeed an algebra (although it need not be closed). We will show that it contains all operators that commute with \( A \). Thus, whenever \( \mathcal{B}_A \) possesses a n. i. s. one also establishes a nontrivial hyperinvariant subspace for \( A \). In fact, we will see that in many instances the commutant \( \{A\}' \) is a proper subalgebra of \( \mathcal{B}_A \). Then in the case that \( A \) is compact, our result is a true extension of that of Lomonosov.

A word or two about terminology. Due to its relationship with the spectral radius, we decided to name \( \mathcal{B}_A \) a spectral algebra (associated with \( A \)). As noted above, every invariant subspace for \( \mathcal{B}_A \) is hyperinvariant for \( A \). If it can then be shown that \( \mathcal{B}_A \) is strictly larger than \( \{A\}' \) then the level of invariant subspace structure has gone beyond hyperinvariance. We dismissed our attempts at naming this phenomenon since they all involved prefixing something to hyper, and that seemed like overkill.

Here is a brief summary of the text. In Section 2 we will discuss the main properties of spectral algebras. In particular, we will show that \( \mathcal{B}_A \) is an algebra and that \( \{A\}' \subseteq \mathcal{B}_A \). In addition, we will study the conditions that make this inclusion proper. In Section 3 we will introduce an ideal \( \mathcal{Q}_A \) in \( \mathcal{B}_A \) whose elements are all quasinilpotent operators. We will show that the presence of \( \mathcal{Q}_A \neq (0) \) and a nonzero compact operator in \( \mathcal{B}_A \) guarantees the existence of a n. i. s. for the whole algebra. The simplest scenario occurs when \( A \) is quasinilpotent, and we will establish several sufficient conditions for \( \mathcal{B}_A \) to contain a nonzero rank one operator (and thus a n. i. s.). In Section 4 we will be interested in \( \mathcal{B}_K \) where \( K \) is a nonzero compact operator. Using the fact that the ideal \( \mathcal{Q}_K \) is nontrivial, we will show that \( \mathcal{B}_K \) always has a n. i. s. Since \( \{K\}' \subseteq \mathcal{B}_K \) we will examine the question when \( \{K\}' \neq \mathcal{B}_K \).
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2. Spectral Algebras

In this section we will establish some basic properties of spectral algebras. First we address the question of closedness of \( B_A \). The following example, which is a modification of an example from [LP00], shows that a spectral algebra need not be closed in the norm topology and, hence, in any of the usual operator topologies.

**Example 2.1.** Let \( S \) be the unilateral forward shift on \( \mathcal{H} \), and let \( S^* \) be its adjoint. Let \( \{ e_n : n \geq 0 \} \) be the orthonormal basis that \( S \) shifts, i.e., \( S e_n = e_{n+1} \), for all \( n \geq 0 \).

Using the notation \( R_m \) for \( R_m(S^*) \) and \( d_m \) for \( d_m(S^*) \) we have that

\[
R_m^2 = \sum_{n=0}^{\infty} d_m^n S^n S^m = \text{diag} (1, 1 + d_m^2, 1 + d_m^2 + d_m^4, \ldots).
\]

For every \( k \in \mathbb{N} \), let \( u_k = \sum_{i=0}^{k} e_i/(i+1) \), and let \( T_k = u_k \otimes e_0 \). It is straightforward to verify that \( T_k \in \mathcal{B}_{S^*} \). On the other hand, \( u_k \rightarrow u \equiv \sum_{i=0}^{\infty} e_i/(i+1) \) and \( T_k \rightarrow T \equiv u \otimes e_0 \) in the norm topology, but \( T \) does not belong to \( \mathcal{B}_{S^*} \). Indeed, \( \| R_m T R_m^{-1} \|_2^2 = \| R_m u \|_2^2 \| R_m^{-1} e_0 \|_2^2 = \sum_{k=1}^{\infty} (1 + d_m^2 + \cdots + d_m^{2k-2})/k^2 \). Since \( d_m < 1 \) the sequence \( 1, d_m^2, d_m^4, \ldots \) is decreasing and it follows that

\[
\| R_m T R_m^{-1} \|_2^2 > \sum_{k=1}^{\infty} \frac{1}{k^2} k d_m^{2k-2} = \sum_{k=1}^{\infty} \frac{1}{k} d_m^{2k-2} = -\ln(1 - d_m^2)/d_m^2 \rightarrow \infty,
\]

as \( m \rightarrow \infty \). Thus \( T \) does not belong to \( \mathcal{B}_{S^*} \) and, consequently, \( \mathcal{B}_{S^*} \) is not closed in the norm topology.

The previous example is indicative of what it might take for a spectral algebra to be closed. The following result gives a precise formulation.

**Lemma 2.2.** Let \( A \in \mathcal{L}(\mathcal{H}) \). The algebra \( \mathcal{B}_A \) is closed in the norm topology if and only if there is a constant \( C \) such that, for every \( T \in \mathcal{B}_A \),

\[
\sup_m \| R_m T R_m^{-1} \| \leq C \| T \|.
\]

**Proof.** If \( \mathcal{B}_A \) is closed in the norm topology then it is a Banach space. For each \( m \), the mapping \( G_m : \mathcal{B}_A \rightarrow \mathcal{L}(\mathcal{H}) \) given by \( G_m(T) = R_m T R_m^{-1} \) is linear and continuous. By the definition of \( \mathcal{B}_A \) the family \( \{ G_m : m \in \mathbb{N} \} \) is pointwise uniformly bounded. The uniform boundedness principle then implies that there is a constant \( C \) such that \( \sup_m \| R_m T R_m^{-1} \| \leq C \| T \| \).

Conversely, suppose that such a scalar \( C \) exists, and let \( \{ T_n \} \) be a sequence of operators in \( \mathcal{B}_A \) converging in norm to the operator \( T \). For a fixed positive integer \( m \), let \( \lim_{n \rightarrow \infty} \| R_m T R_m^{-1} - R_m T_n R_m^{-1} \| = 0 \) so, for all sufficiently large \( i \),

\[
\| R_m T R_m^{-1} \| \leq \| R_m T R_m^{-1} - R_m T_i R_m^{-1} \| + \| R_m T_i R_m^{-1} \| \leq 1 + \| R_m T_i R_m^{-1} \| \leq 1 + C \| T_i \|.
\]
The sequence \( \{\|T_i\|\} \) is convergent, hence bounded, so there is a constant \( M \) such that \( \|T_i\| \leq M \) and, consequently, \( \|R_m T R_m^{-1}\| \leq 1 + CM \). Since this inequality is true independent of \( m \) we conclude that \( T \in B_A \) and the lemma is proved. \( \square \)

Our next task is to show that spectral algebras are quite substantial.

**Proposition 2.3.** Suppose \( A \) is a nonzero operator, \( B \) is a power bounded operator commuting with \( A \), and \( T \) is an operator for which \( AT = BTA \). Then \( T \in B_A \).

**Proof.** It is easy to verify that \( A^2 T = B^2 TA^2 \). Using induction one can prove that \( A^n T = B^n TA^n \), for every \( n \in \mathbb{N} \). The operator \( B \) is power bounded so there is a constant \( C \) such that \( \|B^n\| \leq C \), for each \( n \in \mathbb{N} \). For any vector \( x \in \mathcal{H} \) and any positive integer \( m \), we have that

\[
\|R_m T R_m^{-1} x\|^2 = \sum_{n=1}^{\infty} d_m^n \|A^n T R_m^{-1} x\|^2
\]

\[
= \sum_{n=1}^{\infty} d_m^n \|B^n T A^n R_m^{-1} x\|^2
\]

\[
\leq C^2 \|T\|^2 \sum_{n=1}^{\infty} d_m^n \|A^n R_m^{-1} x\|^2
\]

\[
= C^2 \|T\|^2 \sum_{n=1}^{\infty} d_m^n \langle A^n A^n R_m^{-1} x, R_m^{-1} x \rangle
\]

\[
= C^2 \|T\|^2 \langle R_m R_m^{-1} x, R_m^{-1} x \rangle
\]

\[
= C^2 \|T\|^2 \|x\|^2.
\]

Thus \( T \in B_A \). \( \square \)

From this we deduce an easy consequence.

**Corollary 2.4.** Let \( T \) be an operator such that \( AT = \lambda TA \) for some complex number \( \lambda \) with \( |\lambda| \leq 1 \). Then \( T \in B_A \). In particular \( B_A \) contains the commutant of \( A \).

In fact, the proof of Proposition 2.3 reveals that a stronger assertion is true.

**Proposition 2.5.** Suppose that \( A \) is an operator and \( B \in B \). If \( AT = BTA \) and \( \sup_m \|B^n T\| < \infty \), then \( T \in B_A \).

It follows from Corollary 2.4 that \( \{A\}' \subset B_A \). On the other hand, Example 2.1 shows that the inclusion can be proper, since \( \{A\}' \) is always closed. This leads to the question whether one can have \( B_A = \mathcal{L}(\mathcal{H}) \). It is easy to see that this happens when, for example, \( A \) is a scalar multiple of the identity. The following result gives a characterization of such operators in terms of the sequence \( \{R_m\} \).

**Theorem 2.6.** Let \( A \) be an operator in \( \mathcal{L}(\mathcal{H}) \). Then \( B_A = \mathcal{L}(\mathcal{H}) \) if and only if \( \sup_m \|R_m\| \|R_m^{-1}\| < \infty \). If \( B_A = \mathcal{L}(\mathcal{H}) \) then, for every nonzero operator \( T \), \( \lim \sup_m \|R_m T R_m^{-1}\| > 0 \).
This shows that sup \( m \| R_m \| \| R_m^{-1} \| \) < \( \infty \) then \( B_A = \mathcal{L}(\mathcal{H}) \). Conversely, suppose that \( B_A = \mathcal{L}(\mathcal{H}) \). Then \( B_A \) is norm closed so the proof of Lemma 2.2 shows that, for all \( m \in \mathbb{N} \) and all \( T \in \mathcal{L}(\mathcal{H}) \), \( \| G_m(T) \| \leq C \| T \| \). Here \( G_m \) and \( C \) have the same meaning as in the proof of Lemma 2.2. Thus \( \| G_m \| = \| R_m \| \| R_m^{-1} \| \), and sup \( m \| R_m \| \| R_m^{-1} \| \) < \( \infty \).

As for the last assertion of the theorem, suppose that \( B_A = \mathcal{L}(\mathcal{H}) \) and that there exists a nonzero operator \( T \) such that \( \lim sup_m \| R_m TR_m^{-1} \| = 0 \). Obviously, \( \lim_m \| R_m TR_m^{-1} \| = 0 \). It follows that \( B_A T \subset B_A \cap \mathcal{Q} \). Indeed, if \( X \in B_A \) then \( XT \in B_A (= \mathcal{L}(\mathcal{H})) \). In addition, \( r(XT) = r(R_m XTR_m^{-1}) \leq \| R_m XTR_m^{-1} \| \leq \| R_m XR_m^{-1} \| \| R_m TR_m^{-1} \| \to 0 \), so \( XT \in \mathcal{Q} \). Let \( K \) be a compact operator such that \( K' = KT \) is nonzero. Then \( \mathcal{L}(\mathcal{H}) K' \subset \mathcal{Q} \) which leads to a contradiction since, by Proposition 1.4, this would imply that \( \mathcal{L}(\mathcal{H}) \) has a n.i.s.}

Theorem 2.6 can be used to compute a spectral algebra in the following situation.

**Corollary 2.7.** Let \( p \) be a positive integer, and let \( \lambda \in \mathbb{C} \). If \( A^p = \lambda I \), then \( B_A = \mathcal{L}(\mathcal{H}) \).

**Proof.** In order to exploit the defining property of \( A \) we write

\[
R_m^2 = \sum_{n=0}^{\infty} d_{2n}^m A^n A^n = \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} d_{kp+j}^m A^{kp+j} A^{kp+j}.
\]

Clearly, \( A^{kp} A^{k'p} = |\lambda|^{2k} \), so

\[
R_m^2 = \sum_{j=0}^{p-1} d_{2j}^m A^{kj} A^{kj} \sum_{k=0}^{\infty} d_{kp}^m |\lambda|^{2k}.
\]

Notice that \( d_m < 1/r(A) = 1/|\lambda|^{1/p} \) so the infinite series above converges \( \forall m \in \mathbb{N} \). Thus,

\[
\| R_m \| \| R_m^{-1} \| = \left( \sum_{j=0}^{p-1} d_{2j}^m A^{kj} A^{kj} \right)^{-1} = \left( \sum_{j=0}^{p-1} d_{kp}^m |\lambda|^{2k} \right)^{-1}.
\]

This shows that sup \( m \| R_m \| \| R_m^{-1} \| < \infty \) so that \( B_A = \mathcal{L}(\mathcal{H}) \).}

It appears to be quite difficult to find explicit descriptions of the operators in \( B_A \) for a given operator \( A \). We now illustrate the level of difficulty one should expect by describing a spectral algebra for a particularly simple operator.

**Proposition 2.8.** If \( u \) and \( v \) are unit vectors then \( B_{u \otimes v} = \{ T \in \mathcal{L}(\mathcal{H}) : v \text{ is an eigenvector for } T^* \} \).

**Proof.** Let \( A = u \otimes v \) be a rank one operator, where \( u \) and \( v \) are unit vectors.

One knows that \( r(u \otimes v) = \| \langle u, v \rangle \| \). A calculation shows that, for \( n \in \mathbb{N} \), \( A^n = \langle u, v \rangle^{n-1} u \otimes v \) and \( A^n A^n = r^{2n-2} v \otimes v \). Therefore,

\[
R_m^2 = I + \left( \sum_{n=1}^{\infty} d_{2n}^m r^{2n-2} \right) v \otimes v = I + \frac{d_m^2}{1 - d_m^2 r^2} v \otimes v.
\]
Let $\lambda_m = \sqrt{1 + \frac{d_m^2}{(1 - d_m^2 r^2)}}$ for every $m \in \mathbb{N}$. Notice that $\lambda_m \to \infty$ as $m \to \infty$. Indeed, either $d_m \to 1/r$ or, if $A$ is quasinilpotent, $\lambda_m = \sqrt{1 + m^2}$. If we decompose $\mathcal{H}$ as a direct sum of $\mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is the one dimensional space spanned by $v$ and $\mathcal{H}_2$ is the orthogonal complement of $\mathcal{H}_1$, then the corresponding block matrix of $R_m$ is

$$R_m = \begin{pmatrix} \lambda_m & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$R_m^{-1} = \begin{pmatrix} 1/\lambda_m & 0 \\ 0 & 1 \end{pmatrix},$$

where, in actuality,

$$\begin{pmatrix} \lambda_m & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_m I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}.$$

If $T$ is an arbitrary operator, say $T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, then

$$R_m TR_m^{-1} = \begin{pmatrix} \lambda_m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} 1/\lambda_m & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X & Y \lambda_m \\ Z/\lambda_m & W \end{pmatrix}$$

and it is easy to see that $\sup_m \|R_m TR_m^{-1}\| < \infty$ if and only if $Y = 0$. This means that $\mathcal{H}_2$ is invariant for $T$ or, equivalently, that $\mathcal{H}_1$ is invariant for $T^*$. The observation that this happens if and only if $v$ is an eigenvector for $T^*$ completes the proof.

3. Quasinilpotent Operators

Proposition 1.4 shows that there is a connection between the class $Q$ of quasinilpotent operators and the existence of a n. i. s. for an algebra $A$. In this section we will explore this relationship in the case when $A$ is a spectral algebra. We start by introducing an important subset of $\mathcal{B}_A$. Let

$$Q_A = \{ \mathcal{L}(\mathcal{H}) : \|R_m TR_m^{-1}\| \to 0 \}.$$

Lemma 3.1. $Q_A$ is a two sided ideal in $\mathcal{B}_A$ and every operator in $Q_A$ is quasinilpotent. Furthermore, if $A \in Q$ then $A \in Q_A$.

Proof. Let $T \in Q_A$ and let $X \in \mathcal{B}_A$. Then

$$\|R_m TXR_m^{-1}\| \leq \|R_m TR_m^{-1}\| \|R_m XR_m^{-1}\| \to 0$$

so $Q_A$ is a right ideal. Since the same estimate holds for $XT$ we see that $Q_A$ is a two sided ideal in $\mathcal{B}_A$. On the other hand $r(T) = r(R_m TR_m^{-1}) \leq \|R_m TR_m^{-1}\|$ which shows that if $T \in Q_A$ then it must be quasinilpotent. Finally, if $A \in Q$ then $r(A) = 0$ and the fact that $\|R_m AR_m^{-1}\| \to 0$ follows from Proposition 1.5.

Remark 3.2. The ideal $Q_A$ need not contain every quasinilpotent operator in $\mathcal{B}_A$. Indeed, if $A$ is the unilateral forward shift a calculation shows that $R_m^2 = 1/(1 - d_m^2)$. Since every operator commutes with a scalar multiple of the identity it follows that $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$. On the other hand, $\|R_m TR_m^{-1}\| = \|T\|$ for any $T$ in $\mathcal{L}(\mathcal{H})$, so $Q_A = (0)$. 

Remark 3.3. Combining Remark 3.2 and Example 2.1 we see that algebras $B_A$ and $B_{A^*}$ can have quite different properties.

The following result justifies our interest in $Q_A$.

**Theorem 3.4.** If $Q_A \neq (0)$ and there exists a nonzero compact operator in $B_A$, then $B_A$ has a n. i. s.

**Proof.** Let $K$ be a nonzero compact operator in $B_A$. Without loss of generality we may assume that $QK = 0$ for every $Q \in Q_A$. Indeed, if $QK \neq 0$ for some $Q \in Q_A$, then $QK$ is a compact quasinilpotent operator with the property that $B_A QK \subset Q$ and the result follows from Proposition 1.4.

Let $Q$ be a fixed nonzero operator in $Q_A$ and let $T$ be an arbitrary operator in $B_A$. Then $QT \in Q_A$ and, hence, $QTK = 0$. Since $K \neq 0$ there is a nonzero vector $z$ in the range of $K$. Clearly, $QTz = 0$ so $Tz \in \ker Q$ for all $T \in B_A$. Naturally, the closure of the subspace $\{Tz : T \in B_A\}$ is an invariant subspace for $B_A$. It is nonzero since $z \neq 0$ and the identity operator is in $B_A$. Finally, it is not $\mathcal{H}$ since it is contained in the kernel of a nonzero operator $Q$.

From Theorem 3.4 we deduce some easy consequences.

**Corollary 3.5.** Suppose that $A$ is a quasinilpotent operator, $B$ is a power bounded operator commuting with $A$, and $K$ is a nonzero compact operator satisfying $AK = BKA$. Then $B_A$ has a n. i. s.

**Proof.** By Proposition 2.3, $K$ is in $B_A$. Since $A \in Q$, Lemma 3.1 shows that $A \in Q_A$. The result then follows from Theorem 3.4.

**Corollary 3.6.** Suppose that $A$ is a quasinilpotent operator, $\lambda$ is a complex number, and $K$ is a nonzero compact operator satisfying $AK = \lambda KA$. Then either $B_A$ or $B_{A^*}$ has a n. i. s. In any case, $A$ has a proper hyperinvariant subspace.

**Proof.** If $|\lambda| \leq 1$ Corollary 3.5 implies that $B_A$ has a n. i. s. For $|\lambda| > 1$, we have $A^*K^* = (1/\lambda)K^*A^*$ so the same argument shows that $B_{A^*}$ has a n. i. s. If $\mathcal{M}$ is such a subspace then it is hyperinvariant for $A^*$. It follows that $\mathcal{M}^\perp$ is a proper hyperinvariant subspace for $A$.

**Remark 3.7.** The existence of a proper hyperinvariant subspace was established under a weaker condition (without the assumption that $A$ is quasinilpotent) independently by Scott Brown in [sB79] and Kim, Pearcy, and Shields in [cP78]. By making a stronger hypothesis we obtain an invariant subspace for an effectively larger class of operators. In addition, Corollary 3.5 is, to the best of our knowledge, new.

Clearly, when $A \in Q$, the ideal $Q_A \neq (0)$. Thus, Theorem 3.4 raises a question whether there always is a nonzero compact operator in the algebra $B_A$ for $A \in Q$. Since rank one operators are the most basic examples of compact operators it is natural to try to establish the membership of such operators in $B_A$. The following result shows that the issue is more subtle than it might appear at first glance.

**Proposition 3.8.** Let $A \in \mathcal{L}(\mathcal{H})$. The following are equivalent:
Proposition 3.11. The sequence \{\|R_m\|\} diverges to +\infty.
Proof. The case when $A$ is quasinilpotent is easy since, in that case, $d_m = m$ and, for any $x$,
\[
\|R_m x\|^2 = \sum_{n=0}^{\infty} m^{2n} \|A^n x\|^2 \geq m^2 \|Ax\|^2.
\]
So we assume that $A$ is not quasinilpotent.

For the remainder of the proof $r$ will stand for the spectral radius of $A$. Let $N$ be an arbitrary positive integer. The sequence \( \left\{ \frac{rm}{rm+1} \right\} \) converges to 1, as $m \to \infty$, so there exists $m_0$ such that, for $m \geq m_0$,
\[
\frac{rm}{rm+1} > \frac{1}{2\sqrt{2}}.
\]
We choose $\epsilon > 0$ such that
\[
\epsilon \left( \sum_{n=1}^{N} \frac{2n\|A\|^{n-1}}{r^n} \right) < 1.
\]
Since we are assuming that $r > 0$ there exists $\lambda$ such that $|\lambda| = r$ and $\lambda$ belongs to $\sigma_l(A)$ — the left spectrum of $A$. This means that there exists a unit vector $y \in \mathcal{H}$ such that $\|Ay - \lambda y\| < \epsilon$. Using the triangle inequality and the fact that $|\lambda| \leq \|A\|$ we obtain that $\|A^2y - \lambda^2y\| < 2\epsilon\|A\|$ and, inductively, that $\|A^n y - \lambda^n y\| < n\epsilon\|A\|^{n-1}$. The last inequality, coupled with the fact that $|\lambda| = r$, implies that, for $n \in \mathbb{N}$,
\[
\|A^n y\| > r^n \|y\| - n\epsilon\|A\|^{n-1} = r^n - n\epsilon\|A\|^{n-1}.
\]
Let $m \geq m_0$. Since all the summands in the formula for $\|R_m y\|^2$ are nonnegative and $\|y\| = 1$ we obtain the estimate
\[
\|R_m y\|^2 \geq \sum_{n=1}^{N} \left( \frac{m}{rm+1} \right)^{2n} (r^n - n\epsilon\|A\|^{n-1})^2
\]
\[
\geq \sum_{n=1}^{N} \left( \frac{m}{rm+1} \right)^{2n} (r^{2n} - 2n\epsilon\|A\|^{n-1} r^n)
\]
\[
\geq \sum_{n=1}^{N} \left( \frac{rm}{rm+1} \right)^{2n} - \sum_{n=1}^{N} 2n\epsilon\|A\|^{n-1} \frac{1}{r^n}
\]
\[
\geq \sum_{n=1}^{N} \left( \frac{rm}{rm+1} \right)^{2n} - 1
\]
\[
\geq N \left( \frac{rm}{rm+1} \right)^{2N} - 1
\]
\[
\geq N \cdot \frac{1}{2} - 1.
\]
This implies that, for $m \geq m_0$, $\|R_m\| \geq \frac{N}{2} - 1$ and the result follows. \( \Box \)
4. Compactness and $B_K$

In this section we consider the algebra $B_K$ in the case when $K$ is a compact operator. We will prove that, in this situation, $B_K$ must have a n. i. s. Since this algebra contains the commutant $\{K\}'$ of $K$ we obtain a result that is at least as strong as Theorem 1.1. However, it is a step forward only if $B_K$ is strictly larger than $\{K\}'$. We will examine this relationship in detail.

Before we start, we briefly recall some properties of compact operators. If $K$ is a compact operator its spectrum $\sigma(K)$ is a countable set containing 0 (we are assuming that $H$ is infinite dimensional), and $\sigma(K) \setminus \{0\}$ is either finite or forms a sequence converging to 0. If $\lambda \in \sigma(K) \setminus \{0\}$, then $\lambda$ is an eigenvalue for $K$, and the corresponding eigenspace is finite dimensional. Also, if $r(K) > 0$, then there is an eigenvalue $\lambda$ for $K$ for which $|\lambda| = r(K)$.

The following is the main result of this paper. In fact, it is an open question whether it is the invariant subspace theorem.

**Theorem 4.1.** Let $K$ be a nonzero compact operator on the separable, infinite dimensional Hilbert space $H$. Then $B_K$ has a n. i. s.

**Proof.** We will show that $Q_K \neq (0)$. The result will then follow from Theorem 3.4. Of course, if $K$ is quasinilpotent there is nothing to prove so, for the rest of this proof, we will assume that $r(K) > 0$.

In order to show that $Q_K \neq (0)$ it suffices to exhibit a rank one operator $x \otimes y$ with $\sup_m \|R_m x\| < \infty$ and $\lim_m \|R_m^{-1} y\| = 0$. A vector $y$ with the desired property is supplied by the following lemma.

**Lemma 4.2.** Suppose that $K$ is compact and $r(K) > 0$. Then there is a unit vector $v$ such that $\lim_m \|R_m^{-1} v\| = 0$.

**Proof.** Let $\lambda$ be a complex number in $\sigma(K)$ such that $|\lambda| = r(K)$. Then $X \in \sigma(K^*)$ so there are unit vectors $u$ and $v$ for which $Ku = \lambda u$ and $K^* v = \bar{\lambda} v$. An easy calculation shows that $K (u \otimes v) = (u \otimes v) K$ so that $u \otimes v \in \{K\}' \subset B_A$. It then follows that $\sup_m \|R_m u\| \|R_m^{-1} v\| < \infty$. On the other hand, a straightforward calculation shows that $\|R_m u\| \to \infty$, $m \to \infty$. Since $\sup_m \|R_m u\| \|R_m^{-1} v\| < \infty$ it must follow that $\|R_m^{-1} v\| \to 0$.

So it remains to provide a nonzero vector $x$ with the property that

$$\sup_m \|R_m x\| < \infty. \tag{4.1}$$

To that end, it suffices for $x$ to satisfy

$$\limsup_n \|K^n x\|^{1/n} < r(K). \tag{4.2}$$

Indeed, (4.2) implies that the power series $\sum_{n=0}^\infty \|K^n x\|^2 z^n$ has radius of convergence bigger than $1/r^2$ and, consequently, the series $\sum_n \|K^n x\|^2 / r^{2n}$ converges. Since

$$\|R_m x\|^2 = \sum_{n=0}^\infty \left( \frac{m}{1 + mr} \right)^{2n} \|K^n x\|^2$$
and \( \{ m/(1 + mr) \} \) is an increasing sequence converging to \( 1/r \), we see that (4.2) implies (4.1).

It is not hard to see that, if \( K \) has an eigenvalue \( \lambda \) with the property that \( |\lambda| < r(K) \), then any eigenvector corresponding to \( \lambda \) satisfies (4.2). Thus we may assume that 0 is an isolated point of \( \sigma(K) \). (Of course, 0 \( \in \sigma(K) \) since \( \mathcal{H} \) is infinite dimensional.) Let \( \Gamma \) be a positively oriented circle around the origin such that 0 is the only element of \( \sigma(K) \) inside the circle, and let
\[
P = -\frac{1}{2\pi i} \int_{\Gamma} (K - \lambda I)^{-1} d\lambda.
\]
One knows (cf., [RS55]) that \( P \) is a (not necessarily selfadjoint) projection operator that commutes with \( K \) and that the restriction \( K_0 \) of \( K \) to the invariant subspace \( P\mathcal{H} \) is quasinilpotent. It follows that, if \( x \) is a unit vector in \( PH \), then \( ||K^nx||^{1/n} \rightarrow 0 \). This completes the proof of the theorem.

As mentioned earlier, the presence of proper invariant subspaces for \( B_K \) (\( K \) compact) is an advancement in invariant subspace theory only if \( B_K \neq \{ K \} \). We do not know at the present time if \( B_K \) can equal \( \{ K \} \) for a compact nonzero operator \( K \) on an infinite dimensional space. We do know that the answer is no if \( K \) has positive spectral radius.

**Proposition 4.3.** Let \( K \) be a compact operator on an infinite dimensional Hilbert space such that \( r(K) > 0 \). Then \( B_K \neq \{ K \} \).

**Proof.** Notice that the vectors \( x \) and \( y \) obtained in the proof of Theorem 4.1 satisfy (4.1) and \( K^*y = \bar{\lambda}y \), with \( |\lambda| = r(K) \). Since it was established that \( x \otimes y \in B_K \) it suffices to prove that \( K(u \otimes v) \neq (u \otimes v)K \). This follows from the fact that \( Kx \neq \lambda x \) which is a simple consequence of (4.1).

When \( K \) is a quasinilpotent operator it may not have any eigenvalues so the method employed in the proof of Proposition 4.3 is not available. Therefore, we introduce another technique which can be applied to any operator regardless of its spectral radius. This line of approach consists of solving the operator equation
\[
AX = \lambda XA
\]
for some \( \lambda \neq 1 \), with \( |\lambda| \leq 1 \). If there is a nonzero operator \( X \) and a scalar \( \lambda \) as above that satisfy equation (4.3) we say that \( A \)-commutes with \( X \) and that \( \lambda \) is an extended eigenvalue for \( A \). Such an operator \( X \) is in \( \mathcal{B}_A \) by Corollary 2.4 and it is clear that it does not commute with \( A \), unless \( AX =XA = 0 \).

Equation (4.3) appeared in the work of several authors. In particular, it was shown independently by S. Brown [sB79] and Kim, Pearcy, and Shields [cP78] that, if \( A \) is compact and \( X \) satisfies (4.3) for some complex \( \lambda \), then \( X \) has a nontrivial hyperinvariant subspace. Recently, this equation was the object of study in [BLP01], [BLP01a], [vL97] and, with a somewhat different goal, in [CP01].

On the surface it might appear that finding an operator \( X \) that satisfies equation (4.3) with \( A \) compact does not yield anything new since the existence of a hyperinvariant subspace has already been established for such an operator \( X \).
Therefore, we stop to carefully explain the difference. First, the presence of such a solution implies that $B_A$ properly contains $\{A\}'$. Hence, there exists an operator $X$ in $B_A$ that does not commute with $A$. Furthermore, $B_A$ is an algebra. This means that the operator $X + A$ is in $B_A$, yet it does not commute with either $X$ or $A$. Thus, Theorem 4.1 is indeed a new result. In addition, our techniques yield the following generalization that was not known previously.

**Corollary 4.4.** Suppose $K$ is a nonzero compact operator, $B$ is a power bounded operator commuting with $K$, and $T$ is an operator for which $KT = BTK$. Then $T \in B_K$ so that $T$ has a n. i. s.

**Proof.** The result follows directly from Proposition 2.3 and Theorem 4.1.

The following result provides an application of Corollary 4.4.

**Corollary 4.5.** Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of complex numbers such that $\{\alpha_n\}$ converges to 0 and $\{\beta_n\}$ is bounded with $\sup_n |\beta_n| \leq 1$. For a given orthonormal basis $\{e_n : n \geq 0\}$ let $A$ consist of all bounded operators $T$ whose matrix $(t_{ij})$ relative to this basis satisfies $(\alpha_i - \alpha_j \beta_i)t_{ij} = 0$ for all $i, j \geq 0$. Then $A$ has a n. i. s.

**Proof.** Let $K = \text{diag}(\alpha_n)$, $B = \text{diag}(\beta_n)$. Then $K$ is compact, $B$ is a contraction (hence power bounded) commuting with $K$, and $KT = BTK$, $T \in A$. The result now follows from Corollary 4.4.

In view of Proposition 4.3 it is of interest to solve equation (4.3) in the case when $A$ is a compact, quasinilpotent operator in $L(H)$. One knows (cf., [GK]) that every such operator on a Hilbert space is unitarily equivalent to a model operator — a specific integral operator on the space $L_2^r(0, 1)$ of vector valued functions with values in $\mathbb{C}^r$, for some positive integer $r$. In [BLP01] we have, in collaboration with A. Biswas, initiated the study of compact quasinilpotent operators and the associated spectral algebras, by considering a specific operator. Namely, let $H = L^2(0, 1)$ — the space of square integrable functions on $[0, 1]$ with respect to Lebesgue measure and let $A = V$, the Volterra integral operator on $L^2(0, 1)$, defined by

$$Vf(x) = \int_0^x f(t) \, dt.$$  

In particular, we established in [BLP01] that the set of extended eigenvalues of the Volterra operator $V$ is precisely the set $(0, \infty)$. Since $V$ has trivial kernel, an operator $X$ satisfying $VX = \lambda XV$ (for $\lambda \neq 1$) cannot commute with $V$. Therefore, $B_V \neq \{V\}'$. Based on Proposition 4.3 and our results with the Volterra operator we make the following conjecture.

**Conjecture 4.6.** The commutant of a nonzero compact operator is a proper subalgebra of the associated spectral algebra.

So far we were concerned with showing that our result is indeed a new one, i. e., that $\{K\}' \neq B_K$. Assuming for a moment that this is indeed true, we are
now facing the possibility that this is in actuality the invariant subspace theorem. Namely, this would create a challenge of constructing an operator that belongs to no algebra $B_K$ for any compact operator $K$. A similar situation initially arose in connection with the result of Lomonosov and was eventually settled in [HNRR80] where an operator was exhibited to which the theorem did not apply. Notice that the example furnished in that paper was a so-called quasi-analytic shift. Yet, such an operator $W$ belongs to a spectral algebra associated to a rank one operator. Indeed, let $\{e_n : n \geq 0\}$ be the orthonormal basis that $W$ shifts. Then $e_0$ is an eigenvector for $W^*$ (corresponding to the eigenvalue 0) so, by Proposition 2.8, $W$ belongs to the spectral algebra associated to $u \otimes e_0$ for arbitrary nonzero vector $u$. Thus, it is natural to ask the following question.

**Problem 4.7.** Is there an operator $A$ that is not in $B_K$ for any nonzero compact operator $K$?

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