

A Moment Matrix Approach to Multivariable Cubature

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Abstract. We develop an approach to multivariable cubature based on positivity, extension, and completion properties of moment matrices. We obtain a matrix-based lower bound on the size of a cubature rule of degree $2n + 1$; for a planar measure μ , the bound is based on estimating $\rho(C) := \inf\{\text{rank}(T - C) : T \text{ Toeplitz and } T \geq C\}$, where $C := C^\sharp[\mu]$ is a positive matrix naturally associated with the moments of μ . We use this estimate to construct various minimal or near-minimal cubature rules for planar measures. In the case when $C = \text{diag}(c_1, \dots, c_n)$ (including the case when μ is planar measure on the unit disk), $\rho(C)$ is at least as large as the number of gaps $c_k > c_{k+1}$.

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1. Introduction

Let μ denote a positive Borel measure on \mathbb{R}^d having convergent power moments up to at least degree m . Corresponding to a multi-index $i \equiv (i_1, \dots, i_d) \in \mathbb{Z}_+^d$ with total degree $|i| \equiv i_1 + \dots + i_d \leq m$, let β_i denote the i -th power moment of μ , i. e.,

$$\beta_i = \int_{\mathbb{R}^d} t^i d\mu(t) \equiv \int_{\mathbb{R}^d} t_1^{i_1} \dots t_d^{i_d} d\mu(t_1, \dots, t_d),$$

where $t = (t_1, \dots, t_d) \in \mathbb{R}^d$; by assumption, the latter integral is absolutely convergent. A *cubature rule* for μ of degree m and size N consists of nodes x_1, \dots, x_N

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in \mathbb{R}^d and positive *weights* ρ_1, \dots, ρ_N such that

$$\int_{\mathbb{R}^d} p(t) d\mu(t) = \sum_{k=1}^N \rho_k p(x_k)$$

for each polynomial p in \mathcal{P}_m^d (the complex vector space of polynomials in real variables t_1, \dots, t_d with total degree at most m); note that $\vartheta(d, m) \equiv \dim \mathcal{P}_m^d = \binom{d+m}{m}$.

Two recurrent themes in cubature literature are the estimation of the fewest nodes possible in a cubature rule of prescribed degree, and the construction of rules with the fewest nodes possible (cf., [C1], [Mo1] – [Mo3], [My1] – [My3], [My5], [My6], [P2], [R], [S], [Str1] – [Str4], [SX], [X1] – [X3]). In [R], Radon introduced the technique of constructing minimal cubature rules whose nodes are common zeros of multivariable orthogonal polynomials. In the 1960s and 1970s this approach was refined and extended by many authors, particularly Stroud [Str1] – [Str4], Mysovskikh [My1] – [My3], [My5], [My6], and Möller [Mo1] – [Mo3]. More recently, Xu [X1] – [X3] further extended this approach using multivariable ideal theory, and Putinar [P2] has presented a development of cubature based on operator dilation theory. In the present note, we introduce still another approach to the estimation problem, based on positivity and extension properties of the *moment matrix* $M(\lfloor \frac{m}{2} \rfloor)[\mu]$ that we associate to μ . This approach emerges naturally from a recent study of multivariable truncated moment problems by R. Curto and the first-named author (cf., [CF1], [CF2], [CF3]); for terminology and notation concerning moment matrices, see below and Section 2.

Suppose μ (as above) is *square positive*, i. e., if $f \in \mathcal{P}_{\lfloor \frac{m}{2} \rfloor}^d$ and $f \neq 0$, then $\int |f|^2 d\mu > 0$. For this case, the following well-known result provides a basic lower estimate for the number of nodes N in any cubature rule for μ of degree m (cf., [Str1], [C1], [S], [SX]). Namely, if μ is square positive, then

$$N \geq \vartheta(d, \lfloor \frac{m}{2} \rfloor). \quad (1.1)$$

Following [SX], we say that a cubature rule is *Gaussian* if equality holds in (1.1). This terminology is consistent with classical Gaussian quadrature for Lebesgue measure on $[a, b] \subset \mathbb{R}$; indeed, with $d = 1$ and $m = 2n + 1$, Gaussian quadrature provides a minimal cubature rule with $n + 1$ ($= \vartheta(1, \lfloor \frac{m}{2} \rfloor$) nodes (cf., [Str4]).

For the general case, where μ is not necessarily square positive, the following Radon-Stroud estimate (cf., [C1, Theorem 7.1]) provides a basic lower bound.

Theorem 1.1. ([R], [Str1]) $N \geq \dim \mathcal{P}_{\lfloor \frac{m}{2} \rfloor}^d | \text{supp } \mu$.

Here, for $k > 0$, $\mathcal{P}_k^d | \text{supp } \mu := \{p | \text{supp } \mu : p \in \mathcal{P}_k^d\}$; note that $\dim \mathcal{P}_k^d | \text{supp } \mu \leq \dim \mathcal{P}_k^d$, and in the square positive case, $\dim \mathcal{P}_k^d | \text{supp } \mu = \dim \mathcal{P}_k^d$.

Assume now that μ is square positive, with finite moments of all orders, and $m = 2n + 1$. A fundamental result of Mysovskikh [My5] characterizes the existence of Gaussian rules.

Theorem 1.2. ([My5], cf., [X2, Theorem 5.3]) *A square positive measure μ on \mathbb{R}^d admits a Gaussian rule of degree $2n + 1$ if and only if the orthogonal polynomials of degree $n + 1$ in $L^2(\mu)$ have $\binom{n+d}{d}$ common zeros (which then form the support of such a rule).*

For $d > 1$, Gaussian rules are uncommon. Indeed, Möller [Mo1] – [Mo3] developed a general theory of lower bounds and obtained several types of estimates for the size N of a cubature rule of odd degree $2n + 1$. Some estimates are based on ideal theory and orthogonal polynomials, e.g. [Mo3, Theorem 2] (cf. [C1, Theorem 8.6] [CMS, Theorem 11]). Another type of estimate, valid when μ is *centrally symmetric*, (i. e., $\beta_i = 0$ whenever $|i|$ is odd), shows that

$$N \geq \begin{cases} 2 \dim G_n - 1 & \text{if } n \text{ is even and } 0 \text{ is a node,} \\ 2 \dim G_n & \text{otherwise,} \end{cases}$$

where G_{2k} is the space of even polynomials in $\mathcal{P}_{2k+1}^d|_{\text{supp } \mu}$ and G_{2k+1} is the space of odd polynomials in $\mathcal{P}_{2k+1}^d|_{\text{supp } \mu}$ [Mo3, Theorem 3] (cf. [C1, Theorem 8.3] [CMS, Theorem 13]). These estimates are particularly concrete in the planar case of centrally symmetric measures, where both types of estimates may be expressed as follows.

Theorem 1.3. (Möller [Mo2]) *If μ is a square positive, centrally symmetric measure on \mathbb{R}^2 , then the size N of any cubature rule for μ of degree $2n + 1$ satisfies*

$$N \geq \frac{(n+1)(n+2)}{2} + \lfloor \frac{n+1}{2} \rfloor.$$

It follows immediately from Theorem 1.3 that μ admits no Gaussian rule of degree $2n + 1$; for classes of non-centrally symmetric measures on \mathbb{R}^2 with Gaussian rules of arbitrarily large degree, see Schmid-Xu [SX] (cf. also Schmid [S]). In [Mo2], Möller also characterized the cubature rules that attain the lower bounds of [Mo2] (cf. Section 5 below); subsequently, the theory of lower estimates and minimal rules developed in several directions, e. g., [Mo3], [CS], [X1] – [X3], [S]; many of these developments are discussed in the surveys of Xu [X2], Cools [C1], and Cools et. al. [CMS].

Our moment matrix approach is based on the observation that for a positive Borel measure μ on \mathbb{R}^d with convergent moments $\beta_i = \int t^i d\mu$, $|i| \leq m$, the existence of a cubature rule for μ of degree m is equivalent to the existence of a finitely atomic representing measure ν in the following Truncated Multivariable Moment Problem for $\beta \equiv \beta^{(m)}[\mu] = \{\beta_i\}_{|i| \leq m}$:

$$\beta_i = \int t^i d\nu, \quad |i| \leq m, \quad \nu \geq 0, \quad \text{supp } \nu \subset \mathbb{R}^d. \quad (1.2)$$

Following a line of results beginning with Tchakaloff's Theorem [T], and including generalizations due to Mysovskikh [My1] and Putinar [P1], in [CF5, Theorem 1.4] it was proved that if μ has convergent moments up to at least order $m + 1$, then μ

admits an *inside cubature rule* of degree m , with size $\leq 1 + \dim(\mathcal{P}_m^d | \text{supp } \mu)$. (An inside rule is one for which each node is contained in $\text{supp } \mu$.)

Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $m = 2n$. The moment data $\beta \equiv \beta^{(2n)}[\mu]$ correspond to a *moment matrix* $M(n) \equiv M_{\mathbb{R}^d}(n)[\mu]$ defined as follows. Consider the basis B_n^d for \mathcal{P}_n^d consisting of the degree lexicographic ordering of monomials (for $d = 2$, the ordering is $1, x_1, x_2, x_1^2, x_1x_2, x_2^2, \dots, x_1^n, x_1^{n-1}x_2, \dots, x_2^n$); for $p \in \mathcal{P}_n^d$, let \hat{p} denote the coefficient vector of p relative to B_n^d . $M(n)$ has $\vartheta(d, n)$ rows and columns and is uniquely determined by

$$\langle M(n)\hat{p}, \hat{q} \rangle = \int p\bar{q} d\mu, \quad p, q \in \mathcal{P}_n^d. \quad (1.3)$$

See Section 2 for other descriptions of $M(n)$, which we sometimes denote as $M(n)[\mu]$ to emphasize μ ; since $\mu \geq 0$, then (1.3) immediately implies that $M(n)[\mu] \geq 0$ (cf., [CF2, (3.2), p. 15]).

Our first moment matrix estimate concerns the “even” case, $m = 2n$.

Proposition 1.4. (Cf. Proposition 3.1.) *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $m = 2n$. The size N of any cubature rule for μ of degree m satisfies $N \geq \text{rank } M(n)[\mu]$.*

If μ is square positive, then $M(n)$ is invertible (cf. Proposition 2.9 or (1.3)), so $\text{rank } M(n) = \binom{n+d}{d}$, whence Proposition 1.4 recovers (1.1) for m even. In the general case, we show in Section 2 (Proposition 2.8 below) that $\text{rank } M(n)[\mu] = \dim \mathcal{P}_n^d | \text{supp } \mu$, so Proposition 1.4 recovers the “even” case of the Radon-Stroud lower bound in Theorem 1.1. The following result shows that the estimate established in Proposition 1.4 is sharp.

Theorem 1.5. *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $2n$. Then μ has a cubature rule of degree $2n$ with (minimal) size $N = \text{rank } M(n)[\mu]$ if and only if $M(n)[\mu]$ can be extended to a moment matrix*

$$M(n+1) \equiv \begin{pmatrix} M(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix}$$

*satisfying $\text{rank } M(n+1) = \text{rank } M(n)[\mu]$; equivalently, there is a choice of “new moments” of degree $2n+1$ and a corresponding matrix W , such that $M(n)W = B(n+1)$ (i. e., $\text{Ran } B(n+1) \subset \text{Ran } M(n)$) and $W^*M(n)W$ is a moment matrix block (of degree $2n+2$).*

For planar Lebesgue measure restricted to such basic sets as a square, disk, or triangle, Gaussian rules of degree $2n$, having $(n+1)(n+2)/2$ nodes, exist only for very small values of n (cf. [C1] [C2]). By contrast, the measures studied by Schmid and Xu [SX] (op. cit.), have Gaussian rules of all degrees and are supported on a region of the plane with nonempty interior. Recently, we showed in [CF4] and [CF8] that if μ (as in Theorem 1.5) is supported in a parabola or ellipse in the plane, then μ always admits a Gaussian rule of degree $2n$ with size $N = \text{rank } M(n)[\mu]$.

In the sequel, we refer to a rank-preserving extension as described above as the *flat* moment matrix extension of $M(n)$ determined by $B(n+1)$, denoted by $[M(n); B(n+1)]$ (cf. Theorem 2.2 and Corollary 2.7). In the case of the real line, $d = 1$, a moment matrix is simply a Hankel matrix; in the planar case, $d = 2$, the block $C(n+1)$ is Hankel. [I, Theorem 11.1] gives a formula for the rank of an arbitrary Hankel matrix, and in [I, page 53] rank-preserving Hankel extensions of Hankel matrices are referred to as *singular extensions*. In the case of the complex plane \mathbb{C} that we consider below, a moment matrix block $C(n+1)$ is a Toeplitz matrix, and [I] contains a theory for rank-preserving Toeplitz extensions of Toeplitz matrices, and a formula for the rank of an arbitrary Toeplitz matrix [I, Theorem 15.1].

We prove Theorem 1.5 in Section 3 (Theorem 3.2). The following example illustrates how Theorem 1.5 can be used to construct minimal cubature rules; for certain details of the computational methods that we use, see Section 2.

Example 1.6. We use Theorem 1.5 to describe a family of 6-node (minimal) cubature rules of degree 4 for planar measure $\mu \equiv \mu_2$ restricted to the unit square $S = [0, 1] \times [0, 1]$. We have $\beta_{ij} = \left(\frac{1}{i+1}\right) \left(\frac{1}{j+1}\right)$, so

$$M(2) \equiv M(2)[\mu] = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/3 & 1/4 & 1/3 \\ 1/2 & 1/3 & 1/4 & 1/4 & 1/6 & 1/6 \\ 1/2 & 1/4 & 1/3 & 1/6 & 1/6 & 1/4 \\ 1/3 & 1/4 & 1/6 & 1/5 & 1/8 & 1/9 \\ 1/4 & 1/6 & 1/6 & 1/8 & 1/9 & 1/8 \\ 1/3 & 1/6 & 1/4 & 1/9 & 1/8 & 1/5 \end{pmatrix}.$$

From Theorem 1.5, a rank-preserving moment matrix extension $M(3)$ has the form

$$M(3) = \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix},$$

where

$$B(3) = \begin{pmatrix} 1/4 & 1/6 & 1/6 & 1/4 \\ 1/5 & 1/8 & 1/9 & 1/8 \\ 1/8 & 1/9 & 1/8 & 1/5 \\ u & a & v & b \\ a & v & b & z \\ v & b & z & c \end{pmatrix}$$

has the property that $C \equiv B(3)^* M(2)^{-1} B(3)$ is a Hankel matrix (which is the form of a moment matrix block for $d = 2$, cf. Section 2). The preceding system is too difficult to solve in general, so to simplify the system we assign “correct” values to $a \equiv \beta_{41} = 1/10$, $b \equiv \beta_{23} = 1/12$, and $c \equiv \beta_{05} = 1/6$. With these values, C is Hankel if and only if $C_{31} = C_{22}$ and $C_{42} = C_{33}$, i.e.,

$$\begin{aligned} 6481 - 32400u - 51840v + 388800uv - 311040v^2 - 32400z + 388800vz &= 0, \\ -3995 + 33696v - 388800v^2 + 51840z + 311040vz - 388800z^2 &= 0. \end{aligned}$$

These equations can be solved for u and z in terms of v provided $(126 - \sqrt{7})/1512 \leq v \leq (126 + \sqrt{7})/1512$ (approximately, $0.0815835 \leq v \leq 0.0850832$). For a numerical example, we set $v = (126 + \sqrt{7})/1512$. In $M(3)$, the columns are labelled $1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3$ (cf. Section 2). Since $\text{rank } M(3) = \text{rank } M(2)$ and $M(2) > 0$, we can compute polynomials $q_i(x, y)$ ($1 \leq i \leq 4$) of degree 2 such that in the column space of $M(3)$ we have $X^3 = q_1(X, Y)$, $X^2Y = q_2(X, Y)$, $XY^2 = q_3(X, Y)$, $Y^3 = q_4(X, Y)$. The *variety* of $M(3)$, $\mathcal{V} \equiv \mathcal{V}(M(3))$, is the set of common zeros of $p_1(x, y) = x^3 - q_1(x, y)$, $p_2(x, y) = x^2y - q_2(x, y)$, $p_3(x, y) = xy^2 - q_3(x, y)$, and $p_4(x, y) = y^3 - q_4(x, y)$ (cf. Section 2), and a calculation shows that $\mathcal{V} = \{z_i \equiv (x_i, y_i)\}_{i=0}^5$, where $z_0 \approx (0.940959, 0.0590414)$, $z_1 \approx (0.311018, 0.138127)$, $z_2 \approx (0.00142475, 0.5)$, $z_3 \approx (0.734, 0.5)$, $z_4 \approx (0.311018, 0.861873)$, $z_5 \approx (0.940959, 0.940959)$. Since $\text{card } \mathcal{V} = \text{rank } M(3) = \text{rank } M(2) = 6$, it now follows from Theorem 1.5 and the “real” version of Corollary 2.4 that μ has a (minimal) 6-node cubature rule of degree 4 of the form $\nu = \sum_{i=0}^5 \rho_i \delta_{z_i}$. The densities ρ_i may be computed from the Vandermonde-type equation $V(\rho_0, \rho_1, \dots, \rho_5)^t = (\beta_{00}, \beta_{10}, \beta_{01}, \beta_{20}, \beta_{11}, \beta_{02})^t$, where

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_0y_0 & x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 \\ y_0^2 & y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \end{pmatrix}.$$

Indeed, since $M(2)$ is invertible, the real version of Proposition 2.1 shows that V is invertible, and we find $\rho_0 = \rho_5 \approx 0.0642857$, $\rho_1 = \rho_4 \approx 0.22272727$, $\rho_2 \approx 0.09854228$, $\rho_3 \approx 0.3274317$. Note that each node of ν is inside S , and the same property holds if we use $v = (126 - \sqrt{7})/1512$; we have not examined whether the same holds for every intermediate value of v . \square

We next consider the “odd” case, where μ is a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $m = 2n + 1$. The matrix $M(n)$ (corresponding to moment data $\beta^{(2n)}$) admits a block decomposition $M(n) = (M_{ij})_{0 \leq i, j \leq n}$, where the entries of M_{ij} are the moments β_k of total degree $|k| = i + j$ (cf., Section 2). Since $m = 2n + 1$, we may similarly define blocks $M_{i, n+1}$, $0 \leq i \leq n$, and we set $B(n+1) \equiv B(n+1)[\mu] = (M_{i, n+1})_{0 \leq i \leq n}$. If μ has a cubature rule of degree $2n + 1$, then there is a matrix W such that $M(n)W = B(n+1)$ (cf. Proposition 2.6), in which case $C^\sharp(n+1) \equiv C^\sharp(n+1)[\mu] := W^*M(n)W$ is independent of W satisfying $M(n)W = B(n+1)$ (cf. the proof of Theorem 3.3). Now $C^\sharp(n+1)$ has the size of any d -dimensional moment matrix block of the form $H = M_{n+1, n+1}$. For any positive matrix S of this size, we set

$$\rho(S) = \inf\{\text{rank}(H - S) : H = M_{n+1, n+1} \geq S\}.$$

The following result (which is proved in Section 3 as Theorem 3.3) is our main existence theorem concerning minimal cubature rules of odd degree.

Theorem 1.7. *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $2n+1$. The size N of any cubature rule for μ of degree $2n+1$ satisfies*

$$N \geq N[n, \mu] \equiv \text{rank } M(n)[\mu] + \rho(C^\sharp(n+1)[\mu]).$$

Further, let $H = M_{n+1, n+1}$ be a moment matrix block satisfying $H \geq C^\sharp(\equiv C^\sharp(n+1)[\mu])$ and $\text{rank}(H - C^\sharp) = \rho(C^\sharp)$, and set

$$M_H(n+1) = \begin{pmatrix} M(n) & B(n+1)[\mu] \\ B(n+1)[\mu]^* & H \end{pmatrix}.$$

Then μ admits a cubature rule of degree $2n+1$ with minimal size $N[n, \mu]$ if and only if, for some H as above, $M_H(n+1)$ admits a rank-preserving moment matrix extension $M(n+2)$ (cf., Theorem 1.5).

In Theorem 1.11 (below) we show that for $\mu \equiv \mu_{\mathbb{D}}$, Lebesgue measure on the closed unit disk, the estimate for N in Theorem 1.7 coincides with Möller's estimate in Theorem 1.3. The exact relationship between the estimate in Theorem 1.7 and the lower bound in [Mo2] is an open problem. Indeed, there is an extensive literature concerning cases where Möller's lower bounds can be achieved or cases where the estimates cannot be realized (cf. [Mo1] [Mo3] [CH] [CS] [MP] [S] [SX] [VC] [X3]); by contrast, we have concrete estimates for $\rho(C^\sharp)$ in only relatively few cases (discussed below), so at this point it is difficult to ascertain when the lower bound of Theorem 1.7 is attainable, and also difficult to compare our lower bound to those of Möller in [Mo1] – [Mo3]; we believe the main value of Theorem 1.7 is that it affords an alternate approach to lower estimates and the calculation of cubature rules, based on constructive matrix methods.

Example 1.8. We use Theorem 1.7 to compute a minimal, 4-node, cubature rule of degree 3 for planar measure $\mu \equiv \mu_2$ on the unit square S . We have

$$M(1) \equiv M(1)[\mu] = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/4 \\ 1/2 & 1/4 & 1/3 \end{pmatrix}$$

and

$$B(2) \equiv B(2)[\mu] = \begin{pmatrix} 1/3 & 1/4 & 1/3 \\ 1/4 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/4 \end{pmatrix},$$

whence

$$C^\sharp = \begin{pmatrix} 7/36 & 1/8 & 1/9 \\ 1/8 & 5/48 & 1/8 \\ 1/9 & 1/8 & 7/36 \end{pmatrix}.$$

Since

$$H \equiv \begin{pmatrix} 7/36 & 1/8 & 1/9 \\ 1/8 & 1/9 & 1/8 \\ 1/9 & 1/8 & 7/36 \end{pmatrix}$$

satisfies $H \geq C^\sharp$ and $\text{rank}(H - C^\sharp) = 1$, we have $\rho = 1$, so any cubature rule for μ of degree 3 has at least 4 ($= N[1, \mu]$) nodes. We denote the columns of

$$M(2) \equiv \begin{pmatrix} M(1) & B(2) \\ B(2)^* & H \end{pmatrix}$$

by $1, X, Y, X^2, XY, Y^2$ and observe that $X^2 = -(1/6)1 + X$ and $Y^2 = -(1/6)1 + Y$. For any cubature rule ν of degree 3 for which $M(2)[\nu] = M(2)$, $M(3) \equiv M(3)[\nu]$ is *recursively generated* (see Section 2), so in the column space of $M(3)[\nu]$, with columns labelled as $1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3$, we must have relations $X^3 = -(1/6)X + X^2$, $X^2Y = -(1/6)Y + XY$, $XY^2 = -(1/6)X + XY$, $Y^3 = -(1/6)Y + Y^2$. These relations immediately determine $\beta_{05} = \beta_{50} = 11/72$, $\beta_{14} = \beta_{41} = 7/12$, $\beta_{23} = \beta_{32} = 1/12$. With these values,

$$W \equiv \begin{pmatrix} -1/6 & 0 & 0 & -1/6 \\ 5/6 & 0 & -1/6 & 0 \\ 0 & -1/6 & 0 & 5/6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies $M(2)W = B(3)$, and

$$C \equiv B(3)^*W = \begin{pmatrix} 13/108 & 11/144 & 7/108 & 1/16 \\ 11/144 & 7/108 & 1/16 & 7/108 \\ 7/108 & 1/16 & 7/108 & 11/144 \\ 1/16 & 7/108 & 11/144 & 13/108 \end{pmatrix}$$

is Hankel. Thus

$$M(3) \equiv \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C \end{pmatrix}$$

is a rank-preserving moment matrix extension of $M(2)$. To compute a 4-node (minimal) rule of degree 3 for μ (in accord with Theorem 1.7), we use W to note the following column relations in $M(3)$: $X^3 = -(1/6)1 + (5/6)X$, $X^2Y = -(1/6)Y + XY$, $XY^2 = -(1/6)X + XY$, $Y^3 = -(1/6)1 + (5/6)Y$. Now the variety associated with $x^3 = -(1/6) + (5/6)x$, $x^2y = -(1/6)y + xy$, $xy^2 = -(1/6)x + xy$, $y^3 = -(1/6) + (5/6)y$ consists of 4 points, $z_0 = ((1/6)(3 - \sqrt{3}), (1/6)(3 - \sqrt{3}))$, $z_1 = ((1/6)(3 - \sqrt{3}), (1/6)(3 + \sqrt{3}))$, $z_2 = ((1/6)(3 + \sqrt{3}), (1/6)(3 - \sqrt{3}))$, $z_3 = ((1/6)(3 + \sqrt{3}), (1/6)(3 + \sqrt{3}))$. It follows from the ‘‘real’’ version of Corollary 2.4 that μ admits a cubature rule of the form $\nu = \sum_{i=0}^3 \rho_i \delta_{z_i}$. To compute the densities ρ_i , we set $z_i = (x_i, y_i)$ ($0 \leq i \leq 3$) and let

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0y_0 & x_1y_1 & x_2y_2 & x_3y_3 \end{pmatrix};$$

since $1, X, Y, XY$ is a basis for the column space of $M(2)$, the real version of Proposition 2.1 implies that V is invertible. Since $V(\rho_0, \rho_1, \rho_2, \rho_3)^t = (\beta_{00}, \beta_{10}, \beta_{01}, \beta_{20})^t = (1, 1/2, 1/2, 1/4)^t$, then $\rho_i = 1/4$ ($0 \leq i \leq 3$). \square

Since $\rho(\cdot) \geq 0$ and $\text{rank } M(n)[\mu] = \dim \mathcal{P}_n^d |_{\text{supp } \mu}$ (cf. Proposition 2.8), Theorem 1.7 recovers the “odd” case of the Radon-Stroud estimate in Theorem 1.1. Theorem 1.7 is a moment matrix analogue of Möller’s estimate [Mo1], [Mo2]. In particular, we have the following analogue of Theorem 1.2 concerning the existence of Gaussian rules of odd degree.

Theorem 1.9. *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $2n + 1$, and let $M = M(n)[\mu]$, $B = B(n + 1)[\mu]$. Then μ admits a cubature rule of degree $2n + 1$ with minimal size $\text{rank } M(n)[\mu]$ if and only if*

- (i) $\text{Ran } B(n + 1)[\mu] \subset \text{Ran } M(n)[\mu]$, so that $B = MW$ for some matrix W , and
- (ii) W^*MW (which is independent of W satisfying $B = MW$) has the form of a moment matrix block $C(n + 1)$.

From Theorem 1.9, one can readily recover classical Gaussian quadrature on \mathbb{R} . Indeed, let μ be a square positive Borel measure on an interval $I \subset \mathbb{R}$, with convergent moments $\beta_i = \int_I t^i d\mu$, ($0 \leq i \leq 2n + 1$). $M_{\mathbb{R}}(n)[\mu]$ is the $(n + 1) \times (n + 1)$ Hankel matrix $H(n) \equiv (\beta_{i+j})_{0 \leq i, j \leq n}$. Since $H(n)$ is invertible and $C(n + 1)$ is a 1×1 matrix, the conditions of Theorem 1.9 are satisfied trivially. In the unique flat extension $H(n + 1)$ of $H(n)$, if we label the columns as $1, T, T^2, \dots, T^{n+1}$, then we have a dependence relation $T^{n+1} = c_0 1 + c_1 T + \dots + c_n T^n$. It follows from [CF1] (or Theorem 2.5 below) that the corresponding polynomial $t^{n+1} - (c_0 + c_1 t + \dots + c_n t^n)$, has precisely $n + 1$ real roots, $\{t_i\}_{i=0}^n$ ($\subset I$), and that μ has a minimal cubature rule of degree $2n + 1$ of the form $\nu = \sum_{i=0}^n \rho_i \delta_{t_i}$, where the densities $\rho_i > 0$ can be computed from the Vandermonde equation $V(t_0, \dots, t_n)(\rho_0, \dots, \rho_n)^t = (\beta_0, \dots, \beta_n)^t$.

We prove Theorem 1.9 in Section 3, Theorem 3.4. Condition (i) of Theorem 1.9 is satisfied if μ has convergent moments of degree $2n + 2$, for in that case, $M(n + 1)[\mu] \geq 0$ (cf. Proposition 2.6). For the case when μ is square positive, so that $M(n)[\mu]$ is invertible, Theorem 1.9 seems to give a computationally simpler test for the existence of a Gaussian rule than does Theorem 1.2; indeed, one only needs to be able to compute the moment data and to then check whether or not $C^\sharp(n + 1)[\mu]$ ($= B(n + 1)[\mu]^* M(n)[\mu]^{-1} B(n + 1)[\mu]$) has the form of a moment matrix block $M_{n+1, n+1}$. For planar measures ($d = 2$), it is easy to see that a moment matrix block $M_{n+1, n+1}$ is simply an $(n + 2) \times (n + 2)$ Hankel matrix.

In general, it may be difficult to compute $\rho(\cdot)$. In Section 4 we focus on the case where $\mu \equiv \mu_{\mathbb{D}}$ is planar Lebesgue measure restricted to the closed unit disk \mathbb{D} , and we show that the estimate in Theorem 1.7 for $\mu_{\mathbb{D}}$ coincides with Moller’s estimate in Theorem 1.3. In this case, instead of working with the truncated \mathbb{R}^2 moment problem and a moment matrix corresponding to $\beta^{(2n)}[\mu]$, it is convenient to employ the equivalent *truncated complex moment problem* for measures on the

complex plane \mathbb{C} . More generally, the truncated \mathbb{R}^{2d} moment problem for a real sequence $\beta^{(2n)}$ is equivalent to the truncated \mathbb{C}^d moment problem for a corresponding complex sequence $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+^d, |i|+|j| \leq n}$. This problem concerns the existence of a positive Borel measure μ on \mathbb{C}^d such that $\gamma_{ij} = \int \bar{z}^i z^j d\mu$, ($|i| + |j| \leq 2n$), where $z \equiv (z_1, \dots, z_d) \in \mathbb{C}^d$. Corresponding to $\gamma^{(2n)}$ is the complex moment matrix $M_{\mathbb{C}^d}(n) \equiv M_{\mathbb{C}^d}(n)(\gamma)$ (cf. Section 2). Due to the equivalence of the moment problems for $\beta^{(2n)}$ and $\gamma^{(2n)}$ (cf., [CF4, Proposition 1.12], [CF7, Section 2], [StSz, Appendix]), Theorems 1.7 and 1.9 admit exact analogues when $M(n)[\mu]$ is replaced by $M_{\mathbb{C}^d}(n)[\mu]$. To see this, replace $B(n+1)[\mu]$ by $B_{\mathbb{C}^d}(n+1)[\mu]$ and replace $C^\sharp(n+1)[\mu]$ by $C_{\mathbb{C}^d}^\sharp(n+1)[\mu] \equiv W^* M_{\mathbb{C}^d}(n)[\mu] W$ (for W satisfying $B_{\mathbb{C}^d}(n+1)[\mu] = M_{\mathbb{C}^d}(n)[\mu] W$). We now define $\rho_{\mathbb{C}^d}(\cdot)$ by analogy with $\rho(\cdot)$, but using complex moment matrix blocks $M_{n+1, n+1}$. The equivalence of the moment problem on \mathbb{R}^{2d} for $\beta^{(2n)}$ with the moment problem on \mathbb{C}^d for $\gamma^{(2n)}$ readily implies $\text{rank } M_{\mathbb{C}^d}(n)[\mu] = \text{rank } M(n)[\mu]$ and $\rho_{\mathbb{C}^d}(C_{\mathbb{C}^d}^\sharp(n+1)[\mu]) = \rho(C^\sharp(n+1)[\mu])$. The complex version of Theorem 1.7 now states that the size N of any cubature rule for μ of degree $2n+1$ satisfies

$$N \geq \text{rank } M_{\mathbb{C}^d}(n)[\mu] + \rho_{\mathbb{C}^d}(C_{\mathbb{C}^d}^\sharp(n+1)[\mu]),$$

and the complex version of Theorem 1.9 may be formulated similarly. For measures on the complex plane \mathbb{C} , a moment matrix block $M_{n+1, n+1}$ is simply an $(n+2) \times (n+2)$ Toeplitz matrix; moreover, $C_{\mathbb{C}}^\sharp(n+1)$ always has a weak Toeplitz property: in each diagonal a_1, \dots, a_p , we have $a_1 = a_p$, $a_2 = a_{p-1}$, etc. [CF2, Proposition 2.3].

Returning to the case $\mu \equiv \mu_{\mathbb{D}}$, in Proposition 4.5, for $m = 2n+1$, we show that $C_{\mathbb{C}}^\sharp(n+1)[\mu_{\mathbb{D}}]$ is a positive diagonal matrix, $\text{diag}(c_1, \dots, c_{n+2})$; a gap in $\text{diag}(c_1, \dots, c_{n+2})$ is an occurrence of $c_i > c_{i+1}$. Our main computational result, which follows, shows how to estimate $\rho_{\mathbb{C}^d}(C)$ for a positive diagonal matrix C .

Theorem 1.10. (cf. Theorem 4.1.) *Let $C = \text{diag}(c_1, \dots, c_p)$ be a positive diagonal $p \times p$ matrix. Suppose there is a positive integer q and a strictly increasing sequence $\{n_k\}_{k=1}^q$ of positive integers such that $c_{n_k} > c_{n_{k+1}}$ for $1 \leq k \leq q$. If T is a positive Toeplitz matrix such that $T - C \geq 0$ then $\text{rank}(T - C) \geq q$; thus $\rho_{\mathbb{C}^d}(C)$ is at least as large as the number of gaps in C .*

Proposition 4.5 also shows that $C_{\mathbb{C}}^\sharp(n+1)[\mu_{\mathbb{D}}]$ has precisely $\lfloor \frac{n+1}{2} \rfloor$ gaps. From the preceding discussion, and combining the complex version of Theorem 1.7 with Theorem 1.10, we obtain the following lower estimate for $\mu_{\mathbb{D}}$ cubature rules.

Theorem 1.11. (cf. Theorem 4.4.) $\rho_{\mathbb{C}}(C_{\mathbb{C}}^\sharp(n+1)[\mu_{\mathbb{D}}]) \geq \lfloor \frac{n+1}{2} \rfloor$; the size N of any cubature rule for $\mu_{\mathbb{D}}$ of degree $2n+1$ satisfies $N \geq \frac{(n+1)(n+2)}{2} + \lfloor \frac{n+1}{2} \rfloor$.

Note that $\mu_{\mathbb{D}}$ is centrally symmetric. Theorem 1.11 shows that for $\mu = \mu_{\mathbb{D}}$, the lower estimate in Theorem 1.7 coincides with Möller's estimate in Theorem 1.3.

Whether the above estimate for $\rho_{\mathbb{C}}$ can be extended to general centrally symmetric planar measures (so as to recover Theorem 1.3) is an open question. As we discuss in Section 5, other results of Möller in [Mo2] imply that Theorem 1.11 is not sharp when n is even, since the lower bound for N can be increased by at least 1 in this case. Whether, for n even, we can improve the estimate for $\rho_{\mathbb{C}}(C_{\mathbb{C}}^{\sharp}(n+1)[\mu_{\mathbb{D}}])$ is another open problem.

If $\mu \neq \mu_{\mathbb{D}}$ it can still happen (although not very often) that $C_{\mathbb{C}}^{\sharp}[\mu]$ is diagonal, even in cases where μ is not centrally symmetric. More generally, Theorem 1.10 can be applied indirectly in a variety of cases in which $C \equiv C_{\mathbb{C}}^{\sharp}(n+1)[\mu]$ is not diagonal. One obvious case is when $C = D + T_0$ with D a positive diagonal matrix and T_0 Toeplitz. In this case, $\rho_{\mathbb{C}}(C) = \rho_{\mathbb{C}}(D)$. Indeed, if T is Toeplitz and $T - C \geq 0$, then $\text{rank}(T - C) = \text{rank}((T - T_0) - D)$, whence $\rho_{\mathbb{C}}(C) \geq \rho_{\mathbb{C}}(D)$; conversely, if T is Toeplitz and $T \geq D$, then $T + T_0 \geq C$ and $\text{rank}(T - D) = \text{rank}((T + T_0) - C)$, whence $\rho_{\mathbb{C}}(D) \geq \rho_{\mathbb{C}}(C)$. Next, if the compression of C to the first k rows and columns is of the form $D + T_0$ (as above), then $\rho_{\mathbb{C}}(C) \geq \rho_{\mathbb{C}}(D)$. Moreover, the same conclusion can be obtained if the compression of C is to rows and columns i_1, i_2, \dots, i_k as long as the corresponding compression of any Toeplitz matrix T is still Toeplitz. In fact, it is not hard to see that more is true.

Proposition 1.12. *Let C be a positive $N \times N$ matrix and suppose that U is a unitary operator on \mathbb{C}^N such that, for every $N \times N$ Toeplitz matrix T , the compression of U^*TU to the first k rows and columns is Toeplitz. Let C_k denote the corresponding compression of U^*CU , and suppose that $C_k = D + T_0$, with T_0 a Toeplitz matrix, and D a positive diagonal matrix with at least q gaps. Then $\rho_{\mathbb{C}}(C) \geq q$.*

To illustrate the compression technique, consider an example of $C^{\sharp}(6)$ that we have encountered while studying cubature rules of degree 9 for $\mu_{\mathbb{D}}$ (cf. Section 5). Let $C^{\sharp}(6)$ be of the form

$$\begin{pmatrix} a & z & r & x & v & p & q \\ z & b & z & r & y & v & p \\ r & z & c & z & s & y & v \\ x & r & z & c & z & r & x \\ v & y & s & z & c & z & r \\ p & w & y & r & z & b & z \\ q & p & v & x & r & z & a \end{pmatrix}$$

where $0 \leq a < b < c$. By considering the compression of C^{\sharp} to rows and columns 5, 6, and 7 we see that $\rho_{\mathbb{C}}(C^{\sharp}) \geq 2$.

Section 5 illustrates how moment matrix techniques can be used to construct certain minimal cubature rules. In [R] Radon introduced the method of constructing multivariable cubature rules supported on the common zeros of orthogonal polynomials. Using an approach based on matrix theory, Stroud [Str2], [Str4, Section 3.9, p. 88] constructed a family of $2d$ -node cubature rules of degree 3 in \mathbb{R}^d for a class including centrally symmetric measures; Mysovskih [My1] subsequently showed that these rules are precisely the minimal rules of degree 3 for this class.

In Example 5.1, we compute $\rho_{\mathbb{C}}(C_{\mathbb{C}}^{\sharp}(2)[\mu])$ and characterize the minimal rules of degree 3 for a planar measure $\mu \geq 0$. In Proposition 5.2 we give a new description of the minimal rules of degree 3 in the centrally symmetric case; Example 5.3 illustrates our method with planar measure on the square $C_2 = [-1, 1] \times [-1, 1]$.

In Proposition 5.4, we show that a planar measure μ satisfies $\rho_{\mathbb{C}}(C_{\mathbb{C}}^{\sharp}(3)[\mu]) \leq 1$. Among Radon's results in [R] is the description of certain 7-node minimal rules of degree 5 for a wide class of planar measures (cf., [Str4, Section 3.12]). In Theorem 5.5 we use Proposition 5.4 to completely parametrize the (minimal) 7-node rules of degree 5 for $\mu_{\mathbb{D}}$. In a companion paper by C. V. Easwaran and the authors [EFP] we use moment matrix methods to resolve an open problem of [C2] by showing that among the 10-node (minimal) cubature rules of degree 6 for $\mu_{\mathbb{D}}$, there is no inside cubature rule (although there are many minimal rules with 9 points inside). A 12-node (inside, minimal) rule for $\mu_{\mathbb{D}}$ of degree 7 is cited in [Str4, S2:7-1, pg. 281] (cf. [P]). In Proposition 5.8 we develop a new family of 12-node degree 7 rules for $\mu_{\mathbb{D}}$. Proposition 5.10 gives a new proof that there is no degree 9 rule for $\mu_{\mathbb{D}}$ with as few as 17 points. The first example of a degree 9 rule for $\mu_{\mathbb{D}}$ with as few as 19 nodes is due to Albrecht [A]. In Proposition 5.12 we show how Albrecht's rule (and a related infinite family of 19-node rules) can be derived by a 2-step moment matrix extension $M(5) \rightarrow M(6) \rightarrow M(7)$, where $\text{rank } M(5) = 18$ and $\text{rank } M(6) = \text{rank } M(7) = 19$. All of the preceding examples concern planar measures, but the results of Section 3 apply as well to measures on \mathbb{R}^d . Of course, for $d > 2$ it is considerably more difficult to compute moment matrix extensions than it is for $d = 2$. In Example 5.13 we construct a family of minimal cubature rules of degree 2 for volume measure on the unit ball in \mathbb{R}^3 .

We conclude this section by comparing and contrasting our approach to cubature with some established approaches. In case $\text{supp } \mu$ is symmetric, one effective strategy for constructing a cubature rule is to design a highly symmetric (if sometimes non-minimal) distribution of the nodes, reflecting the symmetry in $\text{supp } \mu$ (cf. [Str4], [HP], [CK], [C1]). By contrast, our approach does not take advantage of symmetry, and is applied in the same manner whether or not $\text{supp } \mu$ displays symmetry; in [CF4, Example 4.12] moment matrices were used to give a complete description of the 5-node (minimal) cubature rules of degree 4 for arclength measure on the parabolic arc $y = x^2$, $0 \leq x \leq 1$, where symmetry is not available; further, techniques from the K -moment problem [CF6] were used to characterize which of these rules are supported inside the arc.

As noted above, Radon [R] pioneered the technique of constructing cubature rules supported on the common zeros of orthogonal polynomials in $L^2(\mu)$. By contrast, in [CF2] a representing measure arises from the spectral measure of a normal operator associated with a flat extension. Thus, in our approach, orthogonal polynomials whose common zeros support a cubature rule emerge as a by-product of the flat extension which establishes the existence of the rule. The analogue of the set of common zeros of orthogonal polynomials is the variety corresponding to a

flat extension $[M(n); B(n+1)]$, determined by $B(n+1) = M(n)W$ (cf. Theorem 2.4). The polynomials which determine the variety of $M(n+1)$ are very easy to compute, for they come from dependence relations in the columns of $M(n)$ and from dependence relations in the columns of $(M(n) \ B(n+1))$, relations that are immediately available from W . (Indeed, the referee has kindly pointed out that W provides the coefficients for a Jackson basis for the space of orthogonal polynomials of degree $n+1$ (cf. [Str4, page 67]).) Once a flat extension is known, it is therefore usually straightforward to compute the nodes and densities of the corresponding cubature rule. The main issue in our approach thus concerns the *existence* of a flat extension $M(n+1)$ or, in the case of a non-“minimal” rule, the existence of a sequence of rank increasing positive extensions $M(n+1), \dots, M(n+k)$, followed by a flat extension $M(n+k+1)$; although a number of concrete existence theorems are known (cf. [CF2] – [CF9], [F3]), much remains to be learned about moment matrix extensions.

2. Moment matrices

Let $\mathbb{C}_r^d[z, \bar{z}]$ denote the space of polynomials with complex coefficients in the indeterminates $z \equiv (z_1, \dots, z_d)$ and $\bar{z} \equiv (\bar{z}_1, \dots, \bar{z}_d)$, with total degree at most r ; thus $\dim \mathbb{C}_r^d[z, \bar{z}] = \eta(d, r) = \binom{r+2d}{2d}$. For $i \equiv (i_1, \dots, i_d) \in \mathbb{Z}_+^d$, let $|i| = i_1 + \dots + i_d$ and let $z^i = z_1^{i_1} \dots z_d^{i_d}$. Given a complex sequence $\gamma \equiv \gamma^{(s)} = \{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+^d, |i|+|j| \leq s}$, the *truncated complex moment problem* for γ entails determining conditions for the existence of a positive Borel measure μ on \mathbb{C}^d such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu, \quad |i| + |j| \leq s. \quad (2.1)$$

A measure μ as in (2.1) is a *representing measure* for γ .

In the sequel we focus on $s = 2n$; in this case, γ determines a *moment matrix* $M(n) \equiv M_{\mathbb{C}^d}(n)(\gamma)$ that we next describe. The size of $M(n)$ is $\eta(d, n)$, with rows and columns denoted by $\{\bar{Z}^i Z^j : i, j \in \mathbb{Z}_+^d, |i| + |j| \leq n\}$, following the degree lexicographic order of the monomials in $\mathbb{C}_n^d[z, \bar{z}]$. (For example, with $d = n = 2$, this order is $1, Z_1, Z_2, \bar{Z}_1, \bar{Z}_2, Z_1^2, Z_1 Z_2, Z_1 \bar{Z}_1, Z_1 \bar{Z}_2, Z_2^2, Z_2 \bar{Z}_1, Z_2 \bar{Z}_2, \bar{Z}_1^2, \bar{Z}_1 \bar{Z}_2, \bar{Z}_2^2$.) The entry of $M(n)$ in row $\bar{Z}^i Z^j$, column $\bar{Z}^k Z^l$ is $\gamma_{k+j, i+l}$, ($|i| + |j|, |k| + |l| \leq n$).

For $p \in \mathbb{C}_n^d[z, \bar{z}]$, $p(z, \bar{z}) = \sum_{r,s \in \mathbb{Z}_+^d, |r|+|s| \leq n} a_{rs} \bar{z}^r z^s$, we set $\hat{p} = (a_{rs})$. The *Riesz functional* $\Lambda \equiv \Lambda_\gamma : \mathbb{C}_{2n}^d[z, \bar{z}] \rightarrow \mathbb{C}$ is defined by $\Lambda(\sum b_{rs} \bar{z}^r z^s) = \sum b_{rs} \gamma_{rs}$. The matrix $M_n^d(\gamma)$ is uniquely determined by

$$\langle M_n^d(\gamma) \hat{f}, \hat{g} \rangle = \Lambda_\gamma(f \bar{g}), \quad (f, g \in \mathbb{C}_n^d[z, \bar{z}]). \quad (2.2)$$

If γ has a representing measure μ , then $\Lambda_\gamma(f \bar{g}) = \int f \bar{g} d\mu$; in particular,

$\langle M^d(n)(\gamma) \hat{f}, \hat{f} \rangle = \int |f|^2 d\mu \geq 0$, so $M^d(n)(\gamma)$ is positive semidefinite in this case.

Corresponding to $p \in \mathbb{C}_n^d[z, \bar{z}]$, $p(z, \bar{z}) = \sum a_{rs} \bar{z}^r z^s$ (as above), we define an element in $\text{Col } M(n)$ by $p(Z, \bar{Z}) = \sum a_{rs} \bar{Z}^r Z^s$; the following result will be used in the sequel to locate the nodes of cubature rules.

Proposition 2.1. ([CF2, (7.4)]) *Suppose μ is a representing measure for $\gamma^{(2n)}$ and for $p \in \mathbb{C}_n^d[z, \bar{z}]$, let $\mathcal{Z}(p) = \{z \in \mathbb{C}^d : p(z, \bar{z}) = 0\}$. Then $\text{supp } \mu \subset \mathcal{Z}(p)$ if and only if $p(Z, \bar{Z}) = 0$ in $\text{Col } M(n)$.*

It follows from Proposition 2.1 that if $\gamma^{(2n)}$ has a representing measure, then $M^d(n)(\gamma)$ is *recursively generated* in the following sense:

$$p, q, pq \in \mathbb{C}_n^d[z, \bar{z}], \quad p(Z, \bar{Z}) = 0 \quad \Rightarrow \quad (pq)(Z, \bar{Z}) = 0. \quad (2.3)$$

We define the *variety* of γ (or the variety of $M(n)(\gamma)$) by $\mathcal{V}(\gamma) = \cap \{\mathcal{Z}(p) : p \in \mathbb{C}_n^d[z, \bar{z}], p(Z, \bar{Z}) = 0\}$. Proposition 2.1 implies that if μ is a representing measure for $\gamma^{(2n)}$, then $\text{supp } \mu \subset \mathcal{V}(\gamma)$ and, moreover, that

$$\text{card } \mathcal{V}(\gamma) \geq \text{card } \text{supp } \mu \geq \text{rank } M_{\mathbb{C}}^d(n)(\gamma), \quad (2.4)$$

(cf., [CF7, (7.6)]).

The following result characterizes the existence of “minimal”, i. e., rank $M(n)$ -atomic, representing measures.

Theorem 2.2. ([CF2, Corollary 7.9 and Theorem 7.10]) *$\gamma^{(2n)}$ has a rank $M_{\mathbb{C}}^d(n)(\gamma)$ -atomic representing measure if and only if $M(n)$ is positive semidefinite and $M(n)$ admits an extension to a moment matrix $M(n+1)$ satisfying $\text{rank } M(n+1) = \text{rank } M(n)$. In this case, $M(n+1)$ admits unique successive rank-preserving positive moment matrix extensions $M(n+2), M(n+3), \dots$ and there exists a rank $M(n)$ -atomic representing measure for $M(\infty)$.*

We refer to a rank-preserving extension $M(n+1)$ of a positive moment matrix $M(n)$ as a *flat extension*; such an extension is positive (cf. Corollary 2.7). For planar moment problems ($d = 1$), the following result describes a concrete procedure for computing the unique rank $M(n)$ -atomic representing measure corresponding to the flat extension $M(n+1)$ of $M(n)(\gamma)$ in Theorem 2.2.

Theorem 2.3 (Flat Extension Theorem). ([CF6, Theorem 2.1]) *Suppose $M(n) = M_{\mathbb{C}}(n)(\gamma)$ is positive semidefinite and admits a flat extension $M(n+1)$, so that $Z^{n+1} = p(Z, \bar{Z})$ in $\text{Col } M(n+1)$ for some $p \in \mathbb{C}_n[z, \bar{z}]$. Then there exist unique successive flat (positive) extensions $M(n+2), M(n+3), \dots$, and $M(n+k)$ is uniquely determined by the column relation $Z^{n+k} = (z^{k-1}p)(Z, \bar{Z})$ in $\text{Col } M(n+k)$ ($k \geq 2$). Let $r = \text{rank } M(n)$. There exist unique scalars a_0, \dots, a_{r-1} such that in $\text{Col } M(n)$, $Z^r = a_0 1 + \dots + a_{r-1} Z^{r-1}$. The analytic polynomial $g_{\gamma}(z) = z^r - (a_0 + \dots + a_{r-1} z^{r-1})$ has r distinct roots, z_0, \dots, z_{r-1} , and γ has a rank $M(n)$ -atomic (minimal) representing measure of the form $\nu \equiv \nu[M(n+1)] = \sum_{i=0}^{r-1} \rho_i \delta_{z_i}$, where the densities $\rho_i > 0$ are uniquely determined by the Vandermonde equation $V(z_0, \dots, z_{r-1})(\rho_0, \dots, \rho_{r-1})^t = (\gamma_{00}, \gamma_{01}, \dots, \gamma_{0, r-1})^t$. The measure $\nu[M(n+1)]$ is the unique representing measure for $M(n+1)$.*

The calculation of g_{γ} in Theorem 2.3 entails iteratively computing moment matrices $M(n+2), M(n+3), \dots, M(r)$, and r may be as large as $(n+1)(n+2)/2$. The following result provides an alternate, frequently more efficient, method

for computing a minimal representing measure corresponding to a flat extension $M(n+1)$.

Theorem 2.4. (Cf. [CF7, Theorem 2.3]) *Suppose $M(n) \equiv M_{\mathbb{C}^d}(n)(\gamma)$ is positive and admits a flat extension $M(n+1)$. Then $\mathcal{V} \equiv \mathcal{V}(M(n+1))$ satisfies $\text{card } \mathcal{V} = r = \text{rank } M(n)$ and $\mathcal{V} \equiv \{z_k\}_{k=0}^{r-1}$ forms the support of the unique representing measure ν for $M(n+1)$, i.e., $\nu = \sum \rho_k \delta_{z_k}$. Let $\{\bar{Z}^{i_m} Z^{j_m}\}_{m=0}^{r-1}$ denote a maximal independent subset of the columns of $M(n)$ and let V be the $r \times r$ matrix whose entry in row m , column k is $\bar{z}_k^{i_m} z_k^{j_m}$. Then V is invertible, and the densities $\rho_0, \dots, \rho_{r-1}$ are uniquely determined by $V(\rho_0, \dots, \rho_{r-1})^t = (\gamma_{i_0, j_0}, \dots, \gamma_{i_{r-1}, j_{r-1}})^t$.*

We now turn to real moment matrices. Let $n \geq 1$ and let

$$1, x_1, \dots, x_d, x_1^2, x_1 x_2, \dots, x_1 x_d, x_2^2, x_2 x_3, \dots, x_2 x_d, \dots, x_d^2, \dots, x_1^n, x_1^{n-1} x_2, \dots, x_d^n$$

denote a degree lexicographic ordering of the monomials in x_1, \dots, x_d up to degree n . This ordering defines an ordered basis B_n^d for \mathcal{P}_n^d ; for $f \in \mathcal{P}_n^d$, let \hat{f} denote the coefficient vector of f relative to B_n^d . By $\beta^{(2n)}$ we mean a real multi-sequence $\{\beta_i : i \in \mathbb{Z}_+^d, |i| \leq 2n\}$. The *real moment matrix* $M(n) \equiv M_{\mathbb{R}^d}(n)(\beta)$ corresponding to $\beta \equiv \beta^{(2n)}$ has size $\vartheta(n, d)$ with rows and columns labelled X^i , $|i| \leq n$, following the above ordering, i. e., $1, X_1, \dots, X_d, \dots, X_1^n, X_1^{n-1} X_2, \dots, X_d^n$. The entry of $M(n)(\beta)$ in row X^i and column X^j is β_{i+j} . Suppose $p \in \mathcal{P}_{2n}^d$, $p = \sum_{|i| \leq 2n} a_i x^i$; we define $\Lambda_\beta(p) = \sum_{|i| \leq 2n} a_i \beta_i$. It follows readily that $M(n)$ is uniquely determined by the relation

$$\langle M(n)\hat{p}, \hat{q} \rangle = \Lambda_\beta(pq), \quad (p, q \in \mathcal{P}_n^d). \quad (2.5)$$

Proposition 2.1, (2.3), and (2.4) admit direct analogues for real moment matrices (cf. [CF7]). We next present an analogue of Theorem 2.2 for real truncated moment problems. In the sequel, whenever $\beta \equiv \beta^{(k)}$ is a d -dimensional multi-sequence $\{\beta_i : |i| \leq k\}$, $\tilde{\beta}$ will denote a multi-sequence *extension* of the form $\{\tilde{\beta}_i\}_{|i| \leq m}$, where $m > k$ and $\tilde{\beta}_i = \beta_i$ for $|i| \leq k$.

Theorem 2.5. (cf. [CF7, Theorem 2.8]) *Let $\beta \equiv \beta^{(2n)}$ be a d -dimensional real multisequence, and let $M \equiv M(n)(\beta)$. If β has a representing measure μ , then $\text{card supp } \mu \geq \text{rank } M$. Further, β admits a rank M -atomic representing measure if and only if M is positive semi-definite and can be extended to a moment matrix of the form $M(n+1)$ such that $\text{rank } M(n+1) = \text{rank } M$. In this case, $M(n+1)$ also has a rank M -atomic representing measure.*

Using Theorem 2.5, one can readily formulate the direct analogue of Corollary 2.4 for real moment matrices; this is what we used in Examples 1.6 and 1.8.

We next cite two auxiliary results that we will use to construct flat extensions of moment matrices. Let \mathcal{H}_1 and \mathcal{H}_2 denote complex Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $\tilde{A} \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator whose operator matrix relative

to this decomposition is of the form

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \quad (2.6)$$

with $A = A^* \in \mathcal{L}(\mathcal{H}_1)$, $C = C^* \in \mathcal{L}(\mathcal{H}_2)$. Theorem 2.5 concerns the case $A = M(n)(\beta)$, $\tilde{A} = M(n+1)(\tilde{\beta})$, so we need a characterization of the case when $\tilde{A} \geq 0$ and $\text{rank } \tilde{A} = \text{rank } A$.

Proposition 2.6. (cf. [Smu] [Epp]) *Suppose \tilde{A} is as in (2.6). Then $\tilde{A} \geq 0$ if and only if $A \geq 0$ and there exists $W \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $B = AW$ and $C \geq W^*AW$. In this case, W^*AW is independent of W satisfying $B = AW$, and when \mathcal{H} is finite dimensional, $\text{rank } \tilde{A} = \text{rank } A + \text{rank}(C - W^*AW)$.*

Corollary 2.7. *Suppose $\mathcal{H} \equiv \mathcal{H}_1 \oplus \mathcal{H}_2$ is finite dimensional, $A \in \mathcal{L}(\mathcal{H}_1)$ is positive, and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then there exists $C \in \mathcal{L}(\mathcal{H}_2)$ such that*

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$$

and $\text{rank } \tilde{A} = \text{rank } A$ if and only if there exists $W \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $B = AW$ and $C = W^*AW$.

Since, from Proposition 2.6, W^*AW is independent of W satisfying $B = AW$, it is clear that a rank-preserving extension \tilde{A} of A (≥ 0) is completely determined by A and B (with $\text{Ran } A \subset \text{Ran } B$). We refer to such an extension \tilde{A} as a *flat extension* of A and we denote it by $\tilde{A} = [A; B]$; Proposition 2.6 or [Smu] imply that $[A; B] \geq 0$.

We conclude this section with two results concerning real moment matrices; of course, these results can be reformulated as well for complex moment matrices.

Proposition 2.8. $\text{rank } M(n)[\mu] = \dim \mathcal{P}_n^d |_{\text{supp } \mu}$.

Proof. Let $\text{Col } M(n)[\mu]$ denote the column space of the matrix $M(n)[\mu]$ and consider the map $\psi : \text{Col } M(n)[\mu] \rightarrow \mathcal{P}_n^d |_{\text{supp } \mu}$ defined by $\psi(\sum_{|i| \leq n} a_i X^i) = \sum a_i x^i |_{\text{supp } \mu}$. Since μ is a representing measure for $\beta^{(2n)}[\mu]$, [CF2, Proposition 3.1] implies that for $p \equiv \sum a_i x^i \in \mathcal{P}_n^d$, $\sum a_i X^i = 0$ in $\text{Col } M(n)[\mu]$ if and only if $p|_{\text{supp } \mu} \equiv 0$. Thus ψ is a well-defined isomorphism, whence $\text{rank } M(n)[\mu] = \dim \text{Col } M(n)[\mu] = \dim \mathcal{P}_n^d |_{\text{supp } \mu}$. \square

Based on Proposition 2.8 we can establish the following result.

Proposition 2.9. *For a positive Borel measure μ on \mathbb{R}^d with convergent moments up to at least order $2n$, the following are equivalent:*

- (i) μ is square positive, i. e., for $p \in \mathcal{P}_n^d$, $p \neq 0$, $\int |p|^2 d\mu > 0$;
- (ii) $M(n)[\mu]$ is invertible (equivalently, $M(n)[\mu] > 0$);
- (iii) $\text{rank } M(n)[\mu] = \binom{n+d}{d}$;
- (iv) $\dim \mathcal{P}_n^d |_{\text{supp } \mu} = \dim \mathcal{P}_n^d$;
- (v) $\text{supp } \mu$ is not contained in the zero set of any nonzero element of \mathcal{P}_n^d .

Proof. (i) \Rightarrow (ii) Assume μ is square positive. For $f \in \mathcal{P}_n^d$, $f \neq 0$, $\langle M(n)[\mu]f, f \rangle = \int |f|^2 d\mu > 0$, whence $M(n)[\mu]f \neq 0$. Thus $M(n)[\mu]$ is invertible, and since $M(n)[\mu] \geq 0$, this is equivalent to $M(n)[\mu] > 0$.

(ii) \Rightarrow (iii) Clear, since the size of $M(n)[\mu]$ is $\binom{n+d}{d}$.

(iii) \Rightarrow (iv) If (iii) holds, then $\dim \mathcal{P}_n^d = \binom{n+d}{d} = \text{rank } M(n)[\mu]$, whence (iv) follows from Proposition 2.8.

(iv) \Rightarrow (v) From (iv) and Proposition 2.8, $\text{rank } M(n)[\mu] = \dim \mathcal{P}_n^d |_{\text{supp } \mu} = \dim \mathcal{P}_n^d = \binom{n+d}{d}$, whence $M(n)[\mu]$ is invertible. The proof of Proposition 2.8 shows that, for $p \in \mathcal{P}_n^d$, $p|_{\text{supp } \mu} = 0$ if and only if $M(n)[\mu]\hat{p} = 0$, so (v) follows.

(v) \Rightarrow (i) If (v) holds then, as in the previous implication, $M \equiv M(n)[\mu]$ is invertible; then $M > 0$, whence $M^{1/2} > 0$. Now, for $p \in \mathcal{P}_n^d$, $p \neq 0$, $\int |p|^2 d\mu = \langle M\hat{p}, \hat{p} \rangle = \langle M^{1/2}\hat{p}, M^{1/2}\hat{p} \rangle > 0$ whence (i) holds. \square

3. Lower bounds for cubature rules

In this section we use moment matrices to provide lower estimates for the size of a cubature rule. We state the results only for measures on \mathbb{R}^d , but all of the ensuing results can be reformulated for the complex case (cf. Section 2). We begin with the proof of Proposition 1.4, which we restate for convenience.

Proposition 3.1. *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $m = 2n$. The size N of any cubature rule for μ of degree m satisfies $N \geq \text{rank } M(n)[\mu]$.*

Proof. Let $\beta = \beta^{(2n)}[\mu]$, i. e., $\beta_i = \int t^i d\mu$, $|i| \leq 2n$. If ν is a cubature rule for μ of degree $2n$, then $\beta^{(2n)}[\nu] = \beta^{(2n)}[\mu]$, whence $M(n)[\nu] = M(n)[\mu]$. Since ν is a representing measure for $\beta^{(2n)}[\nu]$, Theorem 2.5 implies $\text{card supp } \nu \geq \text{rank } M(n)[\nu] = \text{rank } M(n)[\mu]$. \square

We next prove Theorem 1.5, which we restate.

Theorem 3.2. *Let μ be as in Proposition 3.1; then μ has a cubature rule of degree $2n$ with (minimal) size $N = \text{rank } M(n)[\mu]$ if and only if $M(n)[\mu]$ can be extended to a moment matrix $M(n+1)$ satisfying $\text{rank } M(n+1) = \text{rank } M(n)[\mu]$; equivalently, there is a choice of “new moments” of degree $2n+1$ and a corresponding matrix W , such that $M(n)W = B(n+1)$ (i. e., $\text{Ran } B(n+1) \subset \text{Ran } M(n)$) and $W^*M(n)W$ is a moment matrix block (of degree $2n+2$).*

Proof. Since $\mu \geq 0$, then $M(n)[\mu] \geq 0$. It follows from Theorem 2.5 that μ admits a cubature rule of degree $2n$ and size $\text{rank } M(n)[\mu]$ if and only if $M(n)[\mu]$ can be extended to a moment matrix $M(n+1)$ satisfying $\text{rank } M(n+1) = \text{rank } M(n)$. The concrete condition for the flat extension follows from Corollary 2.7. \square

We now consider lower estimates in the “odd” case. We begin by proving Theorem 1.7.

Theorem 3.3. *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $2n + 1$. The size N of any cubature rule for μ of degree $2n + 1$ satisfies*

$$N \geq N[n, \mu] \equiv \text{rank } M(n)[\mu] + \rho(C^\sharp(n+1)[\mu]).$$

Further, let $H = M_{n+1, n+1}$ be a moment matrix block satisfying $H \geq C^\sharp$ ($\equiv C^\sharp(n+1)[\mu]$) and $\text{rank}(H - C^\sharp) = \rho(C^\sharp)$, and set

$$M_H(n+1) = \begin{pmatrix} M(n) & B(n+1)[\mu] \\ B(n+1)[\mu]^* & H \end{pmatrix}.$$

Then μ admits a cubature rule of degree $2n + 1$ with minimal size $N[n, \mu]$ if and only if, for some H as above, $M_H(n+1)$ admits a rank-preserving moment matrix extension $M(n+2)$.

Proof. Suppose ν is a cubature rule for μ of degree $2n + 1$. Then $M \equiv M(n+1)[\nu]$ has the form

$$M = \begin{pmatrix} M(n)[\mu] & B(n+1)[\mu] \\ B(n+1)[\mu]^* & C(n+1)[\nu] \end{pmatrix},$$

where $C(n+1)[\nu]$ is block $M_{n+1, n+1}$ of $M(n+1)[\nu]$. Since $\nu \geq 0$, then $M \geq 0$, so Proposition 2.6 implies that there is a matrix W such that $B(n+1) = M(n)W$ and $C(n+1)[\nu] \geq C^\sharp(n+1)[\mu] (= W^*M(n)W)$. Proposition 2.6 further shows that

$$\begin{aligned} \text{rank } M &= \text{rank } M(n)[\mu] + \text{rank}(C(n+1)[\nu] - C^\sharp(n+1)[\mu]) \\ &\geq \text{rank } M(n)[\mu] + \rho(C^\sharp(n+1)[\mu]). \end{aligned}$$

Now ν is a representing measure for $\beta^{(2n+2)}[\nu]$, so Theorem 2.5 implies that $N \equiv \text{card supp } \nu \geq \text{rank } M$, and the estimate follows.

Next, suppose ν is a (minimal) cubature rule for μ of degree $2n + 1$, with precisely $N[n, \mu]$ nodes. As above,

$$N[n, \nu] = \text{card supp } \nu \geq \text{rank } M(n+2)[\nu] \geq$$

$$\text{rank } M(n+1)[\nu] = \text{rank } M(n)[\mu] + \text{rank}(C(n+1)[\nu] - C^\sharp(n+1)[\mu]) \geq N[n, \mu].$$

Thus, $H \equiv C(n+1)[\nu]$ satisfies $H \geq C^\sharp(n+1)[\mu]$ and $\rho(C^\sharp(n+1)[\mu]) = \text{rank}(H - C^\sharp(n+1)[\mu])$, and clearly $M(n+2)[\nu]$ is a flat extension of $M_H(n+1) (= M(n+1)[\nu])$.

Conversely, suppose $H \equiv M_{n+1, n+1}$ satisfies $H \geq C^\sharp$ and $\text{rank}(H - C^\sharp) = \rho(C^\sharp)$; thus $\text{rank } M_H(n+1) = N[n, \mu]$. If $M_H(n+1)$ admits a flat extension $M(n+2)$, then (using Theorem 2.2) $M_H(n+1)$ admits a representing measure ν with $N[n, \mu]$ nodes, and ν thus acts as a minimal cubature rule for μ of degree $2n + 1$. \square

We next prove Theorem 1.9, a moment matrix analogue of Mysovskikh's criterion.

Theorem 3.4. *Let μ be a positive Borel measure on \mathbb{R}^d with convergent moments up to at least degree $2n + 1$, and let $M = M(n)[\mu]$, $B = B(n + 1)[\mu]$. Then μ admits a cubature rule of degree $2n + 1$ with minimal size $\text{rank } M(n)[\mu]$ if and only if*

- (i) $\text{Ran } B(n + 1)[\mu] \subset \text{Ran } M(n)[\mu]$, so that $B = MW$ for some matrix W , and
- (ii) W^*MW (which is independent of W satisfying $B = MW$) has the form of a moment matrix block $C(n + 1)$.

Proof. Suppose ν is a cubature rule for μ of degree $2n + 1$ with $\text{card supp } \nu = \text{rank } M(n)[\mu]$. Then $M(n + 1)[\nu]$ is a positive moment matrix of the form

$$\begin{pmatrix} M(n)[\mu] & B(n + 1)[\mu] \\ B(n + 1)[\mu]^* & C(n + 1) \end{pmatrix} \equiv \begin{pmatrix} M & B \\ B^* & C \end{pmatrix}.$$

Since ν is a representing measure for $\beta^{(2n+2)}[\nu]$, Theorem 2.5 implies that $\text{rank } M(n)[\mu] = \text{card supp } \nu \geq \text{rank } M(n + 1)[\nu] \geq \text{rank } M(n)[\mu]$, so $M(n + 1)[\nu]$ is a flat extension of the positive moment matrix $M(n)[\mu]$. Now (i) and (ii) follow from Corollary 2.7.

Conversely, suppose (i) and (ii) hold and let

$$\tilde{M} = \begin{pmatrix} M & B \\ B^* & C \end{pmatrix},$$

with $M = M(n)[\mu]$, $B = B(n + 1)[\mu]$, $B = MW$, $C = W^*MW$. Since M is positive, Corollary 2.7 shows that \tilde{M} is a flat extension of M of the form $\tilde{M} = M(n + 1)$. Theorem 2.5 now implies that \tilde{M} admits a rank M -atomic representing measure, which acts as a cubature rule for μ of degree $2n + 1$ and size $\text{rank } M$. \square

Theorem 3.4 is very general in the sense that the matrix $M \equiv M(n)[\mu]$ may be singular. From Proposition 2.9, M is singular if and only if $\text{supp } \mu$ is contained in some algebraic subset of \mathbb{R}^d of degree no more than n . If M is invertible, then $\text{rank } M = \vartheta(n, d) = \binom{n+d}{d}$. Thus, we have the following consequence.

Corollary 3.5. *Let μ be a positive Borel measure on \mathbb{R}^d with moments up to degree $2n + 1$ and suppose $\text{supp } \mu$ is not contained in any algebraic subset of degree not exceeding n . Then μ admits a minimal cubature rule of degree $2n + 1$, with $\binom{n+d}{d}$ nodes, if and only if $B^*M^{-1}B$ has the form of a moment matrix block $C(n + 1)$ (where $M = M(n)[\mu]$ and $B = B(n + 1)[\mu]$).*

For $d = 1$, the condition of Corollary 3.5 is satisfied vacuously since $B^*M^{-1}B$ is a real number. In this case, the resulting $(n + 1)$ -atomic cubature rule of degree $2n + 1$ corresponds to classical Gaussian quadrature. For $d = 2$, the condition of Corollary 3.5 is that $B^*M^{-1}B$ has the form of a Hankel matrix.

4. Estimating ρ when $C_{\mathbb{C}}^{\sharp}$ is diagonal

Theorem 3.3 provides a criterion for a positive Borel measure μ on \mathbb{R}^d to have a “minimal” cubature rule of degree $2n + 1$, a rule with $\text{rank } M(n)[\mu] + \rho(C_{\mathbb{C}}^{\sharp}(n + 1)[\mu])$

nodes. In order to utilize Theorem 3.3 to compute minimal or near-minimal rules, it is necessary to be able to estimate $\rho(C^\sharp(n+1)[\mu])$. In the present section we show how to estimate $\rho_{\mathbb{C}}(C^\sharp(n+1)[\mu])$ in case $C_{\mathbb{C}}^\sharp(n+1)[\mu]$ is a diagonal matrix (Theorem 4.1); in the introduction we indicated how this result can be adapted to certain situations in which $C_{\mathbb{C}}^\sharp(n+1)[\mu]$ is non-diagonal. In Proposition 4.5 we show that $\mu_{\mathbb{D}}$, planar measure on the unit disk, has the property that $C_{\mathbb{C}}^\sharp(n+1)[\mu]$ is diagonal, and, as a consequence, we are able to recover Möller's lower estimate of Theorem 1.3 in the case of $\mu_{\mathbb{D}}$ (Theorem 4.4). In Section 5 we will apply Theorems 4.4 and 4.1 to construct families of minimal or near-minimal cubature rules for $\mu_{\mathbb{D}}$.

Theorem 4.1. *Let C be an $N \times N$ positive diagonal matrix with diagonal entries c_1, c_2, \dots, c_N . Suppose that there exists a positive integer q and a strictly increasing sequence $\{n_k\}_{k=1}^q$ of positive integers such that $c_{n_k} > c_{n_{k+1}}$ for $1 \leq k \leq q$. If T is a positive Toeplitz matrix such that $T - C \geq 0$ then $\text{rank}(T - C) \geq q$.*

In order to prove Theorem 4.1 we first establish some notation. Since T is a positive Toeplitz matrix its entries t_{ij} , $1 \leq i, j \leq N$, can be written as $t_{ij} = t_{j-i}$ for $j \geq i$ and $t_{ij} = \bar{t}_{j-i}$ for $j < i$. Let $a_k = t_0 - c_k$ (≥ 0), $1 \leq k \leq N$. Clearly $a_{n_k} < a_{n_{k+1}}$ for $1 \leq k \leq q$. Let S be the principal submatrix of $T - C$ obtained using rows and columns $\{n_k\}_{k=1}^q$. Let $F = \det S$; since $S \geq 0$, then $F \geq 0$. For $1 \leq k \leq q$, let S_k be the matrix obtained from S by replacing a_{n_j} by $a_{n_{j+1}}$, $k \leq j \leq q$; we also set $S_{q+1} = S$. Since $a_{n_{j+1}} > a_{n_j}$ ($k \leq j \leq q$) and $S \geq 0$, it follows that $S_k \geq 0$, whence $F_k \equiv \det S_k \geq 0$ ($1 \leq k \leq q+1$). Further, for $1 \leq k \leq q+1$ and $1 \leq j \leq q$, let $S_k^{(j)}$ denote the matrix obtained from S_k by deleting the j -th row and column, and let $F_k^{(j)} = \det S_k^{(j)}$; clearly $F_k^{(j)} \geq 0$. The following result compares the values of these determinants.

Lemma 4.2. *$F_k \geq F_{k+1}$, $1 \leq k \leq q$, and $F_1 > F_2$.*

Proof. We will use induction on q . The case $q = 1$ is trivial since the determinants under consideration are just numbers: $F_1 = a_{n_1+1}$, $F_2 = F = a_{n_1}$, and $a_{n_1} < a_{n_1+1}$. As an illustration, we show the case $q = 2$. Now we need to prove that $F_1 > F_2$ and $F_2 \geq F$. This follows from straightforward computation:

$$\left| \begin{array}{cc} a_{n_1+1} & t_{n_2-n_1} \\ t_{n_2-n_1} & a_{n_2+1} \end{array} \right| - \left| \begin{array}{cc} a_{n_1} & t_{n_2-n_1} \\ t_{n_2-n_1} & a_{n_2+1} \end{array} \right| = (a_{n_1+1} - a_{n_1})a_{n_2+1} \geq 0 \quad (4.1)$$

$$\left| \begin{array}{cc} a_{n_1} & t_{n_2-n_1} \\ t_{n_2-n_1} & a_{n_2+1} \end{array} \right| - \left| \begin{array}{cc} a_{n_1} & t_{n_2-n_1} \\ t_{n_2-n_1} & a_{n_2} \end{array} \right| = (a_{n_2+1} - a_{n_2})a_{n_1} \geq 0 \quad (4.2)$$

and it is clear that $a_{n_1+1} - a_{n_1} > 0$, while the positivity of $T - C$ implies that $a_{n_2} \geq 0$ and, therefore, $a_{n_2+1} > 0$.

Suppose that the lemma has been proved for $q - 1$. We show that, in this situation, it is true for q . Let k be an integer such that $1 \leq k \leq q$. Then

$$F_k - F_{k+1} = (a_{n_{k+1}} - a_{n_k})F_k^{(k)}. \quad (4.3)$$

Clearly $a_{n_k+1} - a_{n_k} > 0$ and $F_k^{(k)} \geq 0$. Thus $F_k \geq F_{k+1}$. We will show that $F_1 > F_2$. Consider the sequence $n'_1 < \dots < n'_{q-1}$, where $n'_j = n_{j+1}$. By induction, the corresponding determinants F'_j satisfy $F'_1 > F'_2 \geq \dots \geq F'_{q-1} \geq F'_q$. Now $F'_j = F_{j+1}^{(1)}$ ($1 \leq j \leq q$), so $F_2^{(1)} \geq \dots \geq F_{q+1}^{(1)}$ and $F_2^{(1)} > F_3^{(1)}$. Since $F_1^{(1)} = F_2^{(1)}$ and $F_{q+1}^{(1)} = F^{(1)} \geq 0$, it follows that $F_1^{(1)} > 0$, whence (4.3) implies $F_1 > F_2$. The proof is complete. \square

Using Lemma 4.2 we can now easily prove Theorem 4.1. Indeed, we have just established that

$$F_1 > F_2 \geq F_3 \geq \dots \geq F_q \geq F \geq 0,$$

whence $F_1 > 0$. Let R denote the compression of $T - C$ to rows and columns $n_1 + 1, \dots, n_q + 1$. Due to the Toeplitz structure of T , R coincides with S_1 , whence $\det R = \det S_1 = F_1 > 0$. It now follows that $\text{rank}(T - C) \geq \text{rank} R = q$. Thus Theorem 4.1 is established.

We now begin our analysis of $C_{\mathbb{C}}^{\sharp}(n+1)[\mu_{\mathbb{D}}]$. It is often convenient to view $M(n) \equiv M_{\mathbb{C}}(n)(\gamma)$ as a block matrix, as follows. Given a doubly indexed finite sequence of complex numbers $\gamma^{(2n)} := \{\gamma_{ij} : 0 \leq i + j \leq 2n\}$, with $\gamma_{00} > 0$ and $\gamma_{ij} = \bar{\gamma}_{ji}$, one can form a family of Toeplitz-like rectangular matrices $M[i, j]$, $0 \leq i, j \leq n$, where the first row of $M[i, j]$ is $\gamma_{ij}, \gamma_{i+1, j-1}, \dots, \gamma_{i+j, 0}$ while the first column is $\gamma_{ij}, \gamma_{i-1, j+1}, \dots, \gamma_{0, i+j}$. The complex moment matrix $M(n) \equiv M_{\mathbb{C}}(n)(\gamma)$ is then represented as a block matrix

$$M(n) = \begin{pmatrix} M[0, 0] & M[0, 1] & \dots & M[0, n] \\ M[1, 0] & M[1, 1] & \dots & M[1, n] \\ \dots & \dots & \dots & \dots \\ M[n, 0] & M[n, 1] & \dots & M[n, n] \end{pmatrix}. \tag{4.4}$$

Recall that the rows and columns of $M(n)$ are denoted by the degree lexicographic ordering $\mathcal{E} : 1, Z, \bar{Z}, Z^2, \dots$; the entry in row $\bar{Z}^i Z^j$, column $\bar{Z}^k Z^l$ is $\langle M(n) \bar{z}^k z^l, \bar{z}^i z^j \rangle = \gamma_{k+j, l+i}$, ($0 \leq i + j, k + l \leq n$).

In the case of $M(n)[\mu_{\mathbb{D}}]$ it is useful to describe $M(n)$ relative to a permutation of \mathcal{E} . Notice that \mathcal{E} is ordered in such a way that basis vectors are grouped relative to the degree $i + j$ of $Z^i \bar{Z}^j$. In the new basis we will group row and column vectors using the quantity $i - j$ instead, and within each group monomials will be listed by ascending total degree. For example, when $n = 4$, the ordering is

$$\bar{Z}^4, \bar{Z}^3, \bar{Z}^2, Z\bar{Z}^3, \bar{Z}, Z\bar{Z}^2, 1, Z\bar{Z}, Z^2\bar{Z}^2, Z, Z^2\bar{Z}, Z^2, Z^3\bar{Z}, Z^3, Z^4.$$

We will show that relative to this new ordering $M(n) \equiv M(n)[\mu_{\mathbb{D}}]$ is block diagonal. More precisely, for $-n \leq p \leq 0$, consider the following ordered sets of column vectors of $M(n)$,

$$\mathcal{L}_p = \{\bar{Z}^{-p}, Z\bar{Z}^{-p+1}, \dots, Z^k\bar{Z}^{-p+k}, \dots, Z^{\lfloor (n+p)/2 \rfloor} \bar{Z}^{-p+\lfloor (n+p)/2 \rfloor}\}$$

and

$$\mathcal{N}_{-p} = \{Z^{-p}, Z^{-p+1}\bar{Z}, \dots, Z^{-p+k}\bar{Z}^k, \dots, Z^{-p+\lfloor(n+p)/2\rfloor}\bar{Z}^{\lfloor(n+p)/2\rfloor}\}.$$

Let L_p [resp., N_{-p}] be the subspace spanned by \mathcal{L}_p [resp., \mathcal{N}_{-p}]. We claim that for each p , L_p and N_{-p} are invariant for $M(n)$. For $0 \leq j, k \leq \lfloor(n+p)/2\rfloor$,

$$\langle M(n)z^k\bar{z}^{-p+k}, z^j\bar{z}^{-p+j} \rangle = \gamma_{-p+k+j, -p+k+j} = \langle M(n)\bar{z}^k z^{-p+k}, \bar{z}^j z^{-p+j} \rangle,$$

and note that $\gamma_{-p+k+j, -p+k+j} > 0$. Since, for the disk, $\gamma_{rs} \neq 0$ if and only if $r = s$, then for $0 \leq i, j \leq n$ with $j - i \neq p$,

$$\langle M(n)z^k\bar{z}^{-p+k}, z^j\bar{z}^i \rangle = \gamma_{-p+k+j, k+i} = 0 = \gamma_{k+i, -p+k+j} = \langle M(n)\bar{z}^k z^{-p+k}, \bar{z}^j z^i \rangle.$$

Thus L_p and N_{-p} are invariant for $M(n)$; relative to the reordering of the rows and columns of $M(n)$ into the ordered blocks

$$\mathcal{L}_{-n}, \dots, \mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{N}_1, \dots, \mathcal{N}_n, \quad (4.5)$$

$M(n)$ admits a block decomposition

$$M(n) = M_{-n} \oplus \dots \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \dots \oplus M_n, \quad (4.6)$$

with $M_{-p} = M_p$, $1 \leq p \leq n$. Note that for $1 \leq p \leq n$, M_p is a Hankel matrix whose entries are determined by its top row $(\gamma_{pp}, \dots, \gamma_{p+\lfloor(n-p)/2\rfloor, p+\lfloor(n-p)/2\rfloor})$ and its rightmost column,

$$(\gamma_{p+\lfloor(n-p)/2\rfloor, p+\lfloor(n-p)/2\rfloor}, \dots, \gamma_{p+2\lfloor(n-p)/2\rfloor, p+2\lfloor(n-p)/2\rfloor})^t. \quad (4.7)$$

We also note for future reference that the lower right hand entry of M_p is γ_{nn} if and only if $n-p$ is even. The preceding discussion now leads to the following result; note that since $\text{supp } \mu_{\mathbb{D}}$ is not contained in any algebraic subset, Proposition 2.1 implies that $M(n)[\mu_{\mathbb{D}}]$ is invertible.

Lemma 4.3. *Let $0 \leq i + j, k + l \leq n$. If $\langle M(n)[\mu_{\mathbb{D}}]\bar{z}^i z^j, \bar{z}^k z^l \rangle = 0$, then $\langle M(n)[\mu_{\mathbb{D}}]^{-1}\bar{z}^i z^j, \bar{z}^k z^l \rangle = 0$.*

The following result is essentially Theorem 1.11.

Theorem 4.4. *A cubature rule of degree d and size N for planar Lebesgue measure on \mathbb{D} satisfies the following estimates: if $d = 2n + 1$, then $N \geq \frac{(n+1)(n+2)}{2} + \lfloor \frac{n+1}{2} \rfloor$; if $d = 2n$ then $N \geq \frac{(n+1)(n+2)}{2}$.*

Suppose $d = 2n$; since $M(n)[\mu_{\mathbb{D}}]$ is invertible, Proposition 1.4 implies $N \geq \text{rank } M(n)[\mu_{\mathbb{D}}] = (n+1)(n+2)/2$. Now let $d = 2n + 1$; from Theorem 1.7, to complete the proof in this case it suffices to show that $\rho(C^{\sharp}(n+1)[\mu_{\mathbb{D}}]) \geq \lfloor(n+1)/2\rfloor$. This inequality is an immediate consequence of Theorem 4.1 and the following result.

Proposition 4.5. *$C^{\sharp} \equiv C_{\mathbb{C}}^{\sharp}(n+1)[\mu_{\mathbb{D}}]$ is diagonal. The diagonal entries c_0, c_1, \dots, c_{n+1} satisfy $c_i < c_{i+1}$ if $0 \leq i < \lfloor(n+1)/2\rfloor$ and $c_i > c_{i+1}$ if $n+1 - \lfloor(n+1)/2\rfloor \leq i < n+1$, so there are $\lfloor(n+1)/2\rfloor$ ‘‘gaps’’.*

Proof. First we show that C^\sharp is diagonal. Using the same partitioning as in (4.4), we can write $P \equiv M^{-1} = (P_{ij})_{i,j=0}^n$ and, similarly, we can write B as a block column matrix $(B_i)_{i=0}^n$. Thus,

$$C^\sharp = B^* M^{-1} B = \sum_{i,j} B_i^* P_{ij} B_j.$$

Clearly, it suffices to show that, for all i, j , $B_i^* P_{ij} B_j$ is a diagonal matrix. Note that if i has the same parity as n , then $B_i = 0$ (and similarly for j), so we may assume that i and j have different parity than n . In this case, B_i (which has size $(i+1) \times (n+2)$) can be partitioned into 3 blocks, $B_i = \begin{pmatrix} 0 & D_i & 0 \end{pmatrix}$, where D_i is a nonzero scalar multiple of identity (of size $(i+1) \times (i+1)$) and both zero matrices have size $(i+1) \times (n-i+1)/2$. Now

$$B_i^* P_{ij} B_j = \begin{pmatrix} 0 \\ D_i \\ 0 \end{pmatrix} P_{ij} \begin{pmatrix} 0 & D_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & & \\ & D_i P_{ij} D_j & \\ & & 0 \end{pmatrix}.$$

Thus, it suffices to consider the rectangular $(i+1) \times (j+1)$ block $D_i P_{ij} D_j$. Since both D_i and D_j are scalar multiples of the identity, it remains to prove that the rectangular block P_{ij} has the desired property, namely that its middle portion (deleting the leftmost and rightmost blocks of size $(i+1) \times (n-i+1)/2$) is diagonal. This follows from Lemma 4.3 and the fact that in $M(n)$ the corresponding middle portion of block $M[i, j]$ is diagonal.

One knows (cf. [CF2, Proposition 2.3]) that $c_r = c_{n+1-r}$, $0 \leq r \leq n+1$. Thus, it remains to prove that $c_r < c_{r+1}$, $0 \leq r < \lfloor (n+1)/2 \rfloor$. To that end, we will compute the numbers c_r explicitly. In order to simplify notation we will write γ_k for γ_{kk} . (Of course, $\gamma_{ij} = 0$ for $i \neq j$.) By Corollary 2.7, the matrix

$$M^\sharp = \begin{pmatrix} M(n) & B \\ B^* & C^\sharp \end{pmatrix}$$

has the same rank as $M(n)$. Now M^\sharp coincides with $M(n+1)$ except in block C^\sharp . Nevertheless, since C^\sharp is diagonal, as is $C(n+1)$, M^\sharp also admits a block decomposition relative to (4.5) (with n replaced by $n+1$), of the form

$$M^\sharp = M_{-(n+1)}^\sharp \oplus \cdots \oplus M_{-1}^\sharp \oplus M_0^\sharp \oplus M_1^\sharp \oplus \cdots \oplus M_{n+1}^\sharp. \quad (4.8)$$

The only differences between M^\sharp and $M(n+1)$ occur in columns indexed by $\bar{Z}^i Z^j$ with $i+j = n+1$, and each block \mathcal{L}_p or \mathcal{N}_{-p} contains at most one such vector. Such columns occur in alternate blocks M_{n+1}, M_{n-1}, \dots . If M_k has such a column ($1 \leq k \leq n+1$), then it has exactly one such column, say $\bar{Z}^i Z^j$, with $i+j = n+1$, $0 \leq i, j \leq n+1$. In fact, it is not hard to see that, for $0 \leq r < \lfloor (n+1)/2 \rfloor$, M_{n-2r+1} is an $(r+1) \times (r+1)$ Hankel matrix with the top row $(\gamma_{n-2r+1}, \dots, \gamma_{n-r+1})$ and the rightmost column $(\gamma_{n-r+1}, \dots, \gamma_{n+1})^t$. (When $r = 0$, this means that M_{n+1} is just the real number γ_{n+1} .) The corresponding block M_{n-2r+1}^\sharp differs only in the lower right corner, where γ_{n+1} is replaced by c_{n+1-r} . Since $\text{rank } M^\sharp = \text{rank } M(n)$,

(4.8) implies that the last column of M_{n-2r+1}^\sharp is dependent on the first r columns of M_{n-2r+1}^\sharp , and since $M(n) > 0$, the compression of M_{n-2r+1}^\sharp to first r rows and columns is invertible. It now follows that c_r ($= c_{n+1-r}$) is uniquely determined by the equation $\det M_{n-2r+1}^\sharp = 0$.

Next, we show how to solve the equation $\det M_{n-2r+1}^\sharp = 0$ for c_r . Let r be an integer such that $0 \leq r \leq \lfloor (n+1)/2 \rfloor$ and denote $f_r = \det M_{n-2r+1}$ and $f_r^\sharp = \det M_{n-2r+1}^\sharp$. Since $\gamma_m = \pi/(m+1)$ it is natural to consider matrices

$$H_{pq} = \left(\frac{1}{i+j+p-2} \right)_{i,j=1}^q, \quad p \geq 1,$$

and their determinants $A_{pq} = \det H_{pq}$. Note that $f_r^\sharp = \det M_{n-2r+1}^\sharp = 0$ and that $f_r = \det M_{n-2r+1} = \pi^{r+1} \det H_{n-2r+2, r+1} = \pi^{r+1} A_{n-2r+2, r+1}$. On the other hand, using once again the multilinearity of determinants, we have that $f_r^\sharp - f_r = (c_r - \gamma_{n+1}) \det H_{n-2r+2, r} = (c_r - \gamma_{n+1}) \pi^r A_{n-2r+2, r}$. It now follows that

$$c_r = \gamma_{n+1} - \frac{\pi A_{n-2r+2, r+1}}{A_{n-2r+2, r}}.$$

In order to evaluate the last expression we use the formula from [Pol, Problem 7.1.4]

$$A_{pq} = \frac{[1!2! \dots (q-1)!]^2 (p-1)! p! \dots (q+p-2)!}{(q+p-1)! (q+p)! \dots (2q+p-2)!}.$$

Thus

$$\frac{A_{n-2r+2, r+1}}{A_{n-2r+2, r}} = \frac{[r!(n-r+1)!]^2}{(n+1)!(n+2)!}$$

and

$$c_r = \pi \left(\frac{1}{n+2} - \frac{[r!(n-r+1)!]^2}{(n+1)!(n+2)!} \right).$$

Next we make the comparison between c_r and c_{r+1} . Let r be an integer, $0 \leq r < \lfloor (n+1)/2 \rfloor$. Then

$$\begin{aligned} c_{r+1} - c_r &= \pi \frac{[r!(n-r+1)!]^2}{(n+1)!(n+2)!} - \pi \frac{[(r+1)!(n-r)!]^2}{(n+1)!(n+2)!} \\ &= \pi \frac{[r!(n-r)!]^2}{(n+1)!(n+2)!} [(n-r+1)^2 - (r+1)^2] \\ &= \pi \frac{[r!(n-r)!]^2}{(n+1)!(n+2)!} (n+2)(n-2r). \end{aligned}$$

This shows that $c_{r+1} - c_r > 0$ if and only if $r < n/2$. Since it is easy to verify that $r < \lfloor (n+1)/2 \rfloor$ implies $r < n/2$, the proof is complete. \square

5. Moment matrices and minimal cubature rules: examples

In this section we show how moment matrix techniques from the previous sections can be used to construct minimal or near-minimal cubature rules, and how these techniques can be used to analyze the minimal size of a cubature rule. We begin by analyzing $\rho(C_{\mathbb{C}}^{\sharp}(2)[\mu])$ for a large class of planar measures; this leads to a moment matrix characterization of the existence of minimal rules of degree 3. We next show that $\rho(C_{\mathbb{C}}^{\sharp}(3)[\mu]) \leq 1$ for an arbitrary planar measure μ having moments up to at least degree 5, and we use this result to parameterize the minimal rules of degree 5 for $\mu_{\mathbb{D}}$. We then present a series of additional results concerning $\mu_{\mathbb{D}}$, including a proof of the conjecture of [HP] on the nonexistence of 17 point rules of degree 9 for $\mu_{\mathbb{D}}$, and a moment matrix development of Albrecht's 19 point rule of degree 9 for $\mu_{\mathbb{D}}$. We conclude with an example which illustrates how moment matrix methods can be applied in \mathbb{R}^3 .

We begin by analyzing $\rho \equiv \rho(C_{\mathbb{C}}^{\sharp}(2)[\mu])$ and the structure of minimal degree 3 cubature rules for planar measures μ satisfying $\text{Ran } B_{\mathbb{C}}(2)[\mu] \subset \text{Ran } M_{\mathbb{C}}(1)[\mu]$. The range hypothesis is satisfied, in particular, whenever $\text{supp } \mu$ is not contained in any line (cf. Proposition 2.1), or whenever μ has finite moments up to at least degree 4 (so that $M(2)[\mu] \geq 0$, cf. Proposition 2.6). In the sequel, we write

$$M(1) \equiv M_{\mathbb{C}}(1)[\mu] = \begin{pmatrix} 1 & x & \bar{x} \\ \bar{x} & e & \bar{w} \\ x & w & e \end{pmatrix} (\geq 0), \quad B(2) \equiv B_{\mathbb{C}}(2)[\mu] = \begin{pmatrix} w & e & \bar{w} \\ t & \bar{t} & \bar{s} \\ s & t & \bar{t} \end{pmatrix},$$

$$\text{and } C^{\sharp}(2) \equiv C_{\mathbb{C}}^{\sharp}(2)[\mu] = \begin{pmatrix} a & \bar{b} & \bar{d} \\ b & c & \bar{b} \\ d & b & a \end{pmatrix} (\geq 0).$$

Example 5.1. Suppose $\mu \geq 0$ is a planar Borel measure with convergent moments up to at least degree 3 and suppose $\text{Ran } B(2) \subset \text{Ran } M(1)$. If $a = c$ in $C^{\sharp}(2)$, then clearly $\rho = 0$. In this case, the existence of a minimal cubature rule for μ of degree 3 having exactly $\text{rank } M(1)$ nodes follows from the complex version of Theorem 1.9 (cf. Theorem 2.2); such a rule can be explicitly constructed using [CF2, Theorem 4.7] (cf. Theorem 2.3).

We note that the preceding case ($a = c$) includes the cases when $r \equiv \text{rank } M(1) \leq 2$. Indeed, in these cases there are constants $\alpha, \beta \in \mathbb{C}$ such that

$$\bar{Z} = \alpha 1 + \beta Z \tag{5.1}$$

in $\text{Col } M(1)$, whence $\bar{z} = \alpha + \beta z$ in $\text{supp } \mu$ [CF2, Proposition 3.1]. Multiplying this last relation by various powers of z and \bar{z} and then integrating with respect to μ shows that the following relations hold in the columns of $(M(1) \ B(2))$:

$$\bar{Z}Z = \alpha Z + \beta Z^2, \tag{5.2}$$

$$\bar{Z}^2 = \alpha \bar{Z} + \beta \bar{Z}Z. \tag{5.3}$$

Since $\text{Ran } B(2) \subset \text{Ran } M(1)$, there are scalars $A, B \in \mathbb{C}$ such that in $\text{Col } \begin{pmatrix} M(1) & B(2) \end{pmatrix}$,

$$Z^2 = A1 + BZ. \quad (5.4)$$

Using the definition of $C^\sharp(2)$, we see that (5.1) – (5.4) must hold in the columns of $[M(1); B(2)]$ as well, whence $a = A\bar{w} + B\bar{t} = A(\alpha\bar{x} + \beta e) + B(\alpha e + \beta t) = \alpha(A\bar{x} + Be) + \beta(Ae + Bt) = \alpha t + \beta b = c$.

Suppose now that $a \neq c$, so that $\text{rank } M(1) = 3$. For $\beta \in \mathbb{C}$, $\beta \neq b$, let

$$\alpha = \frac{1}{2}(a + c + \sqrt{(a - c)^2 + 4|\beta - b|^2}) \quad (5.5)$$

and $\delta = d + \frac{(\beta - b)^2}{\alpha - c}$. A calculation shows that

$$T \equiv \begin{pmatrix} \alpha & \bar{\beta} & \bar{\delta} \\ \beta & \alpha & \bar{\beta} \\ \delta & \beta & \alpha \end{pmatrix}$$

satisfies $T \geq C^\sharp(2)$ and that $\text{rank}(T - C^\sharp(2)) = 1$. Thus $\rho = 1$, and any degree 3 cubature rule for μ has at least 4 ($= \text{rank } M(1) + \rho$) nodes (Theorem 1.7).

We next address existence (and construction) of 4-node (minimal) rules of degree 3 for the case $a \neq c$. With T as above, let

$$M(2) = \begin{pmatrix} M(1) & B(2) \\ B(2)^* & T \end{pmatrix}.$$

Proposition 2.6 implies that $M(2) \geq 0$ and that $\text{rank } M(2) = \text{rank } M(1) + \text{rank}(T - C^\sharp(2)) = 4 = \text{rank } [M(2)]_4$ (since $M(1) > 0$ and $\alpha > a$); here, $[M(2)]_4$ denotes the compression of $M(2)$ to the first 4 rows and columns. In $\text{Col } M(2)$ we thus have a linear dependence relation

$$\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2, \quad (5.6)$$

where $A, B, C, D \in \mathbb{C}$ depend on β . Let $p_\beta(z, \bar{z}) = z\bar{z} - (A + Bz + C\bar{z} + Dz^2)$. If, for some $\beta \neq b$, $D = 0$, then [CF4, Theorem 1.2] implies that μ has a unique (minimal) 4-node cubature rule ν_β of degree 3 satisfying $\gamma_{13}[\nu_\beta] = \beta$, and the rule may be constructed as in [CF4, Section 2].

Suppose now that for each $\beta \neq b$, we have $D \neq 0$ in p_β . In this case, it follows from [F2, Theorem 1.3] that μ has a 4-node minimal cubature rule of degree 3 if and only if there exists $\beta \neq b$, such that $\text{card } \mathcal{Z}(p_\beta) \geq 4$ (this occurs, in particular, if $|D| \neq 1$ [F2, Proposition 1.6]); for each such β , a minimal rule can be constructed as in [F2]. (A similar result, involving a pair of real orthogonal polynomials instead of p_β , was obtained by Goit [G] (cf. [Str4, page 99]).) \square

Concerning the last case in Example 5.1, it is an open question whether there always exists some $\beta \neq b$ for which $\text{card } \mathcal{Z}(p_\beta) \geq 4$. In the case when μ is centrally symmetric, we next use the preceding method to show that each $\beta \neq b$ corresponds to a unique 4 node (minimal) rule of degree 3. An equivalent, but

different, parametrization of these minimal rules is due to Stroud [Str2] (cf. [Str4, Theorem 3.9-2]).

Proposition 5.2. *Suppose $\mu \geq 0$ is a centrally symmetric planar measure with moments up to at least degree 3, and suppose $M(1) > 0$, with $\gamma_{00} = 1$. Then $a \neq c$, and for $\beta \neq b$, let $\alpha = (1/2)(a + c + \sqrt{(a-c)^2 + 4|\beta-b|^2})$. Let $q_\beta(z) = (D^2 - s)z^4 + (2AD - r)z^2 + A^2$, where $A = \frac{\alpha\sqrt{c} - \beta(b/\sqrt{c})}{\alpha - a}$, $D = \frac{\beta - \bar{b}}{\alpha - a}$, $r = (A - \bar{A})/\bar{D}$, and $s = D/\bar{D}$. Then q_β has 4 distinct roots, $\{z_i\}_{i=0}^3$, which provide the support for a minimal cubature rule $\mu_\beta \equiv \sum_{i=0}^3 \rho_i \delta_{z_i}$ of degree 3, and any minimal rule of degree 3 is of this form, where the densities ρ_i are uniquely determined by $V(z_0, z_1, z_2, z_3)(\rho_0, \rho_1, \rho_2, \rho_3)^t = (\gamma_{00}, \gamma_{01}, \gamma_{02}, \gamma_{03})^t$.*

Proof. Since $M(1) > 0$, μ is square positive (Proposition 2.9), so Theorem 1.3 implies that each cubature rule for μ of degree 3 has at least 4 ($> \text{rank } M(1)[\mu]$) nodes. Example 5.1 thus implies that $a \neq c$, and we define α as in (5.5). Since μ is centrally symmetric, we have $x = t = s = 0$, and a calculation shows that in (5.6), $A = \frac{\alpha\sqrt{c} - \beta(b/\sqrt{c})}{\alpha - a}$, $B = 0$, $C = 0$, $D = \frac{\beta - \bar{b}}{\alpha - a}$. Further, from the definition of $C^\sharp(2)$, $a = |w|^2$ and $e^2 = c$. Since $M(1) > 0$ then $|w|^2 > e^2$, and consequently $a > c$. Another calculation now shows that $\alpha - a < |\beta - b|$, whence $|D| \neq 1$. It thus follows from [CF4, Corollary 3.4] or [F2, Proposition 1.6] that $M(2)$ has a unique representing measure ν_β , which serves as a minimal cubature rule for μ of degree 3.

To construct ν_β , write (5.6) as $\bar{Z}Z = A1 + DZ^2$. It follows from [CF2, Lemma 3.10] that $\bar{Z}Z = \bar{A}1 + \bar{D}\bar{Z}^2$, whence

$$\bar{Z}^2 = r1 + sZ^2, \quad (5.7)$$

with $r = (A - \bar{A})/\bar{D}$, $s = D/\bar{D}$. [CF4] and [CF2] together imply that $M(2)$ has a unique flat, recursively generated extension $M(4)$. By recursiveness, (5.6) and (5.7) imply that in $\text{Col } M(4)$ we have $A^21 + 2ADZ^2 + D^2Z^4 = \bar{Z}^2Z^2 = rZ^2 + sZ^4$. Now [CF2, Theorem 4.7] implies that $q_\beta(z) \equiv (D^2 - s)z^4 + (2AD - r)z^2 + A^2$ has 4 distinct roots, z_i ($0 \leq i \leq 3$), which comprise $\text{supp } \nu_\beta$. The densities ρ_i ($0 \leq i \leq 3$) of ν_β are uniquely determined by $V(z_0, z_1, z_2, z_3)(\rho_0, \rho_1, \rho_2, \rho_3)^t = (\gamma_{00}, \gamma_{01}, \gamma_{02}, \gamma_{03})^t$, where V denotes the Vandermonde matrix. \square

Example 5.3. To illustrate Proposition 5.2, consider the square $C_2 = [-1, 1] \times [-1, 1]$ with planar measure. Since $\gamma_{00} = 4$, we cannot directly use the formulas of Proposition 5.2, but we can use exactly the same method. We compute $\alpha = (1/2) \left((16/9) + \sqrt{(16/9)^2 + 4|\beta|^2} \right)$ and $\delta = \beta^2/(\alpha - c)$. For a numerical example, let $\beta = 1/10$. We have $A = 2/3$ and $D \approx 0.0560731$. The roots of q_β are $z_0 \approx -0.794525i$, $z_1 = -z_0$, $z_2 \approx -0.840398$, $z_3 = -z_2$ and we use the Vandermonde equations to compute $\rho_0 = \rho_1 \approx 1.05607$, and $\rho_2 = \rho_3 \approx 0.943927$. \square

We now begin the study of $C^\sharp(3)[\mu]$. Let

$$C^\sharp \equiv C^\sharp(3)[\mu] = \begin{pmatrix} a & \bar{b} & \bar{e} & \bar{f} \\ b & c & \bar{d} & \bar{e} \\ e & d & c & \bar{b} \\ f & e & b & a \end{pmatrix} (\geq 0),$$

$$T = \begin{pmatrix} \alpha & \bar{\beta} & \bar{\gamma} & \bar{\delta} \\ \beta & \alpha & \bar{\beta} & \bar{\gamma} \\ \gamma & \beta & \alpha & \bar{\beta} \\ \delta & \gamma & \beta & \alpha \end{pmatrix} (\geq C^\sharp),$$

and $\Delta \equiv T - C^\sharp (\geq 0)$. In the sequel, $[\Delta]_k$ denotes the compression of Δ to its first k rows and columns.

Proposition 5.4. *For a planar measure $\mu \geq 0$, $\rho(C^\sharp(3)[\mu]) \leq 1$.*

Proof. If $a = c$ and $b = d$, then clearly $\rho = 0$. We next consider the case $a = c$, $b \neq d$, and we claim that $\rho = 1$. Choose $\beta \in \mathbb{C}$ such that $|\beta - d| = |\beta - b| (> 0)$, and let $\alpha = a + |\beta - b|$. Then $[\Delta]_1 > 0$, and $[\Delta]_2 \geq 0$ with $\text{rank } [\Delta]_2 = 1$. To insure that $[\Delta]_3 \geq 0$ with $\text{rank } [\Delta]_3 = 1$, we require $\gamma \in \mathbb{C}$ such that

$$\frac{\bar{\gamma} - \bar{e}}{\bar{\beta} - \bar{b}} = \frac{\bar{\beta} - \bar{d}}{\alpha - c} = \frac{\alpha - c}{\beta - d}. \quad (5.8)$$

Since $|\beta - d|^2 = |\beta - b|^2 = (\alpha - a)^2 = (\alpha - c)^2$, (5.8) holds if and only if $\bar{\gamma} = \bar{e} + \frac{(\bar{\beta} - \bar{b})(\bar{\beta} - \bar{d})}{\alpha - c}$. To complete the construction of T with $\Delta \geq 0$ and $\text{rank } \Delta = 1$, we seek $\delta \in \mathbb{C}$ such that

$$\frac{\bar{\delta} - \bar{f}}{\bar{\gamma} - \bar{e}} = \frac{\bar{\gamma} - \bar{e}}{\bar{\beta} - \bar{d}} = \frac{\bar{\beta} - \bar{b}}{\alpha - c} = \frac{\alpha - a}{\beta - b},$$

which reduces to $\bar{\delta} = \frac{(\bar{\gamma} - \bar{e})^2}{\bar{\beta} - \bar{d}} + \bar{f}$; thus $\rho = 1$.

We now consider the case $a \neq c$, $b \neq d$; we will show that $\rho = 1$. To insure that $[\Delta]_3 \geq 0$ and $\text{rank } [\Delta]_3 = 1$, we require $\alpha, \beta, \gamma \in \mathbb{C}$, with $\alpha \geq a, c$, such that

$$(\alpha - a)(\alpha - c) = |\beta - b|^2, \text{ and} \quad (5.9)$$

$$\frac{\bar{\gamma} - \bar{e}}{\bar{\beta} - \bar{b}} = \frac{\bar{\beta} - \bar{d}}{\alpha - c} = \frac{\alpha - c}{\beta - d}. \quad (5.10)$$

To solve (5.10), we choose $\beta \in \mathbb{C}$ such that $\beta \neq b$ and $\beta \neq d$, and we set $\alpha = c + |\beta - d| (> c)$ and $\bar{\gamma} = \bar{e} + \frac{(\bar{\beta} - \bar{b})(\bar{\beta} - \bar{d})}{\alpha - c}$. To assure $\alpha \geq a$, we further require $|\beta - d| \geq a - c$. Now $(\alpha - a)(\alpha - c) = ((\alpha - c) + (c - a))|\beta - d| = (|\beta - d| + c - a)|\beta - d|$, so (5.9) is equivalent to

$$(|\beta - d| + c - a)|\beta - d| = |\beta - b|^2. \quad (5.11)$$

Let $\psi = c - a$ and $r = |\beta - d|/|\beta - b|$; (5.11) is equivalent to $|\beta - b|r^2 + \psi r - |\beta - b| = 0$, with $r > 0$, or $r = \frac{-\psi + \sqrt{\psi^2 + 4|\beta - b|^2}}{2|\beta - b|}$. It readily follows that $|\beta - d| \geq a - c$, as required. To complete the construction of T such that $\Delta \geq 0$ and $\text{rank } \Delta = 1$, it remains to choose δ such that $\bar{\delta} = \frac{(\bar{\gamma} - \bar{e})^2}{\bar{\beta} - \bar{d}} + \bar{f}$.

Finally, in the case $a \neq c$, $b = d$, we take $\alpha = c$, $\beta = b$, $\gamma = e$, and $\delta = f + (c - a)z$, where z is an arbitrary point in the unit circle $|z| = 1$. It is easy to see that

$$\Delta = \begin{pmatrix} c - a & 0 & 0 & (c - a)\bar{z} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (c - a)z & 0 & 0 & c - a \end{pmatrix},$$

so $\text{rank}(\Delta) = 1$ and, consequently, $\rho = 1$.

We note for future reference that, in the last case, if a Toeplitz matrix T satisfies $T \geq C^\sharp$ and $\text{rank}(\Delta) = 1$, then Δ has the above form. Indeed, in this situation,

$$\Delta = \begin{pmatrix} \alpha - a & \bar{\beta} - \bar{b} & \bar{\gamma} - \bar{e} & \bar{\delta} - \bar{f} \\ \beta - b & \alpha - c & \bar{\beta} - \bar{b} & \bar{\gamma} - \bar{e} \\ \gamma - e & \beta - b & \alpha - c & \bar{\beta} - \bar{b} \\ \delta - f & \gamma - e & \beta - b & \alpha - a \end{pmatrix}.$$

The condition that $\text{rank}(\Delta) = 1$ applied to compressions of this matrix to rows and columns 1 and 2 (resp., 2 and 3) yields $(\alpha - a)(\alpha - c) = |\beta - b|^2 = (\alpha - c)^2$. Since $a \neq c$, it follows that $\alpha = c$ and, consequently, $\beta = b$. Turning attention to the compression to rows 3 and 4 and columns 1 and 2, we see that $\gamma = e$. Finally, the compression to rows and columns 1 and 4 shows that $|\delta - f|^2 = (c - a)^2$, and the result follows. \square

In the next series of results we use moment matrix techniques to study minimal cubature rules for $\mu_{\mathbb{D}}$, planar measure on the closed unit disk.

Theorem 5.5. *The minimal cubature rules of degree 5 for the disk are given by the measures*

$$\nu_\kappa \equiv \sum_{i=0}^6 \rho_i \delta_{z_i},$$

where $\kappa \in \mathbb{C}$ satisfies $|\kappa| = 2\pi/9$, $z_0 = 0$, z_i ($1 \leq i \leq 6$) are the 6th roots of $4\kappa/(3\pi)$, $\rho_0 = \pi/4$, and $\rho_i = \pi/8$, ($1 \leq i \leq 6$).

In [R], J. Radon developed a general method for constructing 7-node, degree 5 cubature rules for subsets of the plane satisfying a hypothesis concerning common zeros of orthogonal polynomials. For the weight function $w(x, y) \equiv 1$, Radon's rule for the disk yields the nodes $(0, 0)$, $(\pm\sqrt{2/3}, 0)$, $(\pm\sqrt{1/6}, \pm\sqrt{1/2})$, with weights $\pi/4$ for $(0, 0)$ and $\pi/8$ for each of the other nodes [Str4, page 279]. This rule corresponds to Theorem 5.5 with $d = 2\pi/9$.

Proof of Theorem 5.5. A calculation shows that $C^\sharp = \text{diag}(0, 2\pi/9, 2\pi/9, 0)$, so Theorem 4.4 implies that any cubature rule for $\mu_{\mathbb{D}}$ of degree 5 satisfies $N \geq 7$. In Proposition 5.4 we have $a \neq c$, $b = d = 0$, so by the note at the conclusion of Proposition 5.4 and by the complex version of Theorem 1.7, any minimal rules of degree 5 with as few as 7 nodes would correspond to flat extensions of

$$M_\kappa(3) \equiv \begin{pmatrix} M(2) & B(3) \\ B(3)^* & T_\kappa \end{pmatrix},$$

where

$$T_\kappa = \begin{pmatrix} 2\pi/9 & 0 & 0 & \bar{\kappa} \\ 0 & 2\pi/9 & 0 & 0 \\ 0 & 0 & 2\pi/9 & 0 \\ \kappa & 0 & 0 & 2\pi/9 \end{pmatrix}$$

and $|\kappa| = 2\pi/9$. We now show that for each κ with $|\kappa| = 2\pi/9$, $M_\kappa(3)$ admits a unique flat extension. Such an extension is completely determined by a choice of new moments of degree 7: $\gamma_{34} \equiv x$, $\gamma_{25} \equiv u$, $\gamma_{16} \equiv v$, $\gamma_{07} \equiv w$ (and their conjugates). Direct calculation shows that in the column space of $\begin{pmatrix} M(3) & B(4) \end{pmatrix}$ we have

$$Z^4 = \frac{3}{\pi}\kappa\bar{Z}^2 + \frac{9x}{2\pi}Z^3 \quad (5.12)$$

and

$$Z^3\bar{Z} = \frac{2}{3}Z^2 + \frac{9\bar{x}}{2\pi}Z^3. \quad (5.13)$$

From (5.12), $u = v = 0$ and $w = (9x/(2\pi))\kappa$. From (5.13), $x = 0$, whence $w = 0$ follows from the previous identity. With these choices for the new moments of order 7, it is straightforward to verify that the flat extension $[M(3); B(4)]$ is a moment matrix $M(4)$; indeed, if we express $B(4)$ as $B(4) = M(3)W$, then a calculation shows that $W^*M(3)W$ is Toeplitz.

In $M(4)$ we have column relations $Z^4 = (3/\pi)\kappa\bar{Z}^2$, $Z^3\bar{Z} = (2/3)Z^2$, and $Z^2\bar{Z}^2 = (2/3)Z\bar{Z}$, which readily imply that the variety of $M(4)$ consists of $z_0 = 0$ and the distinct 6-th roots of $4\kappa/(3\pi)$. The result now follows from Corollary 2.4, together with a Vandermonde calculation, which shows that $\rho_0 = \pi/4$ and $\rho_i = \pi/8$ ($1 \leq i \leq 6$). \square

We continue with a cubature rule of degree 4 for the disk.

Proposition 5.6. ([F2]) *The minimal $\mu_{\mathbb{D}}$ cubature rules of degree 4 with the additional property that they also interpolate the $\mu_{\mathbb{D}}$ -moments of $Z^4\bar{Z}$, $Z^3\bar{Z}^2$, $Z^2\bar{Z}^3$, and $Z\bar{Z}^4$, correspond to the 6-atomic measures $\nu_{a,b} = \sum_{k=0}^5 \rho_k \delta_{z_k}$, where $a, b \in \mathbb{R}$ satisfy $a^2 + b^2 = 2\pi^2/27$, $z_0 = 0$, z_k ($1 \leq k \leq 5$) are the 5-th roots of $(4/(3\pi))(a + ib)$, $\rho_0 = \pi/4$, and $\rho_k = 3\pi/20$ ($1 \leq k \leq 5$).*

Proof. We are seeking a flat extension of $M(2)$ in which $B(3)$ has exact moments of degree 5 (namely, 0) except for γ_{05} and $\gamma_{50} (= \bar{\gamma}_{05})$. Using the same method as in the proof of Theorem 5.5, a calculation shows that the values of γ_{05} for which $B(3)^*M(2)^{-1}B(3)$ is Toeplitz correspond precisely to the measures $\nu_{a,b}$. \square

Remark 5.7. Note that if a $\mu_{\mathbb{D}}$ cubature rule as in Proposition 5.6 also interpolates the moment for Z^5 (and hence for \bar{Z}^5), the rule would have degree 5, and its size would be at least 7, so in this sense the rule in Proposition 5.6 is optimal. For certain measures on the disk of the form $\mu \equiv w(x, y) dx dy$, a minimal 6-node rule of degree 4 is referenced in [Str4, S2:4–1, page 278]. For the weight $w(x, y) \equiv 1$, this rule corresponds to $\nu_{a,b}$ with $a = \sqrt{2/27}\pi$, $b = 0$.

We next develop a family of minimal rules of degree 7 for $\mu_{\mathbb{D}}$. By Theorem 1.3 or Theorem 1.11, any such rule has $N \geq 12$, and a 12-node rule is cited in [Str4, pg. 281]. In the sequel, $T(a_1, a_2, \dots, a_n)$ denotes a self-adjoint Toeplitz matrix whose top row is (a_1, a_2, \dots, a_n) .

Proposition 5.8. *Let $\alpha = 81\pi/416$, $\beta = \delta = 0$; for a complex number w satisfying $|w| = 1$, let $\gamma = 3\pi w/416$, $\epsilon = (81\pi/416)(\gamma/\bar{\gamma})$, and $T_w = T(\alpha, \beta, \gamma, \delta, \epsilon)$. Then*

$$M(4) \equiv \begin{pmatrix} M(3)[\mu_{\mathbb{D}}] & B(4)[\mu_{\mathbb{D}}] \\ B(4)[\mu_{\mathbb{D}}]^* & T_w \end{pmatrix}$$

is a rank 12 positive moment matrix, with a flat extension $M(5)$ corresponding (via Theorem 2.3) to a 12-node cubature rule of degree 7 for $\mu_{\mathbb{D}}$.

Proof. Note that $C^\sharp(4)[\mu_{\mathbb{D}}] = \text{diag}(0, 3\pi/16, 7\pi/36, 3\pi/16, 0)$; a calculation shows that $T_w - C^\sharp(4)[\mu_{\mathbb{D}}]$ is positive with rank 2, so by Proposition 2.6, $M(4)$ is positive with rank 12. A further calculation using Proposition 2.6 reveals that in any positive moment matrix extension $M(5)$ of $M(4)$ all moments of degree 9 must equal 0. With these values in $B(5)$, we find the following dependence relations in the column space of $(M(4) \ B(5))$:

$$\begin{aligned} Z^5 &= -\frac{9}{52} wZ + \frac{27}{104} wZ^2\bar{Z} + \frac{81}{104} \frac{w}{\bar{w}} \bar{Z}^3; & Z^4\bar{Z} &= -\frac{9}{52} w\bar{Z} + \frac{81}{104} Z^3 + \frac{27}{104} wZ\bar{Z}^2; \\ Z^3\bar{Z}^2 &= -\frac{9}{52} Z + \frac{105}{104} Z^2\bar{Z} + \frac{3}{104} w\bar{Z}^3; & Z^2\bar{Z}^3 &= -\frac{9}{52} \bar{Z} + \frac{3}{104} \bar{w}Z^3 + \frac{105}{104} Z\bar{Z}^2; \\ Z\bar{Z}^4 &= -\frac{9}{52} \bar{w}Z + \frac{27}{104} \bar{w}Z^2\bar{Z} + \frac{81}{104} \bar{Z}^3; & \bar{Z}^5 &= -\frac{9}{52} \bar{w}\bar{Z} + \frac{81}{104} \bar{w}^2Z^3 + \frac{27}{104} \bar{w}Z\bar{Z}^2. \end{aligned}$$

The preceding system determines a matrix W such that $M(4)W = B(5)$, and a calculation shows that $C(5) \equiv W^*M(4)W$ is Toeplitz. Thus $[M(4); B(5)]$ is a flat moment matrix extension $M(5)$ of $M(4)$, and the existence of a 12-node rule of

degree 7 now follows from Theorem 2.2 (or the complex version of Theorem 1.7); to compute the nodes and densities, we may use Theorem 2.3.

For a numerical example, set $w = 1$, $\gamma = 3\pi/416$, $\epsilon = 81\pi/416$. A calculation shows that the variety of $M(5)$ consists of the following 12 nonzero common solutions to the polynomial equations corresponding to the above dependencies: $z_0 = -\sqrt{(27 - 3\sqrt{29})/52}i \approx -0.4566707613i$, $z_1 = \bar{z}_0$, $z_2 = -\sqrt{(27 + 3\sqrt{29})/52}i \approx -0.9109958036i$, $z_3 = \bar{z}_2$, $z_4 = -\sqrt{3/8}(1 + i) \approx -0.6123724357(1 + i)$, $z_5 = \bar{z}_4$, $z_6 = \sqrt{3/8}(1 - i)$, $z_7 = \bar{z}_6$, $z_8 = -iz_0$, $z_9 = iz_0$, $z_{10} = -iz_2$, $z_{11} = iz_2$. Since $\text{card } \mathcal{V}(M(5)) = 12 = \text{rank } M(4)$, Corollary 2.4 implies that $\{z_i\}_{i=0}^{11}$ forms the support of a minimal (inside) cubature rule for $\mu_{\mathbb{D}}$ of degree 7, with corresponding densities $\rho_0 = \rho_1 = \rho_8 = \rho_9 \approx 0.3870777960$, $\rho_2 = \rho_3 = \rho_{10} = \rho_{11} \approx 0.1656098005$, $\rho_4 = \rho_5 = \rho_6 = \rho_7 \approx 0.2327105669$. \square

It is not difficult to see that the preceding rule cannot be obtained from the rule in [Str4] by means of rotations or reflections.

We next turn our attention to minimal rules of degree 9. In [Mo2] Möller proved that a cubature rule of degree $4k + 1$ for a planar measure with circular symmetry satisfies $N \geq k(2k + 4) + 1$, and Möller provided necessary conditions for the existence of a rule attaining this lower bound in terms of zeros of certain orthogonal polynomials. In [VC, Theorem 4] Verlinden and Cools obtained concrete criteria for the existence of rules attaining Möller's bound and showed, in particular, that a degree 9 rule for $\mu_{\mathbb{D}}$ requires at least 18 nodes [VC, page 404] (cf. [CH] [MP]). An 18-node rule had previously been obtained by Haegemans and Piessens [HP], who conjectured its minimality. These results show that the lower bound in Theorem 1.11 is not sharp for n even. We next give a moment matrix proof of the nonexistence of 17-node rules of degree 9 for $\mu_{\mathbb{D}}$ and a moment matrix characterization of the 18-node rules. We start with the following result.

Proposition 5.9. *Let $a < b < c$ and let $C = \text{diag}(a, b, c, c, b, a) \geq 0$. A Toeplitz matrix T satisfies $\Delta \equiv T - C \geq 0$ and $\text{rank } \Delta = 2 (= \rho_{\mathbb{C}}(C))$, cf., Theorem 4.1) if and only if*

$$T = \begin{pmatrix} c & 0 & 0 & 0 & \epsilon & 0 \\ 0 & c & 0 & 0 & 0 & \epsilon \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ \bar{\epsilon} & 0 & 0 & 0 & c & 0 \\ 0 & \bar{\epsilon} & 0 & 0 & 0 & c \end{pmatrix} \quad (5.14)$$

with $|\epsilon|^2 = (c - a)(c - b)$.

Proof. It is straightforward to check that if T has the indicated form, then Δ is positive with rank 2. For the converse, let $T = T(\alpha, \beta, \gamma, \delta, \epsilon, \varphi)$ be a Toeplitz

selfadjoint matrix such that $\Delta \geq 0$ and $\text{rank } \Delta = 2$. Then

$$\Delta = \begin{pmatrix} \alpha - a & \beta & \gamma & \delta & \epsilon & \varphi \\ \bar{\beta} & \alpha - b & \beta & \gamma & \delta & \epsilon \\ \bar{\gamma} & \bar{\beta} & \alpha - c & \beta & \gamma & \delta \\ \bar{\delta} & \bar{\gamma} & \bar{\beta} & \alpha - c & \beta & \gamma \\ \bar{\epsilon} & \bar{\delta} & \bar{\gamma} & \bar{\beta} & \alpha - b & \beta \\ \bar{\varphi} & \bar{\epsilon} & \bar{\delta} & \bar{\gamma} & \bar{\beta} & \alpha - a \end{pmatrix}.$$

Since $\text{rank } \Delta = 2$ it follows that the compression $D(i_1, i_2, i_3; j_1, j_2, j_3)$ of Δ to rows i_1, i_2, i_3 and columns j_1, j_2, j_3 must be of rank at most 2, and therefore has zero determinant; further, since $\Delta \geq 0$, the determinant of every central compression of Δ is nonnegative. Considering $D(1, 2, 3; 2, 3, 4)$ and $D(2, 3, 4; 3, 4, 5)$ we obtain

$$\begin{aligned} 0 &= \beta^3 + \gamma^2 \bar{\beta} + \delta(\alpha - b)(\alpha - c) - \delta|\beta|^2 - \beta\gamma(\alpha - c) - \beta\gamma(\alpha - b) \\ 0 &= \beta^3 + \gamma^2 \bar{\beta} + \delta(\alpha - c)(\alpha - c) - \delta|\beta|^2 - \beta\gamma(\alpha - c) - \beta\gamma(\alpha - c). \end{aligned}$$

Subtracting these equations yields $\delta(\alpha - c)(c - b) + \beta\gamma(b - c) = 0$. Since $c > b$, we see that

$$\delta(\alpha - c) = \beta\gamma. \quad (5.15)$$

Next we turn our attention to $D(1, 2, 3; 1, 3, 4)$ and $D(2, 3, 4; 2, 4, 5)$. This leads to

$$\begin{aligned} 0 &= \beta^2(\alpha - a) + \gamma|\gamma|^2 + \delta\bar{\beta}(\alpha - c) - \beta\delta\bar{\gamma} - \gamma(\alpha - c)(\alpha - a) - \gamma|\beta|^2 \\ 0 &= \beta^2(\alpha - b) + \gamma|\gamma|^2 + \delta\bar{\beta}(\alpha - c) - \beta\delta\bar{\gamma} - \gamma(\alpha - c)(\alpha - b) - \gamma|\beta|^2. \end{aligned}$$

Subtracting one equation from another yields $\beta^2(b - a) + \gamma(\alpha - c)(a - b) = 0$. Since $b > a$, we obtain

$$\beta^2 = \gamma(\alpha - c). \quad (5.16)$$

Now we notice that

$$\begin{aligned} \begin{vmatrix} \alpha - a & \beta & \gamma \\ \bar{\beta} & \alpha - b & \beta \\ \bar{\gamma} & \bar{\beta} & \alpha - c \end{vmatrix} - \begin{vmatrix} \alpha - a & \beta & \gamma \\ \bar{\beta} & \alpha - c & \beta \\ \bar{\gamma} & \bar{\beta} & \alpha - c \end{vmatrix} &= \begin{vmatrix} \alpha - a & 0 & \gamma \\ \bar{\beta} & c - b & \beta \\ \bar{\gamma} & 0 & \alpha - c \end{vmatrix} \\ &= (c - b) \begin{vmatrix} \alpha - a & \gamma \\ \bar{\gamma} & \alpha - c \end{vmatrix} \geq 0, \end{aligned}$$

since $c > b$ and since the last determinant corresponds to the central compression $D(1, 3; 1, 3)$. Similarly,

$$\begin{aligned} \begin{vmatrix} \alpha - a & \beta & \gamma \\ \bar{\beta} & \alpha - c & \beta \\ \bar{\gamma} & \bar{\beta} & \alpha - c \end{vmatrix} - \begin{vmatrix} \alpha - b & \beta & \gamma \\ \bar{\beta} & \alpha - c & \beta \\ \bar{\gamma} & \bar{\beta} & \alpha - c \end{vmatrix} &= \begin{vmatrix} b - a & \beta & \gamma \\ 0 & \alpha - c & \beta \\ 0 & \bar{\beta} & \alpha - c \end{vmatrix} \\ &= (b - a) \begin{vmatrix} \alpha - c & \beta \\ \bar{\beta} & \alpha - c \end{vmatrix} \geq 0. \end{aligned}$$

Adding the preceding two sequences of equations and inequalities, we see that the leftmost expression of the sum is 0, since it is the difference of $D(1, 2, 3; 1, 2, 3)$ and $D(2, 3, 4; 2, 3, 4)$. Thus,

$$0 = (c - b)[(\alpha - a)(\alpha - c) - |\gamma|^2] + (b - a)[(\alpha - c)^2 - |\beta|^2] \geq 0,$$

and it follows that $(\alpha - a)(\alpha - c) = |\gamma|^2$ and $(\alpha - c)^2 = |\beta|^2$. Combining these relations with (5.16), we see that $(\alpha - c)^2 = |\gamma| |\alpha - c|$. Of course, $\alpha - c \geq 0$. If $\alpha - c > 0$, then $|\gamma| = \alpha - c$, and it would follow that $|\gamma| = \alpha - a$, implying that $a = c$, a contradiction. Thus $\alpha = c$, and consequently $\beta = \gamma = 0$. Using these values, a consideration of $D(4, 5, 6; 1, 2, 3)$ now shows that $\delta = 0$. Similarly, $D(1, 2, 5; 2, 5, 6)$ can be used to deduce that $\varphi = 0$. It follows that T must be of the form as in (5.14) and it is easy to see that $|\epsilon|^2 = (c - a)(c - b)$. \square

Now we use our approach to give a new proof that there is no 17 point rule of degree 9 for planar measure on the disk.

Proposition 5.10. *There is no 17 point rule of degree 9 for $\mu_{\mathbb{D}}$.*

Proof. Applying Theorem 4.4, with $n = 4$, yields $N \geq 17$. Of course, this estimate is based on Theorem 1.7, the invertibility of $M(4)$ (so that $\text{rank } M(4) = 15$), and the inequality $\rho(C^\sharp(5)) \geq 2$ based on Theorem 4.1. Since $C^\sharp(5) = \text{diag}(0, 4\pi/25, 33\pi/200, 33\pi/200, 4\pi/25, 0)$, Proposition 5.9 shows that a Toeplitz matrix T satisfies $T \geq C^\sharp(5)$ and $\text{rank}(T - C^\sharp(5)) = 2$ if and only if $T \equiv T_z = T(33\pi/200, 0, 0, 0, \sqrt{33\pi}z/200, 0)$, where $|z| = 1$. Theorem 1.7 thus implies that $N = 17$ is attainable if and only if there exists z ($|z| = 1$) such that

$$M(5) \equiv \begin{pmatrix} M(4) & B(5) \\ B(5)^* & T_z \end{pmatrix} \quad (5.17)$$

admits a flat extension. Thus, in order to prove that $N > 17$, we have to demonstrate that every $M(5)$ (as above) fails to have a flat extension. For such $M(5)$, we consider the existence of a flat moment matrix extension of the form

$$M(6) \equiv \begin{pmatrix} M(5) & B(6) \\ B(6)^* & C \end{pmatrix}.$$

Note that the first 15 rows of $B(6)$ contain moments of degree up to 10, and are already contained in $M(5)$. We will show that the remaining 6 rows, with moments of degree 11, consist of zeros only.

We start by writing

$$M(6) = \begin{pmatrix} A_1 & A_2 & B_1 \\ A_2^* & A_3 & B_2 \\ B_1^* & B_2^* & C \end{pmatrix}$$

where A_1 is a compression of $M(5)$ to the first 17 rows and columns. By the invertibility of $M(4)$ and the choice of T , it is not hard to see that A_1 is invertible, and thus $M(6)$ is a flat extension of A_1 . By Corollary 2.7 there is a matrix $X =$

$(X_1 \ X_2)$ such that $A_1X = (A_2 \ B_1)$. Clearly, $A_1X_2 = B_1$ and $A_1X_1 = A_2$ so that $X_1 = A_1^{-1}A_2$. Also,

$$\begin{pmatrix} A_3 & B_2 \\ B_2^* & C \end{pmatrix} = X^*A_1X, \quad (5.18)$$

so it follows that $B_2 = X_1^*A_1X_2 = X_1^*B_1 = (A_1^{-1}A_2)^*B_1 = A_2^*A_1^{-1}B_1$. Notice that the first 15 rows of B_1 consist of moments of degree up to 10, so they can be read from $M(5)$. The last two rows contain moments of degree 11 that have yet to be determined, so we set them as

$$\begin{pmatrix} \gamma_{56} & \gamma_{65} & \gamma_{74} & \gamma_{83} & \gamma_{92} & \gamma_{10,1} & \gamma_{11,0} \\ \gamma_{47} & \gamma_{56} & \gamma_{65} & \gamma_{74} & \gamma_{83} & \gamma_{92} & \gamma_{10,1} \end{pmatrix}.$$

Let $h = (1/\sqrt{33})\bar{z}$ and $k = \sqrt{33}\bar{z}$. Then, a calculation shows that

$$B_2 = A_2^*A_1^{-1}B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h\gamma_{56} & h\gamma_{65} & h\gamma_{74} & h\gamma_{83} & h\gamma_{92} & h\gamma_{10,1} & h\gamma_{11,0} \\ k\gamma_{47} & k\gamma_{56} & k\gamma_{65} & k\gamma_{74} & k\gamma_{83} & k\gamma_{92} & k\gamma_{10,1} \end{pmatrix},$$

and since B_2 must be Toeplitz, it follows immediately that $\gamma_{i,11-i} = 0$ for $i \geq 6$. In view of the fact that $\gamma_{ij} = \bar{\gamma}_{ji}$ we see that all moments of degree 11 are 0. Thus, the matrix B_1 is completely determined. Moreover, (5.18) shows that $C = X_2^*A_1X_2$. Since $A_1X_2 = B_1$ we have that $X_2 = A_1^{-1}B_1$, and thus $C = (A_1^{-1}B_1)^*A_1(A_1^{-1}B_1) = B_1^*A_1^{-1}B_1$. It is now easy to establish that C is not Toeplitz since, for example, its $(1,1)$ entry is $33/500$, while the $(2,2)$ entry is $1122/8000$. \square

Corollary 5.11. *A positive rank 18 completion $M(5)$ of $\begin{pmatrix} M(4)[\mu_{\mathbb{D}}] & B(5)[\mu_{\mathbb{D}}] \\ B(5)[\mu_{\mathbb{D}}]^* & \end{pmatrix}$ which has a flat extension $M(6)$ corresponds to an 18-node cubature rule for $\mu_{\mathbb{D}}$ of degree 9, and conversely.*

Proof. The direct implication follows immediately from Theorem 2.2. Conversely, suppose ν is an 18-node rule for $\mu_{\mathbb{D}}$ of degree 9. We claim that $\text{rank } M(5)[\nu] = 18$. By (2.4) and Proposition 2.6, $18 = \text{card supp } \nu \geq \text{rank } M(5)[\nu] = \text{rank } M(4)[\mu_{\mathbb{D}}] + \text{rank } (C(5)[\nu] - C^{\sharp}(5)[\mu_{\mathbb{D}}]) \geq 17$ (since $\text{rank } M(4)[\mu_{\mathbb{D}}] = 15$ and $C^{\sharp}(5)[\mu_{\mathbb{D}}]$ has 2 gaps). If $\text{rank } M(5)[\nu] = 17$, then $\text{rank } (C(5)[\nu] - C^{\sharp}(5)[\mu_{\mathbb{D}}]) = 2$, so Proposition 5.9 implies that $C(5)[\nu]$ has the form of (5.14). For such $M(5)[\nu]$, a calculation similar to that in the proof of Theorem 5.10 shows that in $M(6)[\nu]$, all moments of degree 11 equal 0. From this, a further calculation implies that $\rho(C^{\sharp}(6)[\nu]) \geq 2$, whence $\text{rank } M(6)[\nu] \geq 19$. This contradiction shows that $\text{rank } M(5)[\nu] = 18$; [CF2] now implies that $18 = \text{rank } M(\infty)[\nu] \geq \text{rank } M(6)[\nu] \geq \text{rank } M(5)[\nu] = 18$, so $M(6)[\nu]$ is a flat extension of $M(5)[\nu]$. \square

The first cubature rule of degree 9 for $\mu_{\mathbb{D}}$ with as few as 19 nodes was found by Albrecht [A] (cf. [Str4, S2:9–1, pg. 281]); an infinite family of such rules is described in [HP]. We next present a family of 19-node rules which includes Albrecht's as a special case; a feature of these rules is that they arise from 2-step extensions: starting with a special rank 18 completion $M(5)$ of $\begin{pmatrix} M(4)[\mu_{\mathbb{D}}] & B(5)[\mu_{\mathbb{D}}] \\ B(5)[\mu_{\mathbb{D}}]^* & T_w \end{pmatrix}$, we construct positive extensions $M(6)$ and $M(7)$ satisfying $\text{rank } M(7) = \text{rank } M(6) = 19$.

Proposition 5.12. *For $w \in \mathbb{C}$, $|w| = 1$, let $T_w = T(\alpha, 0, 0, \delta, 0, 0)$, where $\alpha = 128\pi/775$ and $\delta = (4\pi/775)w$. Then $M(5) \equiv \begin{pmatrix} M(4)[\mu_{\mathbb{D}}] & B(5)[\mu_{\mathbb{D}}] \\ B(5)[\mu_{\mathbb{D}}]^* & T_w \end{pmatrix}$ is a rank 18 positive moment matrix which has positive extensions $M(6)$ and $M(7)$ satisfying $\text{rank } M(7) = \text{rank } M(6) = 19$. The unique measure μ_w corresponding to the flat extension $M(7)$ (cf. Theorem 2.3) is a 19-node cubature rule of degree 9 for $\mu_{\mathbb{D}}$. For $w = 1$, μ_w coincides with Albrecht's rule [A].*

Proof. We have $C \equiv C^{\sharp}(5)[\mu_{\mathbb{D}}] = \text{diag}(a, b, c, c, b, a)$ with $a = 0$, $b = 4\pi/25$, $c = 33\pi/200$. Then $\alpha \equiv b^2/(2b-c)$ ($= 128\pi/775 > c > b > a$) satisfies $(\alpha - a)(\alpha - c) = (\alpha - b)^2$; with $\delta = (4\pi/775)\bar{w}$, we set $T_w = T(\alpha, 0, 0, \delta, 0, 0)$. It follows readily that $\Delta \equiv T - C$ satisfies $\Delta \geq 0$ and $\text{rank } \Delta = 3$, whence $M(5)$ is positive with rank 18 (cf. Proposition 2.6). A calculation similar to that in the proof of Theorem 5.10 shows that in any positive moment matrix extension $M(6)$ of $M(5)$, all moments of degree 11 must equal 0. With these values, $C^{\sharp}(6)$ is not Toeplitz, so there is no flat extension $M(6)$; however, if we set $T' = T(\alpha', 0, 0, \delta', 0, 0, \eta')$ with $\alpha' = 3328\pi/24025$, $\delta' = (1264\pi/120125)\bar{w}$, $\eta' = (3328\pi/24025)\bar{w}^2$, then $M(6) \equiv \begin{pmatrix} M(5) & B(6) \\ B(6)^* & T' \end{pmatrix}$ is positive with rank 19. We claim that $M(6)$ has a unique flat extension $M(7)$. Indeed, in any positive moment matrix extension $M(7)$, we require $\text{Ran } B(7) \subset \text{Ran } M(6)$, and a calculation (as in Theorem 5.10) shows that this requirement is satisfied if and only if all moments of degree 13 equal 0; with these values, we see that $C^{\sharp}(7)$ is indeed Toeplitz.

Let μ_w denote the unique measure corresponding to the flat extension $M(7)$ (cf. Theorem 2.3); μ_w is thus a 19-node cubature rule for $\mu_{\mathbb{D}}$ of degree 9. To compute the nodes and densities of μ_w we may use Theorem 2.3 or Corollary 2.4. More simply, note that in $\text{Col } M(5)$ we have the following dependencies: $Z^2\bar{Z}^3 = -(3/10)\bar{Z} + (6/5)Z\bar{Z}^2 + (1/32)\bar{w}Z^5$, $Z\bar{Z}^4 = -(4/5)\bar{w}Z^3 + (4/5)\bar{Z}^3 + \bar{w}Z^4\bar{Z}$, $\bar{Z}^5 = (48/5)\bar{w}Z - (192/5)\bar{w}Z^2\bar{Z} + 32\bar{w}Z^3\bar{Z}^2$. Since $M(7)$ is a positive extension of $M(5)$, the same relations hold in $\text{Col } M(7)$ (cf. [F1]); thus, $\text{supp } \mu_w$ is contained in the common zeros of the polynomials corresponding to these relations. For $w = 1$, we find that there are precisely 19 distinct common zeros, $\{z_k\}_{k=0}^{18}$, which must therefore coincide with $\text{supp } \mu_1$. Let $r = \sqrt{3/5}$, $s = \sqrt{1/5}$, $t = \sqrt{(96 - 4\sqrt{111})/155}$, $u = \sqrt{(96 + 4\sqrt{111})/155}$, $p = \sqrt{(1/3)(72 - 3\sqrt{111})/155}$,

$q = \sqrt{(1/3)(72 + 3\sqrt{111})/155}$. Then $z_0 = 0$, $z_k = \pm r \pm si$, ($1 \leq k \leq 4$), $z_k = \pm(2/\sqrt{5})i$, ($k = 5, 6$), $z_k = \pm t$, ($k = 7, 8$), $z_k = \pm p \pm pi$, ($7 \leq k \leq 12$), $z_k = \pm u$, ($k = 13, 14$), $z_k = \pm q \pm qi$, ($15 \leq k \leq 18$); all points are inside \mathbb{D} . The corresponding densities (computed using Corollary 2.4) are $\rho_0 \approx 0.3422481580$, $\rho_k \approx 0.1278317323$, ($1 \leq k \leq 6$), $\rho_k \approx 0.2617858597$, ($7 \leq k \leq 12$), $\rho_k \approx 0.0769398239$, ($13 \leq k \leq 18$). Albrecht's rule is described in [Str4] in terms of trigonometric functions, but it is easy to see that it coincides with μ_1 . \square

All of the preceding examples concern planar cubature rules, but the main results of Section 3 apply to measures on \mathbb{R}^d ; we conclude with an example in \mathbb{R}^3 .

Example 5.13. We develop a family of minimal cubature rules of degree 2 for volume measure $\mu_{\mathbb{B}}$ in the unit ball \mathbb{B} of \mathbb{R}^3 (cf. [Str4]). Here, $M(1)[\mu_{\mathbb{B}}] = \text{diag}(4\pi/3, 4\pi/15, 4\pi/15, 4\pi/15)$, with rows and columns indexed by $1, X, Y, Z$. To compute a 4-node (minimal) rule of degree 2 we seek new moments $a = \beta_{300}$, $b = \beta_{210}$, $c = \beta_{201}$, $d = \beta_{120}$, $f = \beta_{111}$, $e = \beta_{102}$, $g = \beta_{030}$, $h = \beta_{021}$, $p = \beta_{012}$, $q = \beta_{003}$, so that

$$B(2) \equiv \begin{pmatrix} 4\pi/15 & 0 & 0 & 4\pi/15 & 0 & 4\pi/15 \\ a & b & c & d & f & e \\ b & d & f & g & h & p \\ c & f & e & h & p & q \end{pmatrix}$$

has the property that $B(2)^*M(1)^{-1}B(2)$ is a 3-dimensional moment matrix block $C(2)$. One branch of the solution to $C(2) = B(2)M(1)^{-1}B(2)$ is given by $a = (1125d^2 - 16\pi^2)/(1125d)$, $e = -16\pi^2/(1125d)$, $b = c = h = p = f = 0$, with d , g , and q free variables. With these choices, in the column space of $[M(1); B(2)]$ we find relations $X^2 = (1/5)1 + (1125d^2 - 16\pi^2)/(300d\pi)X$, $XY = 15d/(4\pi)Y$, $XZ = -4\pi/(75d)Z$, $Y^2 = (1/5)1 + 15d/(4\pi)X + 15g/(4\pi)Y$, $YZ = 0$, $Z^2 = (1/5)1 - (4\pi/75)dX + 15q/(4\pi)Z$, and the corresponding variety, $\mathcal{V}(M(2))$, has precisely 4 points,

$$\begin{aligned} p_1 &= (15d/(4\pi), (75g - \sqrt{5(4500d^2 + 1125g^2 + 64\pi^2)})/(40\pi), 0), \\ p_2 &= (15d/(4\pi), (75g + \sqrt{5(4500d^2 + 1125g^2 + 64\pi^2)})/(40\pi), 0), \\ p_3 &= (-4\pi/(75d), 0, (1125dq - \sqrt{72000d^2\pi^2 + 1024\pi^4 + 1265625d^2q^2})/(600d\pi)), \\ p_4 &= (-4\pi/(75d), 0, (1125dq + \sqrt{72000d^2\pi^2 + 1024\pi^4 + 1265625d^2q^2})/(600d\pi)). \end{aligned}$$

To obtain an inside rule, we can choose, for example, $d = 0.5298$, $g = q = 0$, with $p_1 \approx (0.632402, -0.774553, 0)$, $p_2 \approx (0.632402, 0.774553, 0)$, $p_3 \approx (-0.316254, 0, -0.547738)$, $p_4 \approx (-0.316254, 0, 0.547738)$, and corresponding densities $w_1 = w_2 \approx 0.69821$, $w_3 = w_4 \approx 1.39618$. \square

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