ON SPECTRAL RADIUS ALGEBRAS

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Abstract. We show how one can associate a Hermitian operator P to every operator A, and we prove that the invertibility properties of P imply the non-transitivity and density of the spectral radius algebra associated to A. In the finite dimensional case we give a complete characterization of these algebras in terms of P. In addition, we show that in the finite dimensional case, the spectral radius algebra always properly contains the commutant of A.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{L}(\mathcal{H})$ then a subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for T if $T\mathcal{M} \subset \mathcal{M}$ and it is hyperinvariant for T if it is invariant for every operator in the commutant $\{T\}'$ of T. A nontrivial invariant subspace (n. i. s.) is one that is neither $\mathcal{H}$ nor the zero subspace. It was shown in [2] that one can associate the so-called spectral radius algebra $\mathcal{B}_A$ to each operator A. Such an algebra always contains $\{A\}'$ so, when it has a n. i. s. it represents a generalization of the concept of a hyperinvariant subspace. Such is the case when A is compact (cf., [2]) or a certain type of normal operator (see Theorem 3.6 below). Therefore, it is essential to establish that the inclusion $\{A\}' \subset \mathcal{B}_A$ is proper. Otherwise, any invariant subspace result is probably well known and can be obtained in an easier manner. In the other direction, it is important to demonstrate that the algebra is not too "big", i. e., weakly dense or equal to $\mathcal{L}(\mathcal{H})$, which would preclude it having a n. i. s.

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In this paper we introduce a tool that can be used to calibrate the size of $\mathcal{B}_A$ (i. e., determine whether $\mathcal{B}_A$ has a n. i. s., or is dense in, or equal to $\mathcal{L}(\mathcal{H})$). We will show that a Hermitian operator $P = P_A$ can be associated to A in a natural way, and we will see how the properties of P relate to the properties of $\mathcal{B}_A$. Although this approach can be used in Hilbert space of any finite or infinite dimension, we have been able to obtain the complete characterization only in the case when $\mathcal{H}$ is finite dimensional (see Theorem 2.5, Theorem 2.12, and Theorem 4.1 below). We will also address the issue whether $\mathcal{B}_A$ properly contains $\{A\}'$. We will show (see Theorem 4.3 and Theorem 4.8 below) that, when $\mathcal{H}$ is finite dimensional the inclusion in question is proper.

Before we proceed, we briefly review the relevant definitions. Interested readers can find more details in [2]. If $A \in \mathcal{L}(\mathcal{H})$ and $m \geq 1$, we define

$$R_m(A) = R_m := \left(\sum_{n=0}^{\infty} d_m^{2n} A^{*\,n} A^n\right)^{1/2} \text{ where } d_m = \frac{1}{\frac{1}{m} + r(A)}. \tag{1.1}$$

Since $d_m \uparrow 1/r(A)$ (we use the convention $1/0 = \infty$), the sum in (1.1) is norm convergent and the operators R_m are well defined, positive, and invertible. The spectral radius algebra $\mathcal{B}_A$ consists of all operators $T \in \mathcal{L}(\mathcal{H})$ such that $\sup_{m\in\mathbb{N}} \|R_m T R_m^{-1}\| < \infty$. An important subset of $\mathcal{B}_A$ is the collection $\mathcal{Q}_A = \{T \in \mathcal{L}(\mathcal{H}) : \|R_m T R_m^{-1}\| \to 0\}$. The following result (cf., [2, Theorem 3.4]) shows why.

Theorem 1.1. *If $\mathcal{Q}_A \neq (0)$ and there exists a nonzero compact operator in $\mathcal{B}_A$, then $\mathcal{B}_A$ has a n. i. s.*

When A is compact itself, it is always possible to construct a non-zero operator in $\mathcal{Q}_A$, as was proved in [2, Theorem 4.1].

Theorem 1.2. *Let K be a nonzero compact operator on the separable, infinite dimensional Hilbert space $\mathcal{H}$. Then $\mathcal{B}_K$ has a n. i. s.*

Since $d_m \uparrow 1/r(A)$, the sequence R_m^2 is increasing and, hence, R_m^{-2} is a decreasing sequence of positive operators. Consequently, there is a (Hermitian) operator $P = P_A$ such that R_m^{-1} converges strongly to P.

2. Basic properties of P_A

In this section we will establish some basic properties of P_A. We start with two examples in $\mathcal{L}(\mathbb{C}^2)$.

Example 2.1. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}$, $A^{*n}A^n = \begin{pmatrix} 2^{2n} & 0 \\ 0 & 1 \end{pmatrix}$, $R_m^2 = \begin{pmatrix} (1-4d_m^2)^{-1} & 0 \\ 0 & (1-d_m^2)^{-1} \end{pmatrix}$, and $R_m^{-2} = \begin{pmatrix} 1-4d_m^2 & 0 \\ 0 & 1-d_m^2 \end{pmatrix}$, so $P = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix}$.

Example 2.2. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Then, $A^n = \begin{pmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{pmatrix}$ and $A^{*n}A^n = \begin{pmatrix} 2^{2n} & 2^{2n}-2^n \\ 2^{2n}-2^n & 2^{2n}-2\cdot 2^n+2 \end{pmatrix}$. Consequently,

$$R_m^2 = \begin{bmatrix} \dfrac{1}{1-4d_m^2} & \dfrac{1}{1-4d_m^2} - \dfrac{1}{1-2d_m^2} \\ \dfrac{1}{1-4d_m^2} - \dfrac{1}{1-2d_m^2} & \dfrac{1}{1-4d_m^2} - \dfrac{2}{1-2d_m^2} + \dfrac{2}{1-d_m^2} \end{bmatrix},$$

and

$$R_m^{-2} = \frac{1}{\det(R_m^2)} \begin{bmatrix} \dfrac{1}{1-4d_m^2} - \dfrac{2}{1-2d_m^2} + \dfrac{2}{1-d_m^2} & \dfrac{1}{1-2d_m^2} - \dfrac{1}{1-4d_m^2} \\ \dfrac{1}{1-2d_m^2} - \dfrac{1}{1-4d_m^2} & \dfrac{1}{1-4d_m^2} \end{bmatrix}.$$

A calculation shows that $\det(R_m^2) = \dfrac{2}{(1-4d_m^2)(1-d_m^2)} - \dfrac{1}{(1-2d_m^2)^2}$ and, since $d_m \to 1/2$, we obtain that $P = \frac{\sqrt{3}}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

These examples show that P is not invariant under similarity. However, our next result establishes a connection between similar operators.

Proposition 2.3. *Let $B = TAT^{-1}$. Then $P_B^2 \geq \dfrac{1}{\|T\|^2} T P_A^2 T^*$. In particular, $P_A = 0$ if and only if $P_B = 0$.*

Proof. Since A and B are similar, they have the same spectral radius so $d_m(A) = d_m(B)$, and we denote it by d_m. Then, for all $x \in \mathcal{H}$, $\|R_m(B)x\|^2 = \sum d_m^{2n}\|TA^nT^{-1}x\|^2 \leq \|T\|^2\|R_m(A)T^{-1}x\|^2$. Therefore, $I \leq R_m(B)^2 \leq \|T\|^2 T^{*-1}R_m(A)^2T^{-1}$ and the result follows by taking inverses and letting $m \to \infty$. □

Before we proceed we recall a result from [1].

Theorem 2.4. *Let $A \in \mathcal{L}(\mathcal{H})$. Then $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$ if and only if the operator A is similar to a constant multiple of an isometry.*

It is our goal to establish the connection between P_A and $\mathcal{B}_A$. The following result does that in the case when P_A is injective (or equivalently, when it has dense range).

Theorem 2.5. *With $P = P_A$ as before, we have that $P_A\mathcal{L}(\mathcal{H}) \subset \mathcal{B}_A$. Consequently, if the range of P_A is dense in $\mathcal{H}$, then $\mathcal{B}_A$ is dense in $\mathcal{L}(\mathcal{H})$. Furthermore, if the range of P_A is dense in $\mathcal{H}$, then the equality $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$ holds if and only if P_A is invertible in which case $P_A = I$ and $A = 0$.*

Proof. Clearly $R_mP^2R_m \leq R_mR_m^{-2}R_m = I$ so $R_mP^2R_m$ and R_mP are both contractions. If $T \in \mathcal{L}(\mathcal{H})$, then $\|R_mPTR_m^{-1}\| \leq \|R_mP\|\,\|T\|\,\|R_m^{-1}\| \leq \|T\|$. Consequently, $PT \in \mathcal{B}_A$.

If P is invertible it follows that $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$ and, by Theorem 2.4, A must be similar to λV with V an isometry. If $\lambda \neq 0$, a calculation shows that $P_{\lambda V} = 0$ and Proposition 2.3 would imply that $P_A = 0$. Thus, $\lambda = 0$ and $A = 0$, and it is easy to see that $P_0 = I$.

Notice that $\mathcal{B}_A$ contains every rank one operator of the form $Px \otimes y$ and, if the range of P is dense, these rank one operators span a dense subalgebra of $\mathcal{L}(\mathcal{H})$. □

Corollary 2.6. *When $P_A \neq 0$ we always have that $\mathcal{B}_A \neq \{A\}'$.*

Proof. As we have seen, every rank one operator of the form $Px \otimes y$ belongs to $\mathcal{B}_A$. On the other hand $Px \otimes y$ commutes with A only when y is an eigenvector of A^*. (Of course, A is not scalar since such operators have $P = 0$.) □

The following example shows that P_A can have dense range properly contained in $\mathcal{H}$.

Example 2.7. Let $A = \operatorname{diag}\left(\frac{i}{i+1}\right)$. An easy calculation shows that $P = \operatorname{diag}\left(\sqrt{\frac{2i+1}{(i+1)^2}}\right)$, and it is easy to see that the range of this operator is dense, but it is not surjective.

Now we turn our attention to the case $P = 0$. The following theorem gives a characterization in terms of the asymptotic behavior of the sequence R_m.

Theorem 2.8. *$P = 0$ if and only if $\lim_m \|R_m x\| = \infty$ for all nonzero vectors $x \in \mathcal{H}$.*

Proof. Once again, $R_m P^2 R_m \leq R_m R_m^{-2} R_m = I$ so PR_m is a contraction and the same is true of $R_m P$. Thus, we have $\|R_m Px\| \leq \|x\|$, and if $\lim_m \|R_m x\| = \infty$ for all nonzero $x \in \mathcal{H}$ then P must be the zero operator.

Conversely, if there is $x_0 \in \mathcal{H}$ such that $\sup_m \|R_m x_0\|$ is finite, then for each $y \in \operatorname{Ker} P$, $\langle x_0, y\rangle = \lim_m \langle R_m x_0, R_m^{-1} y\rangle = 0$. Thus, x_0 is orthogonal to $\operatorname{Ker} P$ which shows that $P \neq 0$. □

If $Ax = 0$ then $\|R_m x\| = \|x\|$. Thus we obtain

Corollary 2.9. *If $\operatorname{Ker} A \neq (0)$ then $P_A \neq 0$.*

Remark 2.10. Set $\mathcal{E} = \mathcal{E}_A = \{x \in \mathcal{H} : \sup_m \|R_m x\| < \infty\}$. It is easy to see that $\mathcal{E}$ is a linear manifold invariant under $\mathcal{B}_A$. Indeed, it suffices to notice, that for each $T \in \mathcal{B}_A$ there is $M > 0$ so that $\|R_m T x\| \leq M \|R_m x\|$ for all $x \in \mathcal{H}$.

Example 2.1 shows that the converse to Corollary 2.9 is not true. It becomes true, however, when A is quasinilpotent.

Proposition 2.11. *Suppose A is quasinilpotent and one-to-one. Then, $P_A = 0$.*

Proof. Note that if A is quasinilpotent, then $d_m = m$. Consequently, for all $x \in \mathcal{L}(\mathcal{H})$, we have $\|R_m x\|^2 \geq m^2 \|Ax\|^2 \to \infty$ unless $x \in \operatorname{Ker} A$. In view of Theorem 2.8 this finishes the proof. □

Theorem 2.5 shows that, if P is injective, then $\mathcal{B}_A$ is dense in $\mathcal{L}(\mathcal{H})$. In fact, as we will show, the converse is almost true as well.

Theorem 2.12. *Let $P_A \neq 0$ and not injective. Then $\mathcal{B}_A$ has a n. i. s.*

Proof. Since P is not injective, there is a non-zero vector y such that $Py = 0$ and, hence, $\lim_m R_m^{-1} y = 0$. On the other hand, Theorem 2.8 implies that there is a nonzero vector $x \in \mathcal{H}$ such that $\sup_m \|R_m x\|$ is finite. Consequently, the rank one operator $x \otimes y$ belongs to $\mathcal{Q}_A$ and the result follows from Theorem 1.1. □

Theorem 2.5 and Theorem 2.12 show that, when $P_A \neq 0$, there is a nice relationship between P_A and $\mathcal{B}_A$. Namely, if $\operatorname{Ker} P \neq (0)$ then $\mathcal{B}_A$ has a n. i. s.; if $\operatorname{Ker} P = (0)$ but the range of P is a proper, dense subset of $\mathcal{H}$ then $\mathcal{B}_A$ is a proper, weakly dense subalgebra of $\mathcal{L}(\mathcal{H})$; when P is invertible, $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$. Unfortunately, when $P_A = 0$, as the following examples will show, $\mathcal{B}_A$ can have any of the 3 properties above.

Example 2.13. Let $A = S$, the unilateral shift. Since S is an isometry, $\mathcal{B}_S = \mathcal{L}(\mathcal{H})$. On the other hand, $S^{*n} S^n = I$, so $R_m^2 = \sum d_m^{2n}$, $R_m^{-2} = 1 - d_m^2$, and $P = 0$.

Example 2.14. Let A be a weighted shift with weights $\alpha_1 = 1$, $\alpha_n = \sqrt{(n-1)/n}$ for $n \geq 2$. Then $R_m^2 = \operatorname{diag}\left(\sum d_m^{2n} k/(n+k)\right) = \dfrac{k}{d_m^{2k}} \left[-\ln(1-d_m^2) - 1 - d_m^2 - \dfrac{d_m^4}{2} - \cdots - \dfrac{d_m^{2k-2}}{k-1} \right]$ and it is easy to see that $P = 0$. On the other hand, $\mathcal{B}_A$ contains every rank-one operator of the form $e_i \otimes e_j$, where $\{e_n\}$ is the orthonormal basis that A shifts. Indeed, the previous calculation shows that the ith entry on the diagonal of R_m behaves asymptotically (as $m \to \infty$) like $-\dfrac{i \ln(1-d_m^2)}{d_m^{2i}}$, so the jth entry of R_m^{-1} behaves as $-\dfrac{d_m^{2j}}{j\ln(1-d_m^2)}$ and $\|R_m e_i\|\|R_m^{-1}e_j\|$ has the limit i/j. Of course, one knows that A is not similar to an isometry (cf., [3, Theorem 2]), so $\mathcal{B}_A \neq \mathcal{L}(\mathcal{H})$.

Example 2.15. Let A be a weighted shift with weights $\alpha_n = 1/\sqrt{n}$ for $n \geq 1$. Since A is quasinilpotent and injective, $P_A = 0$ by Proposition 2.11. Furthermore, A is compact so, by Theorem 1.2, $\mathcal{B}_A$ has a n. i. s.

The examples above show that it may be hard to characterize operators A for which $P_A = 0$. Surprisingly, if $\mathcal{H}$ is finite dimensional such a characterization is available. We will postpone this discussion until Section 4.

3. Applications

In this section, we will compute P_A and use it to analyze $\mathcal{B}_A$ in some specific cases, namely when the operator A is either a co-isometry, or quasinormal or a backward weighted shift. We start by considering the case when A is a co-isometry. One knows that A is unitarily equivalent to a direct sum $S^* \oplus U$ acting on $\mathcal{K} = \mathcal{G} \oplus \mathcal{G}'$ where S is a forward unilateral shift (of arbitrary multiplicity) on $\mathcal{G}$ and U is a unitary operator on $\mathcal{G}'$. Recall that A is a pure co-isometry if $\mathcal{G}' = (0)$. One knows that $\mathcal{G} = \oplus_{i=1}^{\infty}\mathcal{G}_i$, where each $\mathcal{G}_i$ can be identified with $\operatorname{Ker} S^*$.

Proposition 3.1. *Suppose $A = S^* \oplus U$ as above. Then $P_A = P \oplus 0$ where P is the diagonal operator relative to $\mathcal{G} = \bigoplus_{i=0}^{\infty} \mathcal{G}_i$ given by $P_{S^*}x = \left(\dfrac{x_k}{\sqrt{k+1}}\right)$, $x = (x_k)_{k=0}^{\infty} \in \mathcal{G}$.*

Proof. It is easy to see that $R_m(A) = R_m(S^*) \oplus R_m(U)$, hence $P_A = P_{S^*} \oplus P_U$ A calculation shows that $R_m^2(U) = (1-d_m^2)^{-1}$ so $P_U = 0$. On the other hand, $R_m(S^*)$ is the diagonal operator $\operatorname{diag}(\sum_{i=0}^{n} d_m^{2i})$ so P_{S^*} is the diagonal operator $\operatorname{diag}(1, 1/\sqrt{2}, 1/\sqrt{3}, \dots)$. □

Corollary 3.2. *When A is a pure co-isometry, then $\mathcal{B}_A$ is weakly dense, but not equal to $\mathcal{L}(\mathcal{H})$. Otherwise, $\mathcal{B}_A$ has a n. i. s.*

Proof. This follows from Theorem 2.5 and Theorem 2.12. □

We will now consider the case when A is quasinormal. Recall that an operator A is quasinormal if A commutes with (A^*A). Furthermore, if A is quasinormal, then $r(A) = \|A\|$.

Theorem 3.3. *If A is quasinormal, then $P_A^2 = I - \dfrac{1}{\|A\|^2}(A^*A)$.*

Proof. Using the fact that $A(A^*A) = (A^*A)A$, it is easy to prove by induction that, for all $n \in \mathbb{N}$, $A^{*n}A = (A^*A)^n$. Consequently, $R_m^{-2} = I - d_m^2 A^*A$. The result now follows from the fact that $\lim_{m\to\infty} d_m = 1/r = 1/\|A\|$. □

Remark 3.4. As a consequence of the previous result, if A is quasinormal, then we have

(1) P_A commutes with A.

(2) $\operatorname{Ker} P_A = \operatorname{Ker} P_A^2 = \{x : \|Ax\| = \|A\|\|x\|\}$.

Theorem 3.3 allows us to obtain a result about the spectral radius algebra associated to a quasinormal operator.

Corollary 3.5. *Let A be a quasinormal operator and let $\mathcal{M} = \{x \in \mathcal{H} : \|Ax\| = \|A\|\|x\|\}$. If $\mathcal{M} = (0)$ then $\mathcal{B}_A$ is dense but not equal to $\mathcal{L}(\mathcal{H})$; if $\mathcal{M} = \mathcal{H}$ then $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$. In all other cases, $\mathcal{B}_A$ has a n. i. s.*

Proof. Notice that $\mathcal{M} = \operatorname{Ker} P_A$, so the result follows from Theorem 2.5 and Theorem 2.12. □

As a special case, when A is normal, we obtain one of the main results of [1].

Theorem 3.6. *Let A be a nonzero normal operator on $\mathcal{H}$ with $\|A\| = 1$.*

(a) If A is unitary, then $\mathcal{B}_A = \mathcal{L}(\mathcal{H})$.

(b) If A is completely nonunitary, then $\mathcal{B}_A$ is weakly dense in, but not equal to, $\mathcal{L}(\mathcal{H})$.

(c) If A is neither unitary nor completely nonunitary, then $\mathcal{B}_A$ has a n. i. s.

Let W denote a forward weighted shift (of any multiplicity) with weight sequence $\{\alpha_n\}_{n\in\mathbb{N}}$. Regarding the adjoint W^* we have the following theorem.

Theorem 3.7. *If $r(W^*) = 0$ then P is the projection on $\operatorname{Ker} W^*$ so $\mathcal{B}_{W^*} \neq \{W^*\}'$ has a n. i. s. If $r(W^*) \neq 0$ then $\operatorname{Ran} P \neq \mathcal{H}$ is dense, so $\mathcal{B}_{W^*} \neq \mathcal{L}(\mathcal{H})$ is weakly dense.*

Proof. A straightforward calculation combined with Theorem 2.12 leads to the first assertion. If $r = r(W^*) \neq 0$, then $P = \operatorname{diag}(1, (1 + |\alpha_1|^2/r^2)^{-1/2}, (1 + |\alpha_2|^2/r^2 + |\alpha_1\alpha_2|^2/r^4)^{-1/2}, \dots)$. Since none of the diagonal entries is 0, P is injective, and the result follows from Theorem 2.5. $\square$

4. The finite dimensional case

In this section we consider the case when $\dim \mathcal{H} < \infty$. In this situation the case $P_A = 0$ can be completely characterized. Also, we will show that $\mathcal{B}_A \neq \{A\}'$.

Theorem 4.1. *Suppose that $\dim \mathcal{H} < \infty$ and $A \in \mathcal{L}(\mathcal{H})$. Then $P_A = 0$ if and only if $r(A) > 0$ and every eigenvalue λ of A satisfies $|\lambda| = r(A)$. In this situation, $\mathcal{B}_A$ has a n. i. s. if and only if A is not diagonalizable.*

Proof. Suppose that $P_A = 0$. Then $r = r(A) \neq 0$, otherwise A would have a nontrivial kernel and, by Corollary 2.9, we would have that $P_A \neq 0$. We will show that, if $|\lambda| \neq r$, then $A - \lambda I$ is invertible. To that end, let $T_m = R_m A R_m^{-1}$ and notice that $T_m^* T_m = R_m^{-1} A^* R_m^2 A R_m^{-1} = R_m^{-1}(\frac{1}{d_m^2}(R_m^2 - I))R_m^{-1} = \frac{1}{d_m^2}(I - R_m^{-2}) \to r^2 I$. Since $A \in \mathcal{B}_A$, the sequence T_m is bounded and there is a subsequence T_{m_k} that converges (in norm) to an operator T. Clearly, $T^*T = r^2 I$ so $T - \lambda I$ is invertible. One knows that the set of invertible operators is open, so there is m such that $R_m A R_m^{-1} - \lambda I$ is invertible. Consequently, $A - \lambda I$ must be invertible.

Conversely, suppose that $r(A) > 0$ and every eigenvalue λ of A satisfies $|\lambda| = r(A)$. Notice that $\mathcal{E}$ is an invariant subspace for A so all eigenvalues of the restriction $A|\mathcal{E}$ are the eigenvalues of A. If $P \neq 0$, Theorem 2.8 would imply that $\mathcal{E}$ is a non-zero subspace and there would be a non-zero vector $x_0 \in \mathcal{E}$ such that $Ax_0 = \lambda x_0$ with $|\lambda| = r(A)$. However, an easy calculation shows that $\|R_m x_0\|^2 = \|x_0\|^2/(1 - r^2 d_m^2) \to \infty$, as $m \to \infty$ contradicting $x_0 \in \mathcal{E}$. Thus $P = 0$. Finally, the last assertion follows from Theorem 2.4. □

In the case where $\mathcal{H}$ is finite dimensional, $\mathcal{B}_A$ is either $\mathcal{L}(\mathcal{H})$ or it has a n. i. s. In the former case, Theorem 2.4 gives a complete characterization of A. In the latter, it is of interest to establish that $\mathcal{B}_A$ properly contains the commutant $\{A\}'$. First we tackle the case when A has at least 2 different eigenvalues. In this situation, the conclusion will be an easy consequence of the following result (cf., [2, Corollary 2.4]).

Proposition 4.2. *Let T be an operator such that $AT = \lambda TA$ for some complex number λ with $|\lambda| \leq 1$. Then $T \in \mathcal{B}_A$.*

Proposition 4.2 leads to the following result.

Theorem 4.3. *Let A be a non-zero operator on a finite dimensional Hilbert space, and suppose that the spectrum of A consists of at least 2 points, or of zero alone. In both cases $\mathcal{B}_A \neq \{A\}'$.*

Proof. Denote two distinct eigenvalues of A by α and β, and suppose that $|\alpha| \leq |\beta|$ and $\beta \neq 0$. Then there exist vectors u and v so that $Au = \alpha u$ and $A^*v = \overline{\beta} v$. It is easy to see that $A(u \otimes v) = (\alpha/\beta)(u \otimes v)A$. Therefore, $u \otimes v$ is an extended eigenvector for A corresponding to the extended eigenvalue $\lambda = \alpha/\beta$ so $u \otimes v \in \mathcal{B}_A$. On the other hand, since $\alpha \neq \beta$, $u \otimes v \notin \{A\}'$. When 0 is the only eigenvalue, we take u, v so that $Au = 0$ and v is not an eigenvector for A^*. Then $A(u \otimes v) = 0(u \otimes v)A$ so $u \otimes v \in \mathcal{B}_A$, and it is easy to see that $u \otimes v \notin \{A\}'$. □

So, it remains to consider the case when the spectrum of A consists of a single (nonzero) point. The following theorem (cf., [1, Theorem 2.4]) shows that in order to study the spectral radius algebra, it suffices to look at the similarity models.

Theorem 4.4. *Let $A, B \in \mathcal{L}(\mathcal{H})$ and T be an invertible operator in $\mathcal{L}(\mathcal{H})$ such that $T^{-1}BT = A$. The map $X \mapsto T^{-1}XT$ is an isomorphism from $\mathcal{B}_B$ onto $\mathcal{B}_A$.*

From now on, we will assume that the operators under scrutiny are in the Jordan form. We start with some preliminary results that will be useful in the proof of our main theorem.

Lemma 4.5. *Let $|x| < 1$ and let $s_k(x) = \sum_{n=0}^{\infty} n^k x^n$. Then $s_k(x)$ is a polynomial of degree $k+1$ in $(1-x)^{-1}$, with the leading coefficient equal to $k!$.*

Proof. It is easy to verify that $s_{n+1}(x) = x[s_n(x)]'$ for $n \geq 0$, whence the proof follows by induction. □

The following result is little harder to prove.

Proposition 4.6. *Let B be an $n \times n$ matrix with (i, j) entry $\binom{i+j}{i}$, $0 \leq i, j \leq n-1$. Then* $\det(B) = 1$.

Proof. Let M_k denote the $k \times k$ minor of B consisting of the k bottom rows and k leftmost columns. We will prove by induction that, for each k, $\det(M_k) = 1$. The case $k = 1$ is obvious since the $(n-1, 0)$ entry of B is 1. We make a hypothesis that $\det(M_k) = 1$ and we consider M_{k+1}. Notice that the leftmost column of B (and, hence, of M_{k+1}) has each entry equal to 1. We will apply a sequence of elementary transformations to M_{k+1} that will have the effect of row transformations $R_{n-1} - R_{n-2} \to R_{n-1}$, $R_{n-2} - R_{n-3} \to R_{n-2}, \ldots, R_{n-k} - R_{n-k-1} \to R_{n-k}$. (By $R_m - R_p \to R_m$ we mean that the row p is subtracted from the row m, and the result placed back in the row m. It is well known that these transformations leave the determinant invariant.) Now the leftmost column of the so obtained matrix M'_{k+1} has 1 in the top position and zeros elsewhere, so the determinant of M'_{k+1} equals the determinant of its principal minor M'_k obtained from M'_{k+1} by deleting its top row and the leftmost column. In order to complete the proof, we notice that $M'_k = M_k$. Indeed, by the construction, the (i, j) entry of M'_k equals

$\binom{i+j}{i} - \binom{i+j-1}{i-1} = \binom{i+j-1}{i}$ which is precisely the $(i, j-1)$ entry of M_k. Thus M_k and M_k' have the corresponding entries equal, and the proposition is proved. $\square$

Next we address the asymptotic behavior of $\det(R_m)$.

Theorem 4.7. *Let α be a complex number and let J_α be the $N \times N$ Jordan block with eigenvalue $\alpha \neq 0$. If $A = J_\alpha$ then $\det(R_m)$ has the order of magnitude $(1-\alpha^2 d_m^2)^{-N^2}$.*

Proof. First we consider the case when $\alpha = 1$. Notice that A^n is an upper triangular Toeplitz matrix with the (k, j) entry $\binom{n}{j-k}$ when $0 \leq k \leq j$. Consequently, the (i, j) entry of $A^{*n}A^n$ is $\sum_{k=0}^{\min\{i,j\}} \binom{n}{i-k}\binom{n}{j-k}$. Clearly $\binom{n}{i-k}\binom{n}{j-k}$ is a polynomial in n of degree $i+j$ with the leading coefficient $1/(i!j!)$ so we can write $\binom{n}{i-k}\binom{n}{j-k} = n^{i+j}/(i!j!) + p_{i+j-1}(n)$. It follows that $(R_m^2)_{i,j}$ — the (i, j) entry of R_m^2 —satisfies $(R_m^2)_{i,j} = \sum_{n \geq 0} d_m^{2n} \left(n^{i+j}/(i!j!) + p_{i+j-1}(n)\right)$ and, using Lemma 4.5 and denoting $1/(1-d_m^2)$ by λ_m, we obtain that $(R_m^2)_{i,j} = (i+j)!/(i!j!)\, \lambda_m^{i+j+1} + q_{i+j}(\lambda_m)$, where q_{i+j} is a polynomial of degree up to $i+j$. Therefore, when $m \to \infty$, R_m^2 behaves asymptotically as the Hankel matrix with (i, j) entry $\binom{i+j}{i}\lambda_m^{i+j+1}$. This matrix can be written as a product $L_i B L_{j+1}$ where L_i stands for a diagonal matrix $\operatorname{diag}(\lambda_m^i)$. A calculation shows that its determinant equals $\lambda_m^{N^2} \det(B)$ where B is the Hankel matrix with (i, j) entry $\binom{i+j}{i}$. The result now follows from Proposition 4.6.

In the general case ($\alpha \neq 1$), it is not hard to see that the $(R_m^2)_{i,j}$ behaves asymptotically as $\sum (\alpha d_m)^{2n} n^{i+j}/(i!j!)$. The proof goes as in the previous case, except that λ_m now denotes $1/(1-\alpha^2 d_m^2)$. $\square$

Now we are ready to prove that $\mathcal{B}_A$ properly contains $\{A\}'$.

Theorem 4.8. *Let A be a matrix with the only eigenvalue α. Then $\mathcal{B}_A \neq \{A\}'$.*

Proof. Using Theorem 4.4 there is no loss of generality in assuming that A is a direct sum of Jordan blocks with eigenvalue α, and that the block in the upper left corner is of size $N \times N$, with $N \geq 2$. In fact, it suffices to find an operator $T \in \mathcal{L}(\mathbb{C}^N)$ such that $T \in \mathcal{B}_{J_\alpha} \setminus \{J_\alpha\}'$, so we will assume that A is an $N \times N$ Jordan block with eigenvalue α. Let $e_0, e_1, e_2, \ldots, e_{N-1}$ be the appropriate basis for $\mathbb{C}^N$. It is easy to see that the rank one operator $e_{N-1} \otimes e_{N-1}$ does not commute with A. We will show that it belongs to $\mathcal{B}_A$. By definition of $\mathcal{B}_A$, it suffices to prove that $\|R_m e_{N-1}\| \|R_m^{-1} e_{N-1}\|$ remains bounded as $m \to \infty$. Notice that $\|R_m e_{N-1}\|^2 = \langle R_m^2 e_{N-1}, e_{N-1}\rangle$, so it equals the $(N-1, N-1)$ entry of R_m^2. As we have seen in the proof of Theorem 4.7, this entry behaves asymptotically as $\binom{2N-2}{N-1}\lambda_m^{2N-1}$, where $\lambda_m = (1 - \alpha^2 d_m^2)^{-1}$. Similarly, $\|R_m^{-1} e_{N-1}\|^2$ equals the $(N-1, N-1)$ entry of R_m^{-2}. One knows that this entry can be calculated by dividing the determinant of the cofactor corresponding to the $(N-1, N-1)$ entry of R_m^2 by the determinant of R_m^2. A calculation shows that the minor in question equals $R_m^2(A')$ where A' is the compression of A to the first $N-1$ rows and columns, hence its determinant behaves, by Theorem 4.7, as $\lambda_m^{(N-1)^2}$. Finaly, again by Theorem 4.7, the determinant of R_m^2 behaves as $\lambda_m^{N^2}$. Consequently, $\|R_m e_{N-1}\|^2 \|R_m^{-1} e_{N-1}\|^2$ behaves as $\binom{2N-2}{N-1}\lambda_m^{2N-1}\lambda_m^{(N-1)^2}/\lambda_m^{N^2} = \binom{2N-2}{N-1}$, and the theorem is proved. □

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