

ON POLYNOMIALLY BOUNDED WEIGHTED SHIFTS

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1. Introduction. Let \mathcal{H} be a separable, infinite-dimensional, complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on \mathcal{H} . An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* (notation: $T \in (\text{PB})$) if there exists an $M > 0$ such that

$$(1) \quad \|p(T)\| \leq M \sup\{|p(\zeta)| : |\zeta| = 1\} \quad \forall \text{ polynomial } p,$$

and to be *power bounded* (notation $T \in (\text{PW})$) if (1) holds for every polynomial of the special form $p(\zeta) = \zeta^n$ where n is a positive integer. If $T \in (\text{PB})$ [resp., $T \in (\text{PW})$], then there is a smallest number M which satisfies (1) [resp., (1) restricted]. This number will be called the *polynomial bound* of T [resp., the *power bound* of T] and denoted by $M_{pb}(T)$ [resp., $M_{pw}(T)$]. Also an operator T in $\mathcal{L}(\mathcal{H})$ is said to be *similar to a contraction* (notation: $T \in (\text{SC})$) if there exists an invertible operator S in $\mathcal{L}(\mathcal{H})$ such that $\|S^{-1}TS\| \leq 1$, and to be *completely polynomially bounded* (notation: $T \in (\text{CPB})$) if there exists an $M > 0$ such that one has

$$(2) \quad \|(p_{ij}(T))\| \leq M \sup_{|\zeta|=1} \|(p_{ij}(\zeta))\| \quad \forall n \in \mathbb{N}, \forall \text{ family } \{p_{ij}\}_{i,j=1}^n \text{ of polynomials,}$$

where the operator $(p_{ij}(T))$ on the left side of (2) is an $n \times n$ matrix with operator entries acting, in the usual fashion, on the direct sum of n copies of \mathcal{H} , and $(p_{ij}(\zeta))$ denotes the obvious $n \times n$ complex matrix. If $T \in (\text{CPB})$, then there is a smallest number M satisfying (2), called the *complete polynomial bound* of T and denoted by $M_{cpb}(T)$. It is well-known that

$$(\text{SC}) \subset (\text{CPB}) \subset (\text{PB}) \subset (\text{PW}),$$

and it is a recent and beautiful theorem of Paulsen [5] that $(\text{SC})=(\text{CPB})$ and

$$(3) \quad M_{cpb}(T) = \inf\{\|S\| \|S^{-1}\| : \|S^{-1}TS\| \leq 1\}, \quad T \in (\text{SC}).$$

We use this result throughout the paper without further comment.

Of course one knows from earlier work of Foguel (cf. also [2]) that $(\text{PB}) \neq (\text{PW})$, and it is a difficult and interesting open question, set forth explicitly by Halmos in [3], whether

$$(4) \quad (\text{PB}) \subset (\text{SC}).$$

For more information concerning this circle of ideas, we recommend a perusal of [6].

In this note we add to the information available concerning the viability of (4) by showing that various classes (to be set forth below) of weighted shift operators with operator weights which are subsets of (PW) *do* belong to $(\text{SC})=(\text{CPB})$.

2. Preliminaries. For use throughout the paper, we introduce the following notation and terminology. We write \mathbb{C} for the complex plane, \mathbb{D} for the open unit disc in \mathbb{C} , and \mathbb{T} for $\partial\mathbb{D}$. As usual, we write \mathbb{N} for the set of positive integers, \mathbb{N}_0 for the set of nonnegative integers, and \mathbb{Z} for the set of all integers. If $n \in \mathbb{N} \cup \{\aleph_0\}$ and \mathcal{K} is any complex Hilbert space, we write $\mathcal{K}^{(n)}$ for the (orthogonal) direct sum of n copies of \mathcal{K} . If $T \in \mathcal{L}(\mathcal{K})$ we write $\sigma(T)$ and $|\sigma(T)|$ for the spectrum and spectral radius of T , respectively. If $\{T_\alpha\}$ is a family of operators in $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{W}_{\{T_\alpha\}}$ the von Neumann algebra generated by the family $\{T_\alpha\}$ (i. e., the smallest von Neumann algebra containing all of the T_α). Recall (cf. [1]) that a von Neumann algebra $\mathcal{W} \subset \mathcal{L}(\mathcal{H})$ is finite and of type I if and only if \mathcal{W} can be written as

$$(5) \quad \mathcal{W} = \bigoplus_{n \in \mathbb{N}} \mathcal{W}_n,$$

where each \mathcal{W}_n is an n -homogeneous von Neumann algebra of type I. (This means that each \mathcal{W}_n is, up to unitary equivalence, a tensor product $\mathcal{W}_n = \mathcal{L}(\mathcal{K}_n) \otimes \mathcal{A}_n$ where $\dim \mathcal{K}_n = n$ and \mathcal{A}_n is an abelian von Neumann algebra.) We say that a von Neumann algebra \mathcal{W} which is finite and of type I is of *bounded homogeneity* if all but finitely many of the direct summands \mathcal{W}_n appearing in (5) act on the space (0).

If \mathcal{K} is any complex Hilbert space of dimension at most \aleph_0 and $\{D_n\}_{n=1}^\infty$ is any bounded sequence from $\mathcal{L}(\mathcal{K})$, we denote by $\text{Diag}(D_1, D_2, \dots)$ the operator in $\mathcal{L}(\mathcal{K}^{(\aleph_0)})$ satisfying

$$\text{Diag}(D_1, D_2, \dots)(k_1, k_2, \dots) = (D_1 k_1, D_2 k_2, \dots)$$

for all vectors (k_1, k_2, \dots) in $\mathcal{K}^{(\aleph_0)}$. If $D_n = D$ for all $n \in \mathbb{N}$, we write $\text{Diag}(D_1, D_2, \dots)$ simply as $\text{Diag}(\{D\})$. (Of course, $\text{Diag}(D_1, D_2, \dots)$ is also the direct sum $\bigoplus_{n=1}^\infty D_n$.) Furthermore, if $\{W_n\}_{n=1}^\infty$ is any bounded sequence from $\mathcal{L}(\mathcal{K})$, we denote by $S_{\{W_n\}}$ the operator in $\mathcal{L}(\mathcal{K}^{(\aleph_0)})$ satisfying

$$(6) \quad S_{\{W_n\}}(k_1, k_2, \dots, k_n, \dots) = (0, W_1 k_1, W_2 k_2, \dots, W_n k_n, \dots),$$

for all vectors (k_1, k_2, \dots) in $\mathcal{K}^{(\aleph_0)}$. (In other words, $S_{\{W_n\}}$ is the unilateral operator-weighted shift with weight sequence $\{W_n\}$.) In the special case in which all the weights in (6) coincide with one weight W , we shall denote $S_{\{W_n\}}$ simply as $S_{\{W\}}$. Clearly $S_{\{W\}}$ is unitarily equivalent to the tensor product $S \otimes W$ acting on $\mathcal{H} \otimes \mathcal{K}$ where S is a unilateral shift in $\mathcal{L}(\mathcal{H})$ satisfying $S e_n = e_{n+1}$, $n \in \mathbb{N}$, for some orthonormal basis $\{e_n\}_{n=1}^\infty$ of \mathcal{H} . We first state, without proof, a result from [7] which shows that one can obtain information about the possible validity of (4) by studying operator-weighted shifts.

Proposition 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then $T \in (\text{PW})$ [resp., $T \in (\text{PB})$, $T \in (\text{SC})$] if and only if $S_{\{T\}} \in (\text{PW})$ [resp., $S_{\{T\}} \in (\text{PB})$, $S_{\{T\}} \in (\text{SC})$], and in this case, $M_{pw}(T) = M_{pw}(S_{\{T\}})$ [resp., $M_{pb}(T) = M_{pb}(S_{\{T\}})$, $M_{cpb}(T) = M_{cpb}(S_{\{T\}})$].*

In particular, this result allows us to reformulate the question whether (4) is valid as

Problem 2.2. *If $T \in (\text{PB})$, does $S_{\{T\}}$ necessarily belong to (SC) ?*

A related question, which is addressed by this note, is as follows.

Problem 2.3. *For which bounded sequences $\{W_n\}_{n=1}^\infty$ such that $S_{\{W_n\}} \in (\text{PW})$ does one have $S_{\{W_n\}} \in (\text{SC})$?*

The first result along these lines goes back to Kelley [4].

Theorem 2.4 (cf. [12]). *If $\{\lambda_n\}_{n=1}^\infty$ is any bounded sequence from \mathbb{C} , \mathcal{K} is a Hilbert space, and $S_{\{\lambda_n 1_{\mathcal{K}}\}} \in (\text{PW})$, then $S_{\{\lambda_n 1_{\mathcal{K}}\}} \in (\text{SC})$. In this situation the complete polynomial bound and the power bound of $S_{\{\lambda_n 1_{\mathcal{K}}\}}$ coincide.*

Proof. The first assertion is proved in [12], and the second one can be easily deduced from the mentioned proof. \square

A generalization of this result was obtained recently by the second author.

Theorem 2.5 ([9]). *Suppose $\{W_n\}_{n=1}^\infty \subset \mathcal{L}(\mathcal{H})$ is any bounded sequence of mutually commuting normal operators such that $S_{\{W_n\}} \in (\text{PW})$. Then $S_{\{W_n\}} \in (\text{SC})$.*

Note that if $\{W_n\}_{n=1}^\infty$ is any sequence of mutually commuting normal operators, then there is an abelian von Neumann algebra \mathcal{A} such that $\{W_n\}_{n=1}^\infty \subset \mathcal{A}$, and \mathcal{A} is, by definition, finite of type I and 1-homogeneous. The principal result of this note is the following generalization of Theorem 2.5 whose proof will be given in Section 4.

Theorem 2.6. *Suppose $\{W_n\}_{n=1}^\infty \subset \mathcal{L}(\mathcal{H})$ is a bounded sequence of mutually commuting operators and there exists a von Neumann algebra \mathcal{W} which is finite of type I and of bounded homogeneity such that $\{W_n\} \subset \mathcal{W}$ and $S_{\{W_n\}} \in (\text{PW})$. Then $S_{\{W_n\}} \in (\text{SC})$.*

On the way to proving Theorem 2.6 we establish

Proposition 2.7. *Suppose $M \geq 1$. Then there exists an increasing sequence $\{\omega_n(M)\}_{n=1}^\infty$ of positive numbers such that, for every sequence $\{W_i\}_{i=1}^\infty$ of mutually commuting operators in $\mathcal{L}(\mathcal{K})$, where $\dim \mathcal{K} = n$, $S_{\{W_i\}} \in (\text{PW})$, and $M_{pw}(S_{\{W_i\}}) \leq M$, we have that $S_{\{W_i\}} \in (\text{SC})$ and $M_{cpb}(S_{\{W_i\}}) \leq \omega_n(M)$.*

Observe also that as an immediate consequence of Proposition 2.1, Theorem 2.6, and the definition, we obtain the following

Corollary 2.8. *If W is any n -normal operator in $\mathcal{L}(\mathcal{H})$ such that $W \in (\text{PW})$, then $W \in (\text{SC})$.*

Proof of Corollary 2.8. According to Proposition 2.1, $S_{\{W\}} \in (\text{PW})$, and since, by definition, an n -normal operator belongs to a finite von Neumann algebra of type I which is n -homogeneous, it follows that $\mathcal{W}_{\{W\}}$ is finite of type I and has bounded homogeneity (having no k -homogeneous direct summand such that $k > n$). Thus, by Theorem 2.6, $S_{\{W\}} \in (\text{SC})$, and another application of Proposition 2.1 gives the result. \square

3. The weights are $n \times n$ matrices. In this section we shall prove Proposition 2.7, and for this purpose we shall need some estimates on the norms of certain upper triangular operators acting on a finite dimensional space.

Definition 3.1. We fix $n \in \mathbb{N}$ and let \mathcal{K} be an n -dimensional Hilbert space. Associated with each ordered orthonormal basis \mathfrak{X} of \mathcal{K} is a collection $\mathcal{U}(\mathcal{K}, \mathfrak{X})$ of operators in $\mathcal{L}(\mathcal{K})$ defined as follows: $T \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ if the matrix $\mathfrak{M}_{\mathfrak{X}}(T) = (\alpha_{ij})_{i,j=1}^n$ of T with respect to \mathfrak{X} is in upper triangular form (i. e., if $\alpha_{ij} = 0$ for $i > j$).

Remark 3.2. It is obvious that the set $\mathcal{U}(\mathcal{K}, \mathfrak{X})$ is convex and closed under multiplication.

Lemma 3.3. Suppose $n \in \mathbb{N}$, $\eta > 0$, $\gamma \in (0, 1)$, $K \geq 1$, and \mathfrak{X} is an ordered orthonormal basis for an n -dimensional Hilbert space \mathcal{K} . (i) If $T \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$, $|\det(T)| > \eta$, and $\|T\| \leq K$, then T is invertible and

$$(7) \quad \|T^{-1}\| \leq \frac{nK^{n-1}}{\eta}.$$

(ii) There exists a fixed invertible operator $D = D(\gamma, n)$ on \mathcal{K} such that $\|D\| = 1$, $\|D^{-1}\| = [(K+1-\gamma)(1-\gamma)^{-1}]^{n-1}$, and $\|D^{-1}TD\| < 1$ for every $T \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ satisfying $|\sigma(T)| \leq \gamma$ and $\|T\| \leq K$.

Proof. To prove (i), suppose $T \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ and satisfies $|\det(T)| > \eta$ and $\|T\| \leq K$. Then T is clearly invertible, and the usual formula for the inverse of an $n \times n$ complex matrix, together with the observation that the determinant of such a matrix is the product of its n eigenvalues, gives (7). To prove (ii), let D be the operator whose matrix relative to the basis \mathfrak{X} is $\text{Diag}(k_1, \dots, k_n)$, and make the matricial calculation to show that it suffices to select

$$k_1 = 1, \quad k_2 = \frac{1-\gamma}{K+1-\gamma}, \quad k_3 = \left(\frac{1-\gamma}{K+1-\gamma}\right)^2, \quad \dots, \quad k_n = \left(\frac{1-\gamma}{K+1-\gamma}\right)^{n-1}.$$

□

We will also need the following lemma.

Lemma 3.4. Suppose \mathcal{K}_1 and \mathcal{K}_2 are finite dimensional Hilbert spaces of dimensions k_1 and k_2 , respectively, $A \in \mathcal{L}(\mathcal{K}_1)$, $C \in \mathcal{L}(\mathcal{K}_2)$, and $B : \mathcal{K}_2 \rightarrow \mathcal{K}_1$, is a bounded linear operator. Suppose also that $\gamma, \delta > 0$, $\sigma(A) \subset \{z \in \mathbb{C} : |z| \leq \gamma\}$, and $\sigma(C) \subset \{z \in \mathbb{C} : \gamma + 2\delta \leq |z|\}$. Then the unique operator $X : \mathcal{K}_2 \rightarrow \mathcal{K}_1$ satisfying $AX - XC = B$ also satisfies

$$(8) \quad \|X\| \leq k_1 k_2 (\|C\| + \gamma + \delta)^{k_2-1} (\|A\| + \gamma + \delta)^{k_1-1} \|B\|(\gamma + \delta)/\delta^2.$$

Proof. The integral formula for X is

$$X = \frac{1}{2\pi i} \int_{\Gamma} (A - z)^{-1} B (z - C)^{-1} dz,$$

where Γ is any closed contour in \mathbb{C} such that $\sigma(A)$ lies in the bounded component of $\mathbb{C} \setminus \Gamma$ and $\sigma(C)$ is contained in the unbounded component (cf. [10, Theorem 3.1]). In particular,

we can take Γ to be the circle of radius $\gamma + \delta$ with center at the origin. Then (8) follows from some straightforward estimates on $\|(A - \lambda)^{-1}\|$ and $\|(C - \lambda)^{-1}\|$ that come from Lemma 3.3. \square

Proof of Proposition 2.7. Fix $M \geq 1$. We construct the sequence $\{\omega_n(M)\}_{n=1}^\infty$ by induction on $n = \dim \mathcal{K}$. For $n = 1$, one knows from Theorem 2.4 that we may take $\omega_1(M) = M$. Thus suppose $n > 1$ and an increasing sequence $\omega_1(M), \dots, \omega_{n-1}(M)$ has been found with the desired properties. We define

$$\omega_n(M) = \omega_{n-1}(M)[1 + (4M)^n n^4] 2^n n M^n,$$

and note that $\omega_n(M) > \omega_{n-1}(M)$. With $\dim \mathcal{K} = n$, let $\{W_i\}_{i \in \mathbb{N}} \subset \mathcal{L}(\mathcal{K})$ be as in the statement of the proposition. Since the weights W_i are mutually commuting operators on \mathcal{K} , one knows that there exists an ordered orthonormal basis \mathfrak{X} for \mathcal{K} such that $W_i \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ for every $i \in \mathbb{N}$. The commutativity also implies that we may unambiguously define the doubly indexed sequence $\{\Pi_{ij}\}_{i,j \in \mathbb{N}_0}$ by $\Pi_{ij} = \prod_{k=i}^j W_k$ when $i \leq j$ and $\Pi_{ij} = 1_{\mathcal{K}}$ otherwise. We define a (possibly empty or finite) subsequence $\{n_j\}_{j \in J}$ of \mathbb{N} as follows. Let n_1 be the smallest positive integer such that

$$(9) \quad \left| \prod_{i=1}^{n_1} \det(W_i) \right| < \frac{1}{2^n}.$$

If such an integer n_1 exists, let n_2 be the smallest positive integer larger than n_1 such that

$$\left| \prod_{i=n_1+1}^{n_2} \det(W_i) \right| < \frac{1}{2^n},$$

and so on. First we consider the case in which the index set J is empty, which is equivalent to saying that

$$\left| \prod_{i=1}^n \det(W_i) \right| \geq \frac{1}{2^n}, \quad n \in \mathbb{N}.$$

Since for each $m \in \mathbb{N}$, the product $\Pi_{1,m} \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ and $\det(\Pi_{1,m}) = \prod_{i=1}^m \det(W_i)$, we conclude from Lemma 3.3(i) (applied to the sequence $\{\Pi_{1,m}\}_{m \in \mathbb{N}}$) and the fact that $M_{pw}(S_{\{W_i\}})$ is at most M that the operator

$$D = \text{Diag}(1, W_1, W_2 W_1, W_3 W_2 W_1, \dots)$$

acting on $\mathcal{K}^{(\mathbb{N}_0)}$ is a bounded invertible operator satisfying $\|D\| \leq M$ and $\|D^{-1}\| \leq 2^n n M^{n-1}$. An easy matricial calculation shows that $D^{-1} S_{\{W_i\}} D = S_{\{1_{\mathcal{K}}\}}$, and hence that $S_{\{W_i\}} \in (\text{SC})$ with

$$M_{cpb}(S_{\{W_i\}}) \leq 2^n n M^n < \omega_n(M)$$

in case J is empty.

Next suppose that $J = \{1\}$ and $n_1 = m$. In this case we set

$$D = \text{Diag} (D_1, D_2, \dots)$$

where $D_j = \Pi_{1,j-1}$ for $j \leq m$ and $D_j = \Pi_{m+1,j-1}$ for $j \geq m+1$. (For example, if $m = 4$ we set

$$D = \text{Diag} (1_{\mathcal{K}}, W_1, W_2 W_1, W_3 W_2 W_1, 1_{\mathcal{K}}, W_5, W_6 W_5, \dots).$$

By applying Lemma 3.3(i) to the sequence $\{D_j\}_{j=1}^{\infty}$ we see, as before, that D is an invertible operator in $\mathcal{L}(\mathcal{K}^{(\aleph_0)})$ satisfying $\|D\| \leq M$ and $\|D^{-1}\| \leq 2^n n M^{n-1}$. By making another matricial calculation, we see that $D^{-1} S_{\{W_i\}} D = S_{\{\tilde{W}_i\}}$, where $\tilde{W}_i = 1_{\mathcal{K}}$ for $i \neq m$ and $\tilde{W}_m = \Pi_{1,m} = W_m W_{m-1} \dots W_1$. (Thus, application of the similarity transform $X \rightarrow D^{-1} X D$ to $S_{\{W_i\}}$ has provided us with a weighted shift $S_{\{\tilde{W}_i\}}$ in which all the weights are $1_{\mathcal{K}}$ except \tilde{W}_m , which satisfies $|\det(\tilde{W}_m)| < 1/2^n$.) The argument in this case could now be completed in several different ways, but we choose to proceed in a way that parallels the argument in the case that the index set J is infinite, which is dealt with below. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of \tilde{W}_m arranged in order of increasing modulus, so $|\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_m|$.

We consider first the subcase in which $|\lambda_n| - |\lambda_1| \leq 1/4$, which gives $|\lambda_n| - 1/4 \leq |\lambda_1|$. If $|\lambda_n| \leq 1/4$, then $|\sigma(\tilde{W}_m)| \leq 1/4$. On the other hand, if $|\lambda_n| > 1/4$, then we have

$$(|\lambda_n| - 1/4)^n \leq |\lambda_1| |\lambda_2| \dots |\lambda_n| \leq 1/2^n,$$

so by taking n^{th} roots, we obtain that $|\sigma(\tilde{W}_m)| \leq 3/4$. By applying Lemma 3.3(ii) to \tilde{W}_m , we obtain the existence of an invertible contraction Z in $\mathcal{L}(\mathcal{K})$ such that

$$\|Z^{-1}\| = (4M + 1)^{n-1} \text{ and } \|Z^{-1} \tilde{W}_m Z\| < 1.$$

We define $\hat{D} = \text{Diag} (\{Z\})$ and observe that $(\hat{D})^{-1} S_{\{\tilde{W}_i\}} \hat{D} = S_{\{\hat{W}_i\}}$ where $\hat{W}_i = 1_{\mathcal{K}}$ for $i \neq m$ and $\hat{W}_m = Z^{-1} \tilde{W}_m Z$. Thus $S_{\{\hat{W}_i\}}$ is a contraction, and consequently $S_{\{W_i\}} \in (\text{SC})$ with

$$M_{cpb}(S_{\{W_i\}}) \leq \|D\| \|\hat{D}\| \|D^{-1}\| \|(\hat{D})^{-1}\| \leq 2^n n M^n (4M + 1)^{n-1} < \omega_n(M)$$

in the case under consideration.

Now we consider the other subcase in which $|\lambda_n| - |\lambda_1| > 1/4$. Obviously there must exist some integer i_0 , $1 \leq i_0 \leq n-1$, such that $|\lambda_{i_0+1}| - |\lambda_{i_0}| > 1/4n$, and we now use the elementary lemma from linear algebra that there exists an ordered orthonormal basis \mathfrak{Q} for \mathcal{K} such that $\tilde{W}_m \in \mathcal{U}(\mathcal{K}, \mathfrak{Q})$ and such that the upper triangular matrix $\mathfrak{M}_{\mathfrak{Q}}(\tilde{W}_m)$ has the eigenvalues λ_i on the diagonal in the natural order (i. e., $\lambda_1, \lambda_2, \dots, \lambda_n$). Let \mathcal{N} be the subspace of \mathcal{K} spanned by the first i_0 vectors in the ordered orthonormal basis \mathfrak{Q} , and write the matrix for \tilde{W}_m relative to the decomposition $\mathcal{K} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ as

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $A \in \mathcal{L}(\mathcal{N})$ and $C \in \mathcal{L}(\mathcal{N}^\perp)$. It follows that $\sigma(A) \subset \{z \in \mathbb{C} : |z| \leq |\lambda_{i_0}|\}$ and $\sigma(C) \subset \{z \in \mathbb{C} : |\lambda_{i_0+1}| \leq |z|\}$. In particular,

$$\text{dist}(\sigma(A), \sigma(C)) \geq \frac{1}{4n}.$$

By Lemma 3.4 there exists an operator $X : \mathcal{N}^\perp \rightarrow \mathcal{N}$ such that $AX - XC + B = 0$ and $\|X\| \leq (2M)^n n^4$. This implies that the operator Y in $\mathcal{L}(\mathcal{K})$ whose matrix relative to the decomposition $\mathcal{K} = \mathcal{N} \oplus \mathcal{N}^\perp$ is

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$$

has the property that the matrix for $Y^{-1}\tilde{W}_m Y$ is

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

and it is obvious that $\|Y\|, \|Y^{-1}\| \leq 1 + (2M)^n n^4$. We define $\hat{D} = \text{Diag}(\{Y\})$ and we note that $(\hat{D})^{-1}S_{\{\tilde{W}_i\}}\hat{D} = S_{\{\hat{W}_i\}}$ where $\hat{W}_i = 1_{\mathcal{K}}$ for $i \neq m$ and $\hat{W}_m = Y^{-1}\tilde{W}_m Y$. Obviously for $i \neq m$, each $\hat{W}_i = 1_{\mathcal{K}}$ can be written as $\hat{W}_i = 1_{\mathcal{N}} \oplus 1_{\mathcal{N}^\perp}$, and consequently, for every $i \in \mathbb{N}$, \hat{W}_i may be written as $A_i \oplus C_i$ relative to the decomposition $\mathcal{K} = \mathcal{N} \oplus \mathcal{N}^\perp$. Since $\hat{D}^{-1}D^{-1}S_{\{W_i\}}D\hat{D} = S_{\{A_i \oplus C_i\}}$, which is easily seen to be unitarily equivalent to $S_{\{A_i\}} \oplus S_{\{C_i\}}$, and since both $\dim \mathcal{N}$ and $\dim \mathcal{N}^\perp$ are less than n and the families $\{A_i\}$ and $\{C_i\}$ are (trivially) mutually commuting, it follows from the induction hypothesis (after observing that $S_{\{A_i\}}$ and $S_{\{C_i\}}$ belong to (PW) with $S_{\{W_i\}}$, and that their respective power bounds are not bigger than M) that $S_{\{A_i\}}, S_{\{C_i\}} \in (\text{SC})$, and that $M_{\text{cpb}}(S_{\{A_i\}}) \leq \omega_{i_0}(M)$ and $M_{\text{cpb}}(S_{\{C_i\}}) \leq \omega_{n-i_0}(M)$. Since the numbers $\omega_k(M)$, $1 \leq k \leq n-1$, form an increasing sequence, it follows that $M_{\text{cpb}}(S_{\{A_i\}}), M_{\text{cpb}}(S_{\{C_i\}}) \leq \omega_{n-1}(M)$. Thus $S_{\{A_i \oplus C_i\}}$ and $S_{\{W_i\}}$ belong to (SC) too, and

$$M_{\text{cpb}}(S_{\{W_i\}}) \leq \omega_{n-1}(M)[1 + (2M)^n n^4]^2 2^n n M^n < \omega(n, M),$$

so the proof in the case that \mathbf{J} is a singleton is complete.

The argument in the case in which $\mathbf{J} = \{1, \dots, p\}$ for some $p > 1$ is an obvious variation of the argument in the case $\mathbf{J} = \{1\}$, so we proceed directly to the case $\mathbf{J} = \mathbb{N}$. In that situation we define $D \in \mathcal{L}(\mathcal{K}^{(\mathbb{N}_0)})$ by

$$D = \bigoplus_{i=1}^{\infty} D_{n_i},$$

where

$$D_{n_i} = \bigoplus_{j=n_{i-1}}^{n_i-1} \Pi_{n_{i-1}+1, j}, \quad i \in \mathbb{N},$$

with the understanding that $n_0 = 0$. (Thus, for example, when $n_i = 3i$ for all $i \in \mathbb{N}$, D is the operator

$$\text{Diag} (1_{\mathcal{K}}, W_1, W_2 W_1, 1_{\mathcal{K}}, W_4, W_5 W_4, 1_{\mathcal{K}}, W_7, W_8 W_7, \dots).$$

Another application of Lemma 3.3(i) (to the sequence $\{\Pi_{n_{i-1}+1, j}\}_{i \in \mathbb{N}, n_{i-1} \leq j \leq n_i-1}$) yields $\|D\| \leq M$, $\|D^{-1}\| \leq 2^n n M^{n-1}$, and $D^{-1} S_{\{W_i\}} D = S_{\{\tilde{W}_i\}}$, where this time $\tilde{W}_i = 1_{\mathcal{K}}$ if $i \notin \{n_1, n_2, \dots\}$ and $\tilde{W}_{n_j} = \Pi_{n_{j-1}+1, n_j}$ for $j \in \mathbb{N}$. Moreover, by construction, the sequence $\{\tilde{W}_{n_j}\}$ satisfies

$$\left| \det(\tilde{W}_{n_j}) \right| < \frac{1}{2^n}, \quad j \in \mathbb{N}.$$

For each $j \in \mathbb{N}$, let $\{\lambda_{i,j}\}_{i=1}^n$ be the sequence of eigenvalues of \tilde{W}_{n_j} arranged in order of increasing modulus. Just as in the case $J = \{1\}$ treated earlier, there are two cases to be dealt with. The first is that in which, for every $j \in \mathbb{N}$,

$$|\lambda_{n,j}| - |\lambda_{1,j}| \leq 1/4.$$

In this case, arguing as before, we see that $|\sigma(\tilde{W}_{n_j})| \leq 3/4$ for every $j \in \mathbb{N}$. Upon applying Lemma 3.3(ii) to the sequence $\{\tilde{W}_{n_j}\}_{j \in \mathbb{N}}$, we obtain an invertible contraction Z on \mathcal{K} such that $\|Z^{-1}\| = (4M + 1)^{n-1}$ and $\|Z^{-1} \tilde{W}_{n_j} Z\| < 1$ for all $j \in \mathbb{N}$. If \hat{D} is the operator $\text{Diag}(\{Z\})$ in $\mathcal{L}(\mathcal{K}^{(\mathbb{N}_0)})$ then $(\hat{D})^{-1} S_{\{\tilde{W}_i\}} \hat{D}$ is a contraction, so in this case $S_{\{W_i\}} \in (\text{SC})$ and

$$M_{\text{cpb}}(S_{\{W_i\}}) \leq \|D\| \|\hat{D}\| \|D^{-1}\| \|(\hat{D})^{-1}\| \leq 2^n n M^n (4M + 1)^{n-1} < \omega_n(M).$$

Turning now to the second case, suppose that there exists $j_0 \in \mathbb{N}$ such that $|\lambda_{n, j_0}| - |\lambda_{1, j_0}| > 1/4$. Then there must exist an integer i_0 , $1 \leq i_0 \leq n-1$, such that

$$|\lambda_{i_0+1, j_0}| - |\lambda_{i_0, j_0}| > \frac{1}{4n},$$

and, as before, we choose an ordered orthonormal basis \mathfrak{Y} for \mathcal{K} such that $\tilde{W}_{n_{j_0}} \in \mathcal{U}(\mathcal{K}, \mathfrak{Y})$ and such that the eigenvalues $\{\lambda_{i, j_0}\}_{i=1}^n$ appear on the diagonal of $\mathfrak{M}_{\mathfrak{Y}}(\tilde{W}_{n_{j_0}})$ in natural order. Once again, if we denote by \mathcal{N} the subspace of \mathcal{K} spanned by the first i_0 vectors in the basis \mathfrak{Y} , then we can write $\tilde{W}_{n_{j_0}}$ as the matrix

$$\tilde{W}_{n_{j_0}} = \begin{pmatrix} A_{n_{j_0}} & B_{n_{j_0}} \\ 0 & C_{n_{j_0}} \end{pmatrix}$$

relative to the decomposition $\mathcal{K} = \mathcal{N} \oplus \mathcal{N}^\perp$, where

$$\text{dist}(\sigma(A_{n_{j_0}}), \sigma(C_{n_{j_0}})) \geq \frac{1}{4n}.$$

By Lemma 3.4 there exists an operator $X : \mathcal{N}^\perp \rightarrow \mathcal{N}$ such that $A_{n_{j_0}}X - XC_{n_{j_0}} + B_{n_{j_0}} = 0$ and $\|X\| \leq (2M)^n n^4$. Just as before, this implies that the operator

$$Y = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

on $\mathcal{K} = \mathcal{N} \oplus \mathcal{N}^\perp$ has the properties that

$$Y^{-1}\tilde{W}_{n_{j_0}}Y = \begin{pmatrix} A_{n_{j_0}} & 0 \\ 0 & C_{n_{j_0}} \end{pmatrix}$$

and $\|Y\|, \|Y^{-1}\| \leq 1 + (2M)^n n^4$. We define $\hat{D} = \text{Diag}(\{Y\})$ and note that $(\hat{D})^{-1}S_{\{\tilde{W}_i\}}\hat{D} = S_{\{\hat{W}_i\}}$ where $\hat{W}_i = 1_{\mathcal{K}}$ for $i \notin \{n_1, n_2, \dots\}$ and $\hat{W}_{n_j} = Y^{-1}\tilde{W}_{n_j}Y$ for all $j \in \mathbb{N}$. Since \hat{W}_{n_j} commutes with $\hat{W}_{n_{j_0}}$ for all $j \in \mathbb{N}$ and $\sigma(A_{n_{j_0}}) \cap \sigma(C_{n_{j_0}})$ is empty, it follows easily that for each $j \in \mathbb{N}$, \hat{W}_{n_j} is an orthogonal direct sum $A_j \oplus C_j$ (relative to the decomposition $\mathcal{K} = \mathcal{N} \oplus \mathcal{N}^\perp$). Thus, using the induction hypothesis and arguing as in the case $J = \{1\}$ we arrive easily at the conclusion that $S_{\{W_i\}} \in (\text{SC})$ and that

$$M_{\text{cpb}}(S_{\{W_i\}}) \leq \omega_{n-1}(M)[1 + (2M)^n n^4]^2 2^n n M^n < \omega_n(M).$$

By the principle of mathematical induction, the sequence $\{\omega_n(M)\}_{n=1}^\infty$ with the desired properties exists, and the proposition is proved. \square

4. The weights are n-normal operators.. In this section we shall prove Theorem 2.6. To move in this direction, suppose $\{W_i\}_{i=1}^\infty$ is a bounded sequence of mutually commuting operators in $\mathcal{L}(\mathcal{H})$ such that the von Neumann algebra $\mathcal{W}_{\{W_i\}}$ generated by the sequence $\{W_i\}_{i=1}^\infty$ is finite of type I and of bounded homogeneity. Then, by definition, there exists a positive integer N such that $\mathcal{W}_{\{W_i\}}$ can be written as a direct sum

$$\mathcal{W}_{\{W_i\}} = \bigoplus_{k=1}^N \mathcal{W}_k$$

where for $1 \leq k \leq N$, \mathcal{W}_k is a k -homogeneous, finite, type I von Neumann algebra (perhaps acting on the space of dimension 0). In any event, for $i \in \mathbb{N}$, W_i has a direct sum decomposition

$$(10) \quad W_i = W_{i,1} \oplus \cdots \oplus W_{i,N},$$

and it is clear that for $1 \leq k \leq N$, the sequence $\{W_{i,k}\}_{i=1}^\infty$ must generate the k -homogeneous von Neumann algebra \mathcal{W}_k . With the sequence $\{W_i\}_{i=1}^\infty$ as above, let us also suppose that the operator $S_{\{W_i\}} \in (\text{PW})$. Our task is to prove that $S_{\{W_i\}} \in (\text{SC}) = (\text{CPB})$. From (10) we have that

$$S_{\{W_i\}} = S_{\{W_{i,1} \oplus \cdots \oplus W_{i,N}\}},$$

and consequently (as is easily seen) $S_{\{W_i\}}$ is unitarily equivalent to the (finite) direct sum

$$(11) \quad S_{\{W_{i,1}\}} \oplus \cdots \oplus S_{\{W_{i,N}\}}.$$

Since obviously $S_{\{W_{i,j}\}} \in (\text{PW})$ for $1 \leq j \leq N$, and since the direct sum in (11) is finite, it suffices to show that $S_{\{W_{i,j}\}} \in (\text{CPB})$ for $1 \leq j \leq N$. Thus in order to prove Theorem 2.6, it suffices to prove the following special case.

Theorem 4.1. *Suppose $\{W_i\}_{i=1}^\infty$ is a sequence of mutually commuting operators in $\mathcal{L}(\mathcal{H})$ such that $S_{\{W_i\}} \in (\text{PW})$ and there exists a von Neumann algebra \mathcal{W} which is finite, type I, and n -homogeneous (for some $n \in \mathbb{N}$) such that $\{W_i\} \subset \mathcal{W}$. Then $S_{\{W_i\}} \in (\text{CPB})$.*

Suppose next that \mathcal{W} is an n -homogeneous von Neumann algebra of type I acting on the Hilbert space \mathcal{H} . Then, as is well known, there exists an abelian von Neumann algebra \mathcal{A} acting on \mathcal{H} such that \mathcal{W} is spatially isomorphic to the von Neumann algebra $M_n(\mathcal{A})$ consisting of all $n \times n$ operator matrices (A_{ij}) with entries from \mathcal{A} acting on the Hilbert space $\mathcal{H}^{(n)}$ in the usual fashion. If we apply this spatial isomorphism to the n -homogeneous von Neumann algebra \mathcal{W} of Theorem 4.1, then we may suppose, without loss of generality, that

$$W_k = \left(A_{ij}^{(k)} \right) \in M_n(\mathcal{A}) \subset \mathcal{L}(\mathcal{H}^{(n)}), \quad k \in \mathbb{N},$$

where the entire family of operators $\{A_{ij}^{(k)}\}_{1 \leq i, j \leq n, k \in \mathbb{N}}$ lies in the abelian von Neumann algebra \mathcal{A} . In particular, the $A_{ij}^{(k)}$ are all mutually commuting normal operators. The problem of proving Theorem 4.1 is thus reduced to showing that the operator $S_{\{(A_{ij}^{(k)})\}} \in (\text{CPB})$, and this operator may be regarded as an $\aleph_0 \times \aleph_0$ matrix with mutually commuting normal operators (acting on \mathcal{H}) as entries. Since there exists a compact Hausdorff space X and a C^* -isomorphism ρ of \mathcal{A} onto $C(X)$, the operator $S_{\{(A_{ij}^{(k)})\}}$ acting on $(\mathcal{H}^{(n)})^{(\aleph_0)}$ may be “exchanged” (in a sense to be made precise below) for the “operator” $S_{\{(a_{ij}^{(k)})\}}$ where for all $1 \leq i, j \leq n$ and $k \in \mathbb{N}$, $a_{ij}^{(k)} = \rho(A_{ij}^{(k)})$, and this latter “operator” may be tested for complete polynomial boundedness pointwise on X . But for any fixed x_0 in X , the operator $S_{\{(a_{ij}^{(k)}(x_0))\}}$ is a weighted shift whose weights are mutually commuting $n \times n$ complex matrices, and thus Proposition 2.7 is available to give sufficient conditions for complete polynomial boundedness.

This completes our outline of the main ideas to be used in the proof of Theorem 4.1, so we turn to the necessary details.

Lemma 4.2. *Suppose \mathcal{A} is an abelian von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$, $\mathfrak{E} = \{e_n\}_{n=1}^\infty$ is a fixed orthonormal basis for \mathcal{H} , and $T = (N_{ij})_{i,j=1}^\infty$ is an operator matrix that defines a bounded operator on $\mathcal{H}^{(\aleph_0)}$ in the usual way, where $N_{ij} \in \mathcal{A}$ for $1 \leq i, j < \infty$. Let X be a compact Hausdorff space such that $\rho : \mathcal{A} \rightarrow C(X)$ is a C^* -isomorphism, and write $n_{ij} = \rho(N_{ij})$ for all i, j . Then for each $x \in X$, there exists an operator $T(x) \in \mathcal{L}(\mathcal{H})$ such that the matrix for $T(x)$ relative to \mathfrak{E} is $(n_{ij}(x))_{i,j=1}^\infty$ and, moreover, $\|T\| = \sup_{x \in X} \|T(x)\|$.*

Proof. This lemma follows from three fundamental facts, all of which are well known. We content ourselves with stating these, and leave the epsilontics to the reader: (1) since T is bounded, the nondecreasing sequence of norms of the finite k -sections $(N_{ij})_{i,j=1}^k$ of T is bounded and has supremum $\|T\|$; (2) a matrix $(n_{ij}(x_0))_{i,j=1}^\infty$ is the matrix of a bounded operator if and only if the nondecreasing sequence of norms of the finite k -sections $(n_{ij}(x_0))_{i,j=1}^k$ is bounded, and $\|(n_{ij}(x_0))_{i,j=1}^\infty\| = \sup_k \|(n_{ij}(x_0))_{i,j=1}^k\|$; (3) (cf. [1]) the norm

of a k -normal operator $(N_{ij})_{i,j=1}^k$ is given by

$$\left\| (N_{ij})_{i,j=1}^k \right\| = \sup_{x \in X} \left\| (n_{ij}(x))_{i,j=1}^k \right\|. \quad \square$$

Lemma 4.3. *Suppose $T = (N_{ij})_{i,j=1}^\infty \in \mathcal{L}(\mathcal{H}^{(\mathbb{N}_0)})$ is as in Lemma 4.2 and has the further property that every row and column of the matrix $(N_{ij})_{i,j=1}^\infty$ contains only finitely many nonzero entries. Then, with the notation as in Lemma 4.2, for every polynomial p , $p(T)$ has a matrix $(M_{ij})_{i,j=1}^\infty$ with (commuting, normal) entries from \mathcal{A} , and for every $x \in X$, $(p(T))(x)$ (meaning the counterpart of $T(x)$ in Lemma 4.2, but for the operator $p(T)$ in place of T) $= p(T(x))$. In particular, $\|p(T)\| = \sup_{x \in X} \|p(T(x))\|$.*

Proof. An easy matricial calculation shows that if p is any polynomial, then the matrix $(M_{ij})_{i,j=1}^\infty$ for $p(T)$ has the properties that 1) each row and column contains only finitely many nonzero entries, and 2) the M_{ij} , $1 \leq i, j < \infty$, are all mutually commuting normal operators. Thus, in view of Lemma 4.2, it suffices to show that for each x in X , $(p(T))(x) = p(T(x))$. We sketch the argument in case $p(z) = z^2$; the general result then follows easily by an induction argument on the degree of p . With $T^2 = (M_{ij})_{i,j=1}^\infty$, fix $i_0, j_0 \in \mathbb{N}$ and $x \in X$. Then M_{i_0, j_0} is a finite sum

$$M_{i_0, j_0} = \sum_k N_{i_0, k} N_{k, j_0},$$

and thus the (i_0, j_0) entry of $(T^2)(x)$ is

$$\begin{aligned} (\rho(M_{i_0, j_0}))(x) &= \left(\rho \left(\sum_k N_{i_0, k} N_{k, j_0} \right) \right) (x) = \left(\sum_k \rho(N_{i_0, k}) \rho(N_{k, j_0}) \right) (x) = \\ &= \sum_k (\rho(N_{i_0, k})(x)) (\rho(N_{k, j_0})(x)), \end{aligned}$$

which is obviously the (i_0, j_0) entry of $(T(x))^2$ by Lemma 4.2. Thus the proof is complete. \square

With this next lemma, our preparation to prove Theorem 4.1 will be complete.

Lemma 4.4. *Let $T = (N_{ij})_{i,j=1}^\infty \in \mathcal{L}(\mathcal{H}^{(\mathbb{N}_0)})$ be as in Lemmas 4.2 and 4.3, suppose $n \in \mathbb{N}$, and let (p_{ij}) be an $n \times n$ matrix whose entries are polynomials. Then the operator matrix $(p_{ij}(T))$ acting on $(\mathcal{H}^{(\mathbb{N}_0)})^{(n)}$ satisfies*

$$(12) \quad \|(p_{ij}(T))\| = \sup_{x \in X} \|(p_{ij}(T(x)))\|$$

in the notation of Lemma 4.2, where, of course, the operator matrix $(p_{ij}(T(x)))$ acts on $\mathcal{H}^{(n)}$.

Proof. For simplicity of notation we give the argument only in the case $n = 2$. The validity of the more general argument will be clear from this. Thus let (p_{ij}) be a 2×2 matrix of

polynomials, and note that it follows from Lemma 4.3 that each of $p_{11}(T), \dots, p_{22}(T)$ is an infinite matrix having entries from \mathcal{A} (and having only finitely many nonzero entries in each row and column). For definiteness, we write $p_{kl}(T) = (M_{ij}^{kl})_{i,j=1}^{\infty}$, $k, l = 1, 2$, where for all i, j, k, l , $M_{ij}^{kl} \in \mathcal{A}$. Thus, $(p_{ij}(T))_{i,j=1}^2$ may be written as the doubly infinite matrix

$$\begin{pmatrix} (M_{ij}^{11}) & (M_{ij}^{12}) \\ (M_{ij}^{21}) & (M_{ij}^{22}) \end{pmatrix},$$

which is obviously unitarily equivalent to the operator matrix

$$A = \begin{pmatrix} M_{11}^{11} & M_{11}^{12} & M_{12}^{11} & M_{12}^{12} & \cdots \\ M_{11}^{21} & M_{11}^{22} & M_{12}^{21} & M_{12}^{22} & \cdots \\ M_{21}^{11} & M_{21}^{12} & M_{22}^{11} & M_{22}^{12} & \cdots \\ M_{21}^{21} & M_{21}^{22} & M_{22}^{21} & M_{22}^{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Lemma 4.2 applies to A , so in the notation of that lemma

$$\|A\| = \sup_{x \in X} \|A(x)\|,$$

where the matrix $A(x)$ is given by

$$A(x) = \begin{pmatrix} \rho(M_{11}^{11})(x) & \rho(M_{11}^{12})(x) & \cdots \\ \rho(M_{11}^{21})(x) & \rho(M_{11}^{22})(x) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly $A(x)$ is unitarily equivalent to the operator matrix

$$\begin{pmatrix} (\rho(M_{ij}^{11})(x)) & (\rho(M_{ij}^{12})(x)) \\ (\rho(M_{ij}^{21})(x)) & (\rho(M_{ij}^{22})(x)) \end{pmatrix},$$

which is, in turn,

$$\begin{pmatrix} (p_{11}(T))(x) & (p_{12}(T))(x) \\ (p_{21}(T))(x) & (p_{22}(T))(x) \end{pmatrix},$$

which by Lemma 4.3 is identical with

$$\begin{pmatrix} p_{11}(T(x)) & p_{12}(T(x)) \\ p_{21}(T(x)) & p_{22}(T(x)) \end{pmatrix}.$$

Thus,

$$\|(p_{ij}(T))_{i,j=1}^2\| = \|A\| = \sup_{x \in X} \|A(x)\| = \sup_{x \in X} \|(p_{ij}(T(x)))_{i,j=1}^2\|,$$

as was to be shown, and the proof is complete. \square

These lemmas essentially constitute a proof of a general principle, which is interesting in its own right, and goes as follows.

Theorem 4.5. *Suppose \mathcal{K} is a complex Hilbert space of dimension no greater than \aleph_0 , $\{N_{ij}\}_{i,j=1}^\infty$ is a family of mutually commuting normal operators on \mathcal{K} , and the operator matrix $T = (N_{ij})_{i,j=1}^\infty$ defines a bounded operator on $\mathcal{K}^{(\aleph_0)}$ and has the additional property that every row and column of T contains only finitely many nonzero entries. Let \mathcal{A} be an abelian von Neumann algebra containing the operators N_{ij} , $1 \leq i, j < \infty$, and let $\rho : \mathcal{A} \rightarrow C(X)$ be the Gelfand transform. Then $T \in (\text{PW})$ [resp., $T \in (\text{PB})$, $T \in (\text{CPB})$] if and only if, in the notation of Lemma 4.2, $T(x) \in (\text{PW})$ [resp., $T(x) \in (\text{PB})$, $T(x) \in (\text{CPB})$] for each x in X and $\sup_{x \in X} M_{pw}(T(x))$ [resp., $\sup_{x \in X} M_{pb}(T(x))$, $\sup_{x \in X} M_{cpb}(T(x))$] is finite.*

Proof. We give the argument for complete polynomial boundedness; the other arguments are similar and simpler. To show that $T \in (\text{CPB})$ it is necessary and sufficient to show that there exists a constant $K > 0$ such that for every $n \in \mathbb{N}$ and for every $n \times n$ matrix (p_{ij}) with polynomial entries,

$$(13) \quad \|(p_{ij}(T))\| \leq K \sup_{t \in \mathbb{T}} |(p_{ij}(t))|$$

(where, of course, $(p_{ij}(T))$ acts on $(\mathcal{K}^{(\aleph_0)})^{(n)}$ and $(p_{ij}(t))$ acts on $\mathbb{C}^{(n)}$). But, by Lemma 4.4,

$$(14) \quad \|(p_{ij}(T))\| = \sup_{x \in X} |(p_{ij}(T(x)))|.$$

Thus, if (13) holds for all $n \in \mathbb{N}$ and all $n \times n$ matrices (p_{ij}) , then it follows from (14) that for $x \in X$, $T(x) \in (\text{CPB})$ and $\sup_{x \in X} M_{cpb}(T(x)) \leq K$. On the other hand, if $T(x) \in (\text{CPB})$

for each $x \in X$ and $\sup_{x \in X} M_{cpb}(T(x)) \triangleq K$, then it follows from (14) that (13) holds, and the proof is complete. \square

Proof of Theorem 4.1 (and theorem 2.6). As we saw earlier, if $\{W_i\}$ satisfies the hypotheses of Theorem 4.1, then up to simultaneous unitary equivalence, we may suppose that there are an abelian von Neumann algebra \mathcal{A} and a family $\{A_{ij}^{(k)}\}$, $1 \leq i, j \leq n$, $k \in \mathbb{N}$, of (mutually commuting, normal) operators in \mathcal{A} such that W_k may be identified with the operator $(A_{ij}^{(k)})_{i,j=1}^n$ acting on $\mathcal{H}^{(n)}$. Thus $S_{\{W_i\}}$ may be identified with the operator $S_{\{(A_{ij}^{(k)})\}}$ acting on $(\mathcal{H}^{(n)})^{(\aleph_0)}$, and $S_{\{(A_{ij}^{(k)})\}} \in (\text{PB})$ and also clearly satisfies the hypotheses of Theorem 4.5. Let X be a compact Hausdorff space, and let $\rho : \mathcal{A} \rightarrow C(X)$ be the Gelfand transform. Then, by Theorem 4.5, it suffices to show that for each x in X , the operator $(S_{\{(A_{ij}^{(k)})\}})(x)$, which is clearly $S_{\{(a_{ij}^{(k)}(x))\}}$ by Lemma 4.1, where $a_{ij}^{(k)} = \rho(A_{ij}^{(k)})$, belongs to (CPB) and satisfies $\sup_{x \in X} M_{cpb}(S_{\{(a_{ij}^{(k)}(x))\}}) < +\infty$. But also by Theorem 4.5, since $S_{\{(A_{ij}^{(k)})\}} \in (\text{PB})$, we have $S_{\{(a_{ij}^{(k)}(x))\}} \in (\text{PW})$ for every x in X and $\sup_{x \in X} M_{pw}(S_{\{(a_{ij}^{(k)}(x))\}}) < +\infty$. Thus, employing the notation of Proposition 2.7, we may set $M = \sup_{x \in X} M_{pw}(S_{\{(a_{ij}^{(k)}(x))\}})$, and

conclude from that proposition that for each x in X , $S_{\{(a_{ij}^{(k)}(x))\}} \in (\text{CPB})$ and satisfies

$$M_{cpb}(S_{\{(a_{ij}^{(k)}(x))\}}) \leq \omega_n(M).$$

(The fact that the $(a_{ij}^{(k)}(x))$, $n \in \mathbb{N}$, are mutually commuting is a consequence of the mutual commutativity of the $(A_{ij}^{(k)})$, $k \in \mathbb{N}$. The result now follows from Theorem 4.5. \square

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We close by mentioning some open questions which at present the authors are unable to resolve:

1. Improve Proposition 2.7 by making it possible to delete the hypothesis that the operators $\{W_i\} \subset \mathcal{L}(\mathcal{H})$ are mutually commuting.
2. Improve Theorem 2.6 by making it possible to delete the hypothesis that the von Neumann algebra \mathcal{W} be of bounded homogeneity.
3. Improve Theorem 4.5 by making it possible to remove the hypothesis that $T = (N_{ij})_{i,j=1}^{\infty}$ has only finitely many nonzero entries in each row and column. (But in this connection, see [8, Proposition 2.2].)
4. Let (P) be the following proposition: Every operator of the form $\bigoplus_{n=1}^{\infty} M_n$, where for $n \in \mathbb{N}$, M_n acts on a space of dimension n , that belongs to (PB) also belongs to (CPB). Does (P) imply that (PB) \subset (CPB)?

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