SOME REMARKS ON THE OPERATOR OF FOIAS AND WILLIAMS

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ABSTRACT. In this paper we study the Foias-Williams operator

$$T(H_g) = \begin{pmatrix} S^* & H_g \\ 0 & S \end{pmatrix}$$

where $g \in L^\infty$, and H_g is a Hankel operator with symbol g. We exhibit a relationship between the similarity of $T(H_g)$ to a contraction and the rate of decay of $\{|g_n|\}_{n=0}^\infty$, the absolute values of the Fourier coefficients of the symbol g.

Let $\mathcal{H}$ denote a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* if there exists an $M \geq 1$ such that

$$\|p(T)\| \leq M \sup\{|p(\zeta)| : |\zeta| = 1\}$$

for all polynomials p. We denote the class of all polynomially bounded operators by (PB). Also, an operator T is similar to a contraction (notation: $T \in$(SC)) if there exists a bounded invertible operator L such that $\|LTL^{-1}\| \leq 1$. Halmos [4] raised the question whether every polynomially bounded operator is similar to a contraction; this question is still open.

While there is a number of results dealing with sufficient conditions for a polynomially bounded operator to be similar to a contraction (cf. [9], [6], [5]), there are very few publications dedicated to the search for a counterexample. In [3] (see also [2]) Foias and Williams have studied the operators of the form

$$T(X) = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$$

acting on $H^2 \oplus H^2$, the direct orthogonal sum of two copies of the Hardy space H^2 (to be defined below). Here S is a forward unilateral shift on H^2 and $X \in \mathcal{L}(H^2)$. In particular, they conjectured that there exists a Hankel operator H_g with symbol g (to be defined below) such that the operator $T(H_g) \in$ (PB) $\setminus$ (SC).

In this paper we continue this line of investigation and we show that the membership in the aforementioned classes depends on the rate at which the sequence $\{g_n\}_{n\in\mathbb{N}_0}$ of the Fourier coefficients of g tends to 0.

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Before stating the main results, however, we briefly introduce some notation and terminology. As usual $\mathbb{N}$ [resp., $\mathbb{N}_0$] denotes the set of positive [resp., non-negative] integers, $\mathbb{C}$ is the complex plane and $\mathbb{T}$ is the unit circle in $\mathbb{C}$. L^∞ is the algebra of all bounded Lebesgue-measurable functions on $\mathbb{T}$, and H^∞ is its subalgebra consisting of those functions whose negative Fourier coefficients vanish. Similarly, H^2 is the space of square-integrable functions on $\mathbb{T}$ whose negative Fourier coefficients vanish. Finally, ℓ^2 is the Hilbert space of all sequences $(x_n)_{n\in\mathbb{N}_0}$ such that $\sum_{n\in\mathbb{N}_0} |x_n|^2 < \infty$ and $\|(x_n)\|_{\ell^2} = \left\{ \sum_{n\in\mathbb{N}_0} |x_n|^2 \right\}^{1/2}$. Furthermore, $z^n = e^{int}$, $n \in \mathbb{N}_0$, is the standard orthonormal basis of H^2, e_n, $n \in \mathbb{N}_0$, is the standard basis of ℓ^2 (e_n is a sequence whose only non-zero term is in the nth position), and there exists a Hilbert space isomorphism $V : H^2 \to \ell^2$ such that $Vz^n = e_n$, $n \in \mathbb{N}_0$. In what follows we shall identify H^2 with ℓ^2 (and z^n with e_n) without further comment. If $g \in L^\infty$, H_g is the Hankel operator with symbol g i.e., H_g is a bounded linear operator on H^2 satisfying $H_g(z^n) = S^{*n}g$. In this situation, using the aforementioned identification between H^2 and ℓ^2, H_g is represented by the matrix $(g_{i+j})_{i,j=0}^\infty$, where g_n denotes the nth Fourier coefficient of g.

Lemma 1. *Suppose $g \in H^\infty$ and the series $\sum_{n=1}^\infty n^2|g_n|^2$ does not converge. Then $T(H_g)$ is not similar to a contraction. In particular, this happens when $|g_n| = (n+1)^{-3/2}$.*

Proof. By [2, Theorem 5.3], $T(X) \in$(SC) if and only if $\exists f \in H^2$ such that the operator W defined on polynomials by

$$Wz^n = S^*(A_{n+1} \cdot 1 - S^{*n} f), \quad n \in \mathbb{N}_0$$

extends to a bounded linear operator on H^2. Here

$$A_n = \sum_{j=0}^{n-1} S^{*(n-1-j)} X S^j, \quad n \in \mathbb{N}.$$

In particular, when $X = H_g$, the Hankel operator with symbol $g \in L^\infty(\mathbb{T})$,

$$A_n = \sum_{j=0}^{n-1} S^{*n-1-j} H_g S^j = \sum_{j=0}^{n-1} S^{*n-1} H_g = nS^{*n-1} H_g,$$

and therefore

$$Wz^n = S^* \left((n+1)S^{*n} H_g \cdot 1 - S^{*n} f \right) = (n+1)S^{*n+1} H_g \cdot 1 - S^{*n+1} f, \quad n \in \mathbb{N}_0.$$

Since $H_g 1 = g$ we have that $Wz^n = (n+1)S^{*n+1}g - S^{*n+1}f$. Clearly, such an operator W, in the basis $\{z^n\}_{n\in\mathbb{N}_0}$ is represented by a matrix

$$(< Wz^r, z^s >)_{r,s=0}^\infty$$

and it is easy to see that

$$< Wz^r, z^s >= (r+1) < g, z^{r+s+1} > - < f, z^{r+s+1} >= (r+1)g_{r+s+1} - f_{r+s+1}.$$

In other words, W is represented by a matrix

$$A = \left(\, (r+1)g_{r+s+1} - f_{r+s+1} \, \right)_{r,s=0}^{\infty}, \tag{1}$$

and for W to be bounded on H^2 it is necessary that the columns of A be square summable i.e.,

$$\sum_r |(r+1)g_{r+s+1} - f_{r+s+1}|^2 < \infty, \quad \forall s \in \mathbb{N}_0.$$

In particular we have that $\sum_r |(r+1)g_r - f_r|^2 < \infty$, which in view of $f \in H^2$ implies that $\sum_r r^2 |g_r|^2 < \infty$, and the lemma is proved. $\square$

The main result of this paper is that $-\frac{3}{2}$ is indeed a limiting point. More precisely, we have the following

Theorem 2. *Suppose that $\alpha > \frac{3}{2}$. Then there exists a function $g \in H^\infty$ such that $|g_n| = \frac{1}{(n+1)^\alpha}$, $n \in \mathbb{N}_0$ and such that, with $f = 0$, the matrix (1) is bounded on ℓ^2.*

Before we can prove Theorem 2 we need to recall a few results. First, one knows that there exists a sequence $\{\epsilon_n\}_{n \in \mathbb{N}_0} \subset \{-1, 1\}$ and a constant C such that

$$\sup_{z \in \mathbb{T}} \left| \sum_{n=0}^{N} \epsilon_n z^n \right| \le C\sqrt{N}, \quad \forall N \in \mathbb{N}. \tag{2}$$

Since the numbers ϵ_n were independently introduced by Shapiro [8] and Rudin [7], they are generally referred to as the Rudin-Shapiro coefficients.

Next, we recall that for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ the Schur product of A and B is defined as $A * B = (a_{ij} b_{ij})$. A matrix M is said to be a Schur multiplier if $M * X$ defines a bounded operator on ℓ^2 for every $X \in \mathcal{L}(\ell^2)$. The following useful test comes from [1]:

Theorem 3. *Let M be a matrix with $\lim_k m_{jk} = 0 = \lim_j m_{jk}$, and suppose that*

$$\sum_{j,k=1}^{\infty} |m_{j,k} - m_{j,k+1} - m_{j+1,k} + m_{j+1,k+1}| < \infty.$$

Then M is a Schur multiplier.

Proof of Theorem 2. Given $\alpha > \frac{3}{2}$, we define $g_n = \frac{\epsilon_n}{(n+1)^\alpha}$, $n \in \mathbb{N}_0$, where ϵ_n denotes the nth Rudin-Shapiro coefficient. In this situation, with $f = 0$, the matrix (1) becomes

$$A = \left(\, (r+1) \frac{\epsilon_{r+s}}{(r+s+1)^\alpha} \, \right)_{r,s=0}^{\infty}.$$

Since $\alpha > \dfrac{3}{2}$, $\exists\beta > 0$ such that $\alpha = 2\beta + \dfrac{3}{2}$. Using the Schur product we can rewrite $A = M * B$ where

$$M = \left(\frac{r+1}{(r+s+1)^{1+\beta}} \right), \quad B = \left(\frac{\epsilon_{r+s}}{(r+s+1)^{\frac{1}{2}+\beta}} \right).$$

First we show that B defines a bounded linear operator on ℓ^2. We notice that B is the Hankel operator corresponding to the function $h(z) = \sum_{k=0}^{\infty} \frac{\epsilon_k}{(k+1)^{\frac{1}{2}+\beta}} z^k$. Therefore it suffices to show that $h \in H^\infty$ because in that case $\|B\| \le \|h\|_\infty$.

An easy application of the Abel summation by part allows us to rewrite

$$h(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \epsilon_j z^j \right) \left[\frac{1}{(k+1)^{\frac{1}{2}+\beta}} - \frac{1}{(k+2)^{\frac{1}{2}+\beta}} \right]$$

and therefore

$$\|h\|_\infty \le \sup_{z\in\mathbb{T}} \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} \epsilon_j z^j \right| \frac{(k+2)^{\frac{1}{2}+\beta} - (k+1)^{\frac{1}{2}+\beta}}{(k+1)^{\frac{1}{2}+\beta}(k+2)^{\frac{1}{2}+\beta}}.$$

By the mean value theorem

$$(k+2)^{\frac{1}{2}+\beta} - (k+1)^{\frac{1}{2}+\beta} = \left(\frac{1}{2} + \beta\right)(k+1+\theta)^{\frac{1}{2}+\beta-1}$$

for some $0 < \theta < 1$. Thus, employing (2),

$$\|h\|_\infty \le \sum_{k=0}^{\infty} C\sqrt{k} \left(\frac{1}{2} + \beta\right) \frac{(k+1+\theta)^{\beta-\frac{1}{2}}}{(k+1)^{\frac{1}{2}+\beta}(k+2)^{\frac{1}{2}+\beta}}$$

and it is easy to see that this series converges, from which it follows that B is a bounded operator.

Finally, we show that M is a Schur multiplier. We apply Theorem 3 with $m_{jk} = \dfrac{j+1}{(j+k+1)^{1+\beta}}$. Clearly, both limits are 0. Therefore, it remains to show the convergence of the double series

$$\sum_{j,k=1}^{\infty} \left| \frac{j+1}{(j+k+1)^{1+\beta}} - \frac{j+1}{(j+k+2)^{1+\beta}} - \frac{j+2}{(j+k+2)^{1+\beta}} + \frac{j+2}{(j+k+3)^{1+\beta}} \right|.$$

One knows that if a function $f(x, y)$ has continuous partial derivatives of the second order, then

$$f(x,y) - f(x,y+1) - f(x+1,y) + f(x+1,y+1) = \frac{\partial^2 f}{\partial x \partial y}(x+\theta, y+\eta), \tag{3}$$

for some $0 < \theta, \eta < 1$. Using (3) with $f(x, y) = \dfrac{x+1}{(x+y+1)^{1+\beta}}$, we obtain that the series above can be written as

$$(\beta+1) \sum_{j,k=1}^{\infty} \left| \frac{(\beta+1)(j+\theta+1) - (k+\eta)}{(j+k+\theta+\eta+1)^{\beta+3}} \right|.$$

Since this double series is obviously convergent, the theorem is proved. □

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