

## VECTORS

### 1. SCALARS AND VECTORS

*Scalar.* A scalar is a quantity that requires only a sign and a magnitude to be totally described. For example: length (m), surface (m<sup>2</sup>), volume (m<sup>3</sup>), work (J), mass (kg), mass flow rate (kg/s), density (kg/m<sup>3</sup>), etc.

*Vector.* A vector is a quantity that requires magnitude, direction and sense to be described. In order to express a vector quantity is necessary to establish a frame of reference. Examples of vector quantities are: position, velocity, acceleration, force, etc.

### 2. VECTOR OPERATIONS

*Multiplication or Division of a Vector by a Scalar.* Given a scalar  $a$  and a vector  $\mathbf{A}$ , the multiplication (or division) of the vector by the scalar is obtained by multiplying (or dividing) each element of the vector by the scalar. For example,

$$\begin{aligned}\vec{A} &= 2\hat{i} - 6\hat{j} + 3\hat{k} & a &= -3.5 \\ \frac{\vec{A}}{a} &= -0.571\hat{i} + 1.714\hat{j} - 0.857\hat{k}\end{aligned}$$

*Vector Addition.* To add two (or more) vectors is necessary to add each component of the first vector with the corresponding component the second vector. For example,

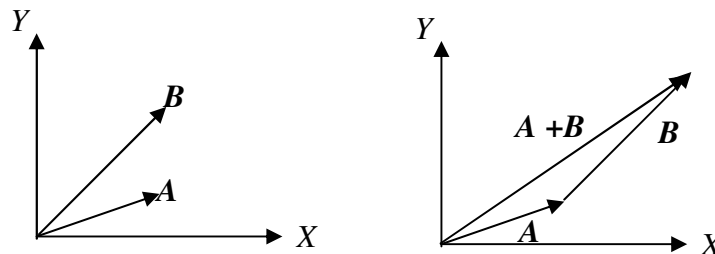


Figure 2.1. Graphic vector addition.

$$\begin{aligned}\vec{A} &= 3\hat{i} + \hat{j} & \vec{B} &= 4\hat{i} + 4\hat{j} \\ \vec{A} + \vec{B} &= 7\hat{i} + 5\hat{j}\end{aligned}$$

*Vector Subtraction.* The difference of two vectors is done by subtracting each component of the subtrahend vector from the corresponding component of the minuend vector of the corresponding components of each vector, For example

$$\begin{aligned}\vec{A} &= 3\hat{i} + \hat{j} & \vec{B} &= 4\hat{i} + 4\hat{j} \\ \vec{A} - \vec{B} &= -\hat{i} - 3\hat{j}\end{aligned}$$

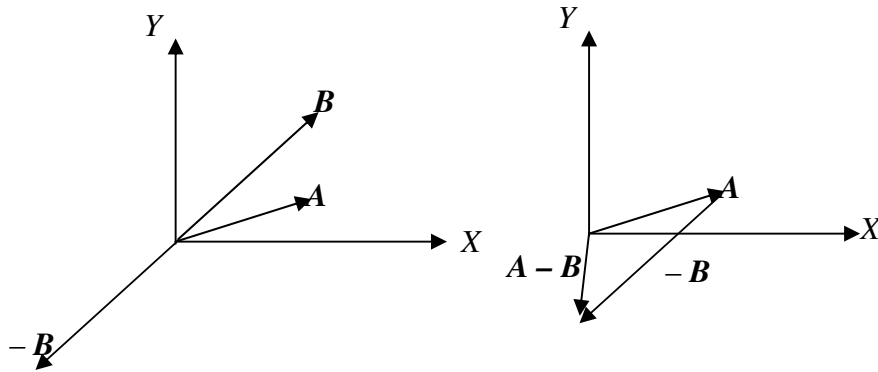
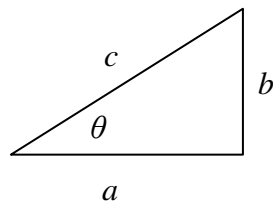


Figure 2.2. Graphic vector subtraction.

### 3. VECTOR ADDITION OF FORCES

A force is a vector quantity since requires magnitude, direction and sense to be completely described. The addition of forces is carried out according to the parallelogram law. The subject of statics frequently requires determining the resultant force of a system of known forces or finding the orthogonal or projected components of a known resultant. These two tasks require invariably the use of trigonometric laws. The majority of the problems that require solving any of the two questions mentioned above can be successfully treated using right triangle relations:



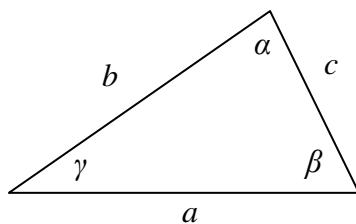
$$c = (a^2 + b^2)^{1/2}$$

$$\sin \theta = b/c$$

$$\cos \theta = a/c$$

$$\tan \theta = b/a$$

However, the sine and cosine laws can also be applied and are sometimes required.



$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

#### Homework No. 2.1

2-14, 2-16, 2-17, 2-24, 2-29.

As shown below the use of sine and cosine laws is restricted to two vectors or to resolve a vector in two components therefore the use of this relationships can become tedious for cases where there are more than two vector quantities acting.

#### 4. ADDITION OF A SYSTEM OF COPLANAR FORCES

As briefly seen previously, any vector quantity can be expressed in terms of its magnitude, direction and sense or in terms of its components (vector components). In the case of Cartesian coordinates a vector quantity is totally described by three components (in the most general case), namely  $x$ -,  $y$ -, and  $z$ - components. Thus a force vector  $\vec{F}$  with three components is expressed as

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

where  $F_x$ ,  $F_y$ , and  $F_z$ , represent the components of  $\mathbf{F}$  in the  $x$ -,  $y$ -, and  $z$ - directions, respectively, and  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  are the unit vectors for the  $x$ -,  $y$ -, and  $z$ - directions, respectively.

Thus, in order to obtain the resultant force of a set of forces it is convenient to express each force in terms of the Cartesian components and then add (or subtract) the components of all the forces in each direction. The resultant force will be simply the result of that summation. The magnitude of the resultant force (or any force expressed in terms of its Cartesian components) can be obtained by using the Pythagorean Theorem and the angle that expresses the direction of the force can be obtained by means of the trigonometric relations. Figure 2.3 shows schematically these concepts.

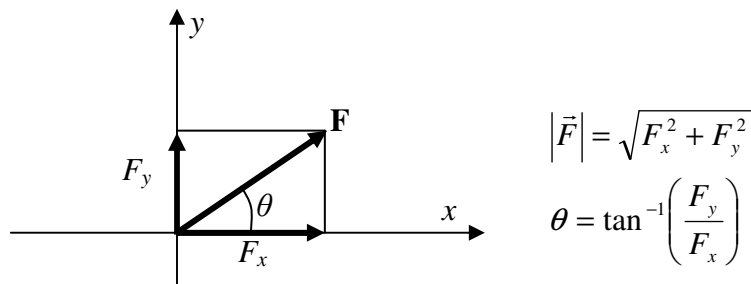


Figure 2.3. Determination of the magnitude and direction of a vector

As can be inferred, this approach is more practical than the use of the sine and cosine laws to obtain the resultant of a set of vectors in several steps.

#### Homework No. 2.2

2-46, 2-51, 2-57, 2-58.

#### 5. CARTESIAN VECTORS

The analysis of three-dimensional problems that we will be facing in this course is greatly simplified by the use of vector algebra. At this point we need to formally introduce the Cartesian system of coordinates ( $x$ ,  $y$ ,  $z$ ).

*Right Handed Coordinate System.* The right handed coordinate system is the most common Cartesian coordinate system. In this system the thumb of the right hand dictates the positive direction of the  $z$ -axis while the fingers curl in from the positive  $x$ -axis towards positive  $y$ -axis. This is shown in Figure 2.4.

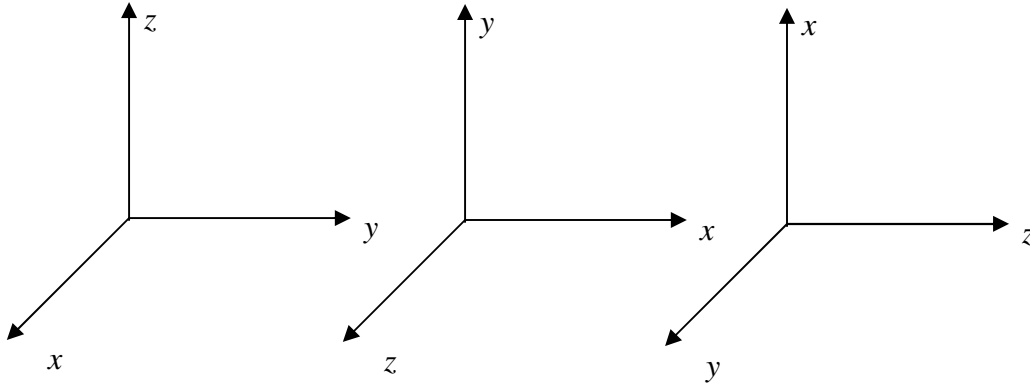


Figure 2.4. Several combinations of the right handed Cartesian coordinate system.

*Rectangular Components of a Vector.* As seen previously, a vector  $\vec{A}$  can have up to three components, each component corresponding to every axis of the system. Thus, a vector  $\vec{A}$  composed of three components is expressed as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

where  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  are the unit vectors in the  $x$ -,  $y$ -, and  $z$ - directions, respectively.

*Unit Vectors.* Some operations in vector algebra require the determination of the direction of a certain vector quantity in vector representation. This direction is described by a unit vector and can be determined as

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} \quad (2.1)$$

where  $|\vec{A}|$  represents the magnitude of the vector.

As can be observed in Eq. 2.1, a unit vector is dimensionless. Then, a unit vector of a vector described by  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$  is

$$\hat{A} = \frac{A_x \hat{i} + A_y \hat{j} + A_z \hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \quad (2.2)$$

*Direction of a Cartesian Vector.* The orientation of vector  $\vec{A}$  is defined by the coordinate direction angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . These angles are measured between the base of the vector and the positive  $x$ -,  $y$ -, and  $z$ - axes located at the base of the vector. Figure 2.5 presents shows this concept.

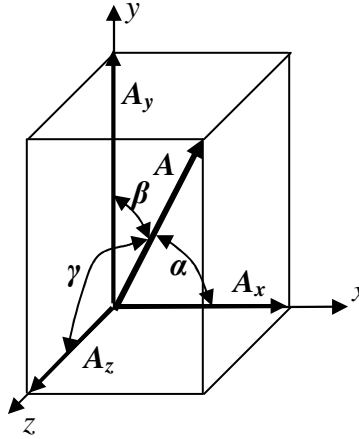


Figure 2.5. Graphic description of the coordinate direction angles.

The determination of  $\alpha$ ,  $\beta$ , and  $\gamma$  is achieved by considering the projection of  $\vec{A}$  onto the  $x$ -,  $y$ -, and  $z$ - axes such that

$$\cos \alpha = \frac{A_x}{|\vec{A}|} \quad \cos \beta = \frac{A_y}{|\vec{A}|} \quad \cos \gamma = \frac{A_z}{|\vec{A}|} \quad (2.3)$$

These quantities are named *direction cosines* of  $\vec{A}$ . The direction cosines can be determined easily by finding a vector unit for  $\vec{A}$  as expressed in Eq. 2.2. Comparing Eqs. 2.2 and 2.3 can be directly established that the unit vector of  $\vec{A}$  is precisely formed by the direction cosines, therefore

$$\hat{A} = \frac{A_x \hat{i} + A_y \hat{j} + A_z \hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\hat{A} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \quad (2.4)$$

The magnitude of a vector is equal to the square root of the sum of the squares of each of its components and since the magnitude of  $\hat{A}$  is 1, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (2.5)$$

If the vector  $\vec{A}$  lies in a known octant, this equation can be used to determine one of the coordinate direction angles if the other two are known.

Also, when the magnitude and coordinate direction angles of  $\vec{A}$  are known, the vector can be expressed as

$$\vec{A} = |\vec{A}| \hat{A} = |\vec{A}| \cos \alpha \hat{i} + |\vec{A}| \cos \beta \hat{j} + |\vec{A}| \cos \gamma \hat{k} \quad (2.6)$$

## 6. ADDITION AND SUBTRACTION OF CARTESIAN VECTORS

The addition or subtraction of two vectors that are expressed in terms of Cartesian coordinates is achieved by adding (or subtracting) term by term the corresponding components of each vector, for example:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} + (A_z + B_z) \hat{k}$$

$$\vec{R}' = \vec{A} - \vec{B} = (A_x - B_x) \hat{i} + (A_y - B_y) \hat{j} + (A_z - B_z) \hat{k}$$

### Homework No. 2.3

2-70, 2-73, 2-79, 2-83.

## 7. POSITION VECTORS

*Position Vector.* A position vector  $\vec{r}$  is a vector that locates a point in space with respect to another point. For example, the position vector directed from  $P_1(x_1, y_1, z_1)$  to  $P_2(x_2, y_2, z_2)$  is given by

$$\vec{r} = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}$$

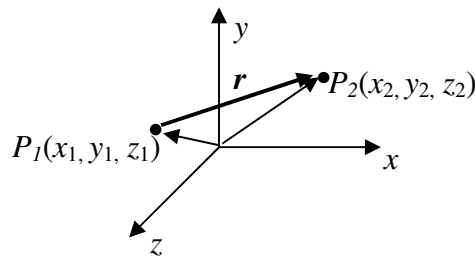


Figure 2.5. Graphic description of a position vector.

The position vector of a point with respect to the origin of coordinates  $O$  is a particular case described above. Then, a position vector of a point  $P(x, y, z)$  with respect to  $O$  is expressed as

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

## 8. FORCE VECTOR ALONG A LINE

In some problems the direction of a force is specified by two points through which its line of action passes. In these cases the vector force can be described by multiplying the unit vector of the position vector of these two points times the magnitude of the force exerted. Then

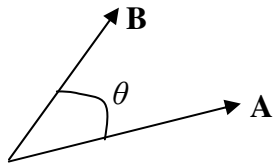
$$\vec{F} = F \left( \frac{\vec{r}}{|\vec{r}|} \right)$$

### Homework No. 2.4

2-95, 2-97, 2-107, 2-109.

## 2.9 DOT PRODUCT

Finding the angle between a pair of vectors with two components, namely  $x$  and  $y$  can be readily done using trigonometric relations. However in a three–dimension case this problem is complicated due to the difficulties to visualize the vectors. This problem can be easily solved using vector algebra, particularly the dot or scalar product. The dot product of two vectors  $\vec{A}$  and  $\vec{B}$  is defined as



$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \quad (2.7)$$

$$0^\circ \leq \theta \leq 180^\circ$$

*Laws of Operation.* The following rules govern the dot product operations.

1. Commutative law:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

2. Multiplication by a scalar:

$$a(\mathbf{A} \cdot \mathbf{B}) = (a\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B}) a$$

3. Distributive law:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})$$

Eq. (2.7) can be employed to determine the dot product between Cartesian unit vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  by noting that these unit vectors are orthogonal to each other.

$$\begin{array}{lll} \hat{i} \cdot \hat{i} = 1 & \hat{i} \cdot \hat{j} = 0 & \hat{i} \cdot \hat{k} = 0 \\ \hat{j} \cdot \hat{j} = 1 & \hat{j} \cdot \hat{k} = 0 & \hat{k} \cdot \hat{k} = 1 \end{array}$$

Again, using Eq. (2.7) and considering that two vectors  $\mathbf{A}$  and  $\mathbf{B}$  are described by

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

it is possible to determine that

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

is expressed finally as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad (2.8)$$

The definitions introduced above show that the dot product can be used to determine the angle formed between two vectors or intersecting lines and the components of a vector parallel and perpendicular to a line.

The computation of the angle between two vectors can be obtained by presenting Eq. (2.7) as

$$\theta = \cos^{-1} \left( \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \right) \quad 0^\circ \leq \theta \leq 180^\circ$$

On the other hand, the component of vector  $\vec{A}$  parallel to or collinear with a segment of line  $xx'$  is given by

$$A_{\parallel} = |\vec{A}| \cos \theta = \vec{A} \cdot \hat{u}$$

where  $\hat{u}$  is the unit vector of the segment of line  $xx'$ .

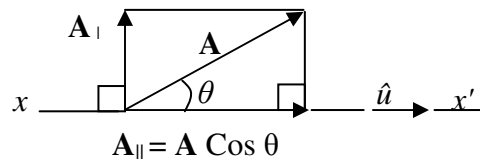


Figure 2.6. Vector projected on a line.

That is the scalar projection of a vector  $\vec{A}$  on a segment of line is obtained from the dot product of the vector  $\vec{A}$  and a unit vector defining the direction of that segment of line. The sign of the operation indicates the direction of vector  $\vec{A}$  with respect to the unit vector. Positive sign corresponds to vectors with same directions.

In order to obtain the component perpendicular to  $A_{\parallel}$  is necessary to remember that

$$\vec{A} = A_{\parallel} + A_{\perp}$$

Therefore  $A_{\perp} = \vec{A} - A_{\parallel}$ . Since the components of  $\vec{A}$  are known and the components of  $A_{\parallel}$  can be obtained from  $(A \cos \theta) \hat{u}$ , the determination of  $A_{\perp}$  is straightforward.

### Homework No. 2.5

2-117, 2-127, 2-129, 2-130.