RESULTANT OF A SYSTEM OF FORCES

4.1 MOMENT OF A FORCE – SCALAR FORMULATION

The *moment* of a force about a point or axis provides a measure of the tendency of the force to cause a body to rotate about the point or axis. In the case of figure 4.1 the force F_x acts perpendicular to the handle of the wrench and is located at a distance d_y from the point O.

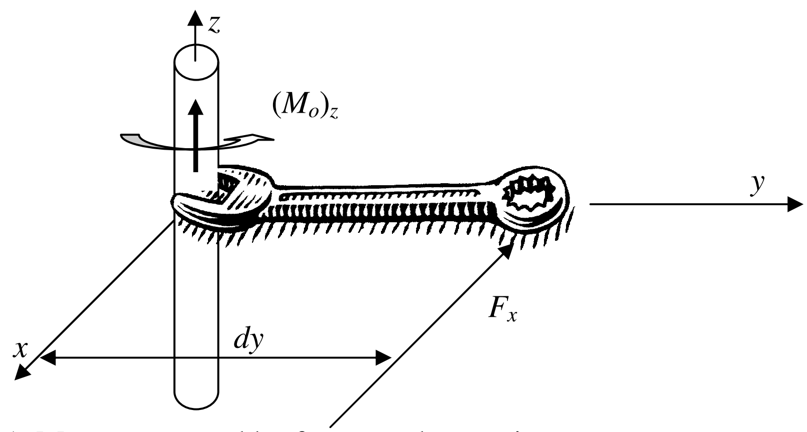


Figure 4.1. Moment created by force F_x about axis z.

The force tends to cause the pipe to spin around the z axis. The larger the force or the distance d_y the greater the spinning effect. This tendency of rotation caused by F_x is called *torque*, *moment of a force* or simply *moment*. Note that the *moment axis* (z) is perpendicular to the plane formed by the vector of the force and the vector of the distance d_y.

If now we apply the force F_z to the wrench as shown in figure 4.2, we can see that this force tends to rotate the pipe around axis x. Notice that it may not be possible to actually turn the pipe around axis x but still, F_z creates a tendency for rotation and thus, the moment $(M_o)_x$ is produced. Again, the force vector, F_z, and the distance (position vector), d_y, are in the same plane and the moment $(M_o)_x$ is perpendicular to this plane.

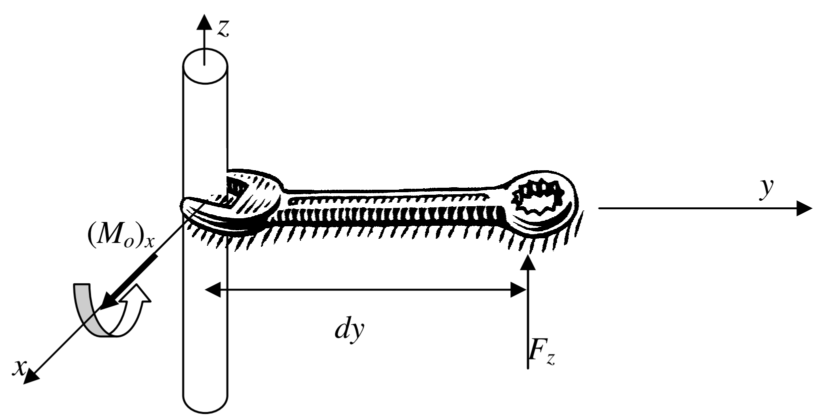


Figure 4.2. Moment created by force F_z about axis x.

Finally, if a force F_y is applied to the wench, as shown in figure 4.3, no moment is produced about point *O*. This is because the line of action of the force passes exactly through *z*–axis and therefore the distance that creates the moment is zero.

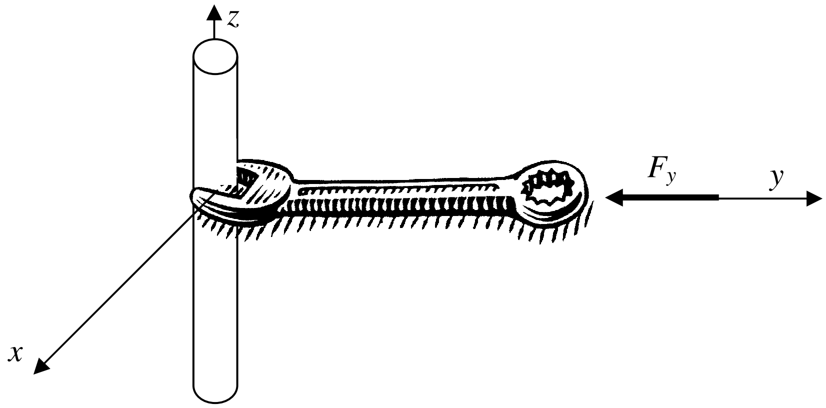


Figure 4.3. Moment created by force F_y is zero.

By generalizing the concepts studied above we can say that the moment M_o about the point *O* or about the axis passing through *O* is equal to the magnitude of the force *F* times the perpendicular distance *d* comprised between the point where the force is applied and the point *O* (or axis *O*). This concept is schematically shown in figure 4.4.

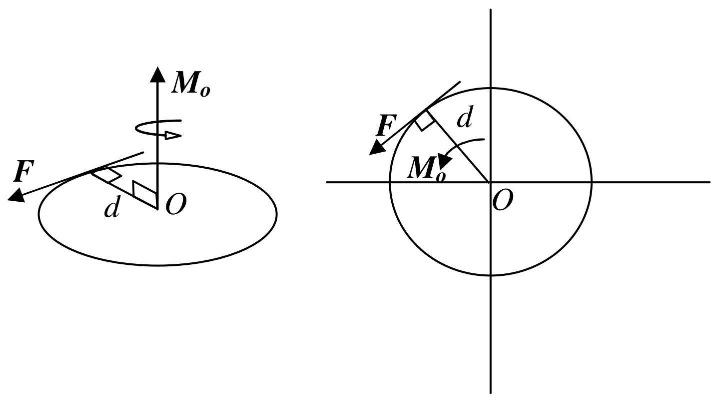


Figure 4.4. Schematic of the generalization of the concept for the moment of a force.

We have defined how to determine the moment of a force about an axis or a point and we have also established that the moment produced by a force about an axis is perpendicular to the plane formed by the vector force and the position vector from the force to the axis. Therefore the Moment of a force has both magnitude and direction and thus it is also a vector quantity.

The magnitude of the moment of a force that has a *moment arm* of *d* units is

$$M_O = F\,d \tag{4.1}$$

The *moment arm* is the perpendicular distance from the axis at point *O* to the line of action of the force. The units for the moment are N·m in the SI system and lb·ft in the English system.

The direction of the Moment M_O is specified using the right hand rule. That is, the fingers of the right hand are curled such that they follow the sense of rotation which would occur if the force could rotate about point *O* (figure 4.4). The *thumb* then *points* along the *moment axis* so that gives the direction and sense of the moment vector.

Resultant Moments of a System of Coplanar Forces. If a system of forces lies in the plane *x–y*, then, the moment produced by each force about point *O* will be directed along axis *z*.

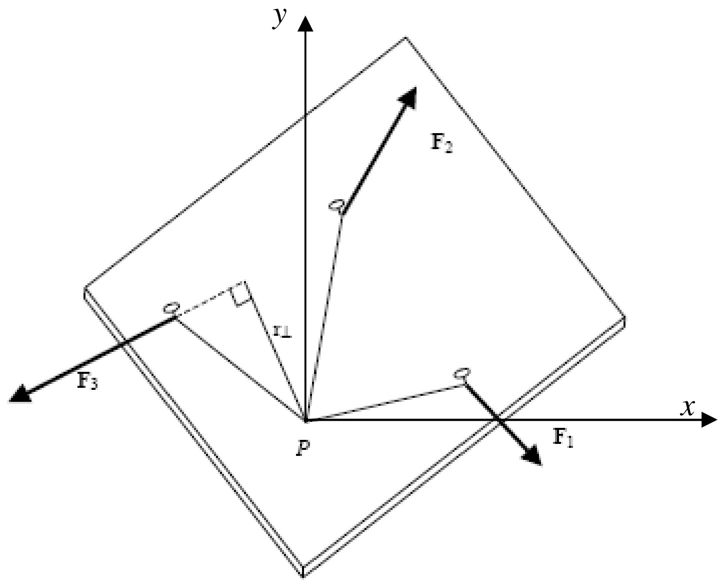


Figure 4.5. System of coplanar forces in the *x* – *y* plane producing a vector moment directed in the *z* direction.

Consequently, the resultant moment $\boldsymbol{M_{RO}}$ of the system can be determined by simply adding the moments of all the forces *algebraically* since all the moment vectors are collinear.

$$\curvearrowleft + M_{RO} = \sum F\, d \tag{4.2}$$

The counterclockwise curl written at the left of the equation expresses that by the scalar sign convention, the moment of any force will be positive if it is directed along the $+z$ axis, whereas a negative moment is directed along the $-z$ axis.

In summary: A moment is a turning effect. A moment always acts about a given point and is either clockwise or anticlockwise in nature. The moment about a point *O* caused by a particular force *F* is defined as the force *F* multiplied by the perpendicular distance from the force's line of action to the point.

Homework No. 4.1:

4–9, 4–18, 4–22, 4–33, 4–35.

4.2 CROSS PRODUCT

In the previous chapter we learn that the dot product results in the projection of one vector onto another one. The dot product is also known as scalar product because the result of this vector operation is a scalar. The *cross product* of two vectors $\vec{A}$ and $\vec{B}$ yields a third vector **C** that is perpendicular to $\vec{A}$ and $\vec{B}$. This operation is expressed as

$$\vec{C} = \vec{A} \times \vec{B}$$

Magnitude. The *magnitude* of $\vec{C}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between their tails ($0^o \le \theta \le 180^o$). Thus, $C = AB \sin\theta$.

Direction. Vector $\vec{C}$ has a direction that is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ such that $\vec{C}$ is specified by the right hand rule.

Then, knowing the magnitude and direction of $\vec{C}$ we can express

$$\vec{C} = \vec{A} \times \vec{B} = (AB \sin\theta)\hat{u}_C \tag{4.3}$$

In the previous expression, the scalar $AB \sin\theta$ defines the magnitude of $\vec{C}$ and the unit vector $\hat{u}_c$ defines the *direction* of $\vec{C}$. Figure 4.6 illustrates the concept of the cross product defined above.

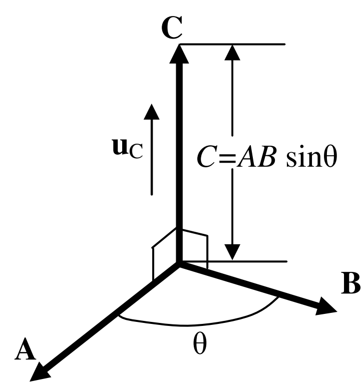


Figure 4.6. The cross product of two vectors yields a third vector perpendicular to a plane containing the two original vectors.

Laws of Operation

1. Commutative law is not valid

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

But
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

This becomes clear if the right hand rule is followed.

2. Multiplication by a scalar

$$a(\vec{A}\times\vec{B}) = (a\vec{A})\times\vec{B} = \vec{A}\times(a\vec{B}) = (\vec{A}\times\vec{B})a$$

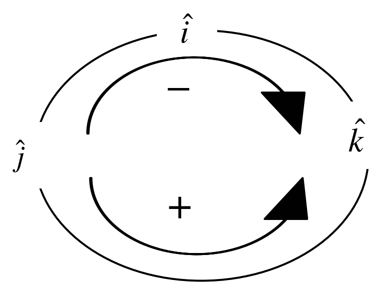
3. The distributive law

$$\vec{A}\times(\vec{B}+\vec{D}) = (\vec{A}\times\vec{B}) + (\vec{A}\times\vec{D})$$

In cross product operations is very important to maintain the proper order of the vectors since they are not commutative.

Cartesian Vector Formulation. To determine the cross product of Cartesian unit vectors the right hand rule can be used if it is kept in mind that the magnitude of a unit vector is "1" and that the cross product results in a third vector perpendicular to the two original vectors, thus

$$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{i} = -\hat{k} \qquad \hat{i}\times\hat{i} = 0$$
$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{j} = -\hat{i} \qquad \hat{j}\times\hat{j} = 0$$
$$\hat{k}\times\hat{i} = \hat{j} \qquad \hat{i}\times\hat{k} = -\hat{j} \qquad \hat{k}\times\hat{k} = 0$$



The calculation of the cross product of two vectors is achieved by multiplying term by term and assigning the proper direction to the resulting values, as shown next.

$$\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \qquad \vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$
$$\vec{A}\times\vec{B} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right)\times\left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right)$$
$$\vec{A}\times\vec{B} = \left[(a_xb_x)\hat{i}\times\hat{i} + (a_xb_y)\hat{i}\times\hat{j} + (a_xb_z)\hat{i}\times\hat{k}\right] +$$
$$+\left[(a_yb_x)\hat{j}\times\hat{i} + (a_yb_y)\hat{j}\times\hat{j} + (a_yb_z)\hat{j}\times\hat{k}\right] +$$
$$+\left[(a_zb_x)\hat{k}\times\hat{i} + (a_zb_y)\hat{k}\times\hat{j} + (a_zb_z)\hat{k}\times\hat{k}\right]$$
$$\vec{A}\times\vec{B} = \left[(a_xb_y)\hat{k} - (a_xb_z)\hat{j}\right] + \left[-(a_yb_x)\hat{k} + (a_yb_z)\hat{i}\right] + \left[(a_zb_x)\hat{j} - (a_zb_y)\hat{i}\right]$$

Finally the last expression can be rearranged as

$$\vec{A}\times\vec{B} = \left(a_yb_z - a_zb_y\right)\hat{i} - \left(a_xb_z - a_zb_x\right)\hat{j} + \left(a_xb_y - a_yb_x\right)\hat{k} \tag{4.4}$$

A more convenient and conventional form to express Eq. (4.4) is

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \quad (4.5)$$

4.3 MOMENT OF A FORCE–VECTOR FORMULATION

The moment of a force $\vec{F}$ about an axis that passes through point *O* is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. This moment can be determined using vector cross product:

$$\vec{M}_O=\vec{r}\times\vec{F} \quad (4.6)$$

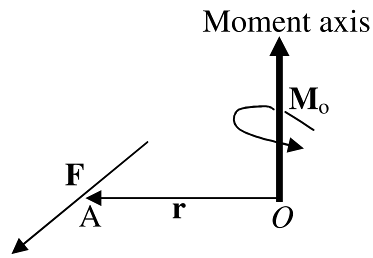


Figure 4.7. Moment of force **F** about axis *O*.

In Eq. (4.6) $\vec{r}$ represents a position vector going from *O* to any point lying on the line of action of $\vec{F}$.

Magnitude. The magnitude of the moment M_o is given by the definition of the cross product of two vectors, Eq. (4.3), that is $M_o = Fr\sin\theta$. The angle θ is measured between the tails of the two vectors, as shown in figure 4.8.

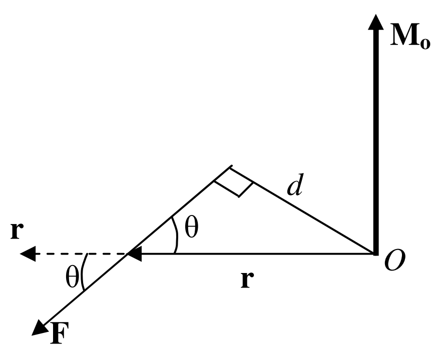


Figure 4.8. Cross product of vector force and position vector results in the moment vector.

Direction. As mentioned above, the direction and the sense of $\vec{M}_0$ are determined by the right hand rule.

Principle of Transmissibility. The principle of transmissibility states that a force may be applied at any point on its given line of action without altering the resultant effects of the force *external* to the rigid body on which acts.

When only the resultant external effects of a force are to be investigated the force may be treated as a *sliding* vector, and it is necessary and sufficient to specify the magnitude, direction and line of action of the force.

Cartesian Vector Formulation. If a force $\vec{F}$ and a vector $\vec{r}$ are expressed in Cartesian coordinates the moment vector produced by the force can be found as

$$\vec{M}_O = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{4.7}$$

In the previous equation r_x, r_y, and r_z are the components of the position vector drawn from the point O to any point on the line of action of the force $\vec{F}$ whose components are F_x, F_y, and F_z. After expanding the determinant, the resulting expression for the moment of the force is

$$\vec{M}_O = (r_y F_z - r_z F_y)\hat{i} - (r_x F_z - r_z F_x)\hat{j} + (r_x F_y - r_y F_x)\hat{k} \tag{4.8}$$

Thus, each of the components of the moment expressed in Eq. (4.8) corresponds to one direction in the Cartesian system of coordinates defined.

Resultant Moment of a System of Forces. If a system of forces is acting on a body as shown in figure 4.9, the resultant moment of the forces about the point O can be determined by vector addition from successive application of Eq. (4.8). This resultant can be expressed as

$$\vec{M}_{RO} = \sum \left(\vec{r} \times \vec{F} \right) \tag{4.9}$$

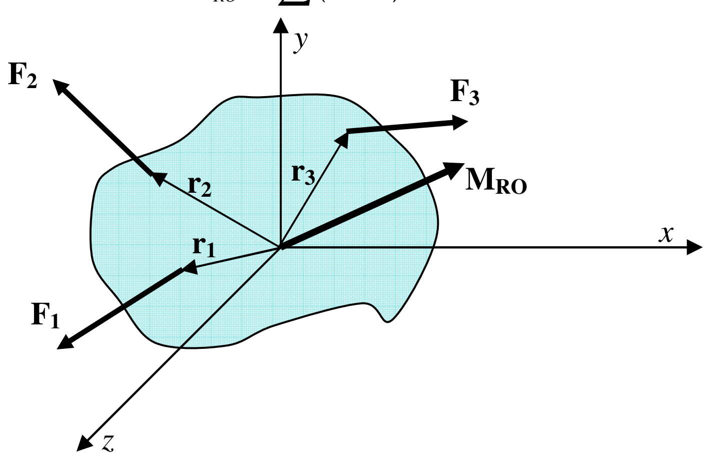


Figure 4.9. Resultant Moment of a system of forces acting on a rigid body.

4.4 PRINCIPLE OF MOMENTS

The *principle of moments* often named also *Varignon's theorem* since it was developed by the French mathematician Varignon (1654–1722) states that:

The moment of a force about a point is equal to the sum of the moments of the force's components about the point.

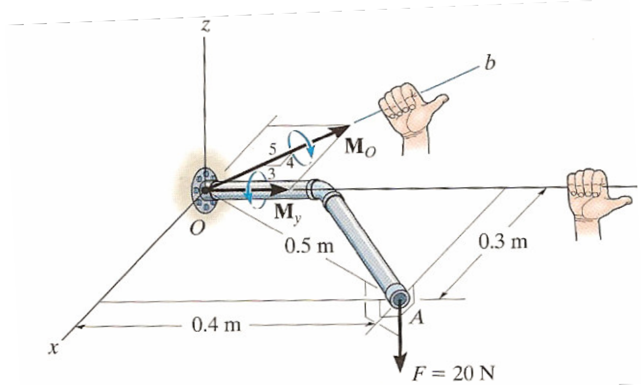
Homework No. 4.2:
4–39, 4–46, 4–47, 4–49, 4–50.

4.5 MOMENT OF A FORCE ABOUT A SPECIFIED AXIS

As studied before, when the moment of a force is computed about a point, the moment and its axis are always perpendicular to the plane containing the force and the moment arm. Some times it is necessary to determine the component of this moment along a specified axis that passes through the point. This problem can be resolved using scalar or vector analysis. These two options are described in the following pair of examples.

Scalar Analysis

The force of 20 N that is applied to the pipe on the *x*–*y* plane, as shown in the figure, produces a total moment M_o. The force is directed in the negative *z*–direction and it is applied at a location of 0.3 m in the *x*–direction and 0.4 m the *y*–direction. This point of application produces a double effect. 1) Due to the location in the *x*–direction the force tends to unscrew the pipe, while 2) due to the moment arm in the *y*–direction, the force generates a tendency for the pipe to rotate around the *x*–axis. This is shown below.



F = 20 N. Force produces a moment with two components:
1) Moment in the *y*–direction: $M_y = F\,d_x$ = (20 N) (0.3 m) = 6 N–m (+)

2) Moment in the *x*–direction: $M_x = F\,d_y$ = (20 N) (0.4 m) = 8 N–m (–)

Thus, the moment that produces tendency to unscrew the pipe is M_y

Vector Analysis

This same problem can be analyzed using vector algebra directly. First we need to determine the position vector form the origin to point *A* and also notice that the force is expressed in vector form as $\vec{F} = -20k$. Thus, $\vec{r} = 0.3\, i + 0.4\, j$. Then the cross product is

$$\vec{M} = \vec{r} \times \vec{F} = (0.3\hat{i} + 0.4\hat{j}) \times (-20\hat{k})$$
$$\vec{M} = 6\hat{j} - 8\hat{i}\ [\text{N - m}]$$

Therefore, we can also see that the force generates a vector moment with two components.

Homework No. 4.3:
4–54, 4–58, 4–65, 4–66, 4–67.

4.6 MOMENT OF A COUPLE

A *couple* is defined as two parallel forces that have the same magnitude, have opposite directions, and are separated by a perpendicular distance *d* (figure 4.10). Since the resultant force is zero, the only effect of the couple is to produce rotation or tendency of rotation in a specific direction.

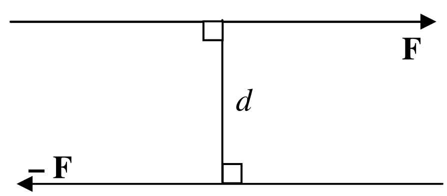


Figure 4.10. Schematic of the definition for a *couple*.

The moment produced by a couple is named *couple moment* and its value can be determined by finding the sum of the moments of both couple forces at any arbitrary point, as schematically presented in figure 4.11 and expressed mathematically as

$$\vec{M} = \vec{r}_A \times (-\vec{F}) + \vec{r}_B \times \vec{F}$$

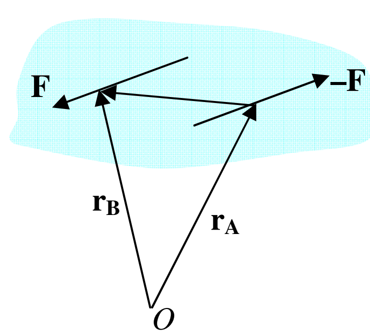


Figure 4.11. Determination of the moment of a couple in space.

The computation of a couple moment can be greatly simplified if instead of calculating the sum of the moments of both forces about a given point, the moment about a point lying on the line of action of one of the forces is determined.

Equivalent Couples. Two couples are said to be equivalent if they produce the same moment. Since the moment produced by a couple is always perpendicular to the plane containing the couple forces, it is therefore necessary that the forces of equal couples lie in the same plane or in parallel planes.

Resultant Couple Moment. Since couple moments are free vectors, they may be applied to any point *P* on a body and added vectorially.

Homework No. 4.4:
4–75, 4–86, 4–93, 4–98, 4–102.

4.7 EQUIVALENT SYSTEM

A force has the effect of translating and rotating a body, and the amount by which it does depends on where and how the force is applied. A methodology to reduce a system of forces and couple moment acting on a body to a single resultant force and couple moment acting at a specified point *O* is discussed next.

Point O is on the Line of Action of the Force.

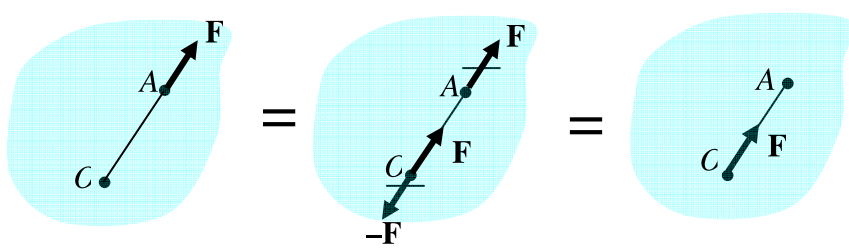


Figure 4.12. Reduction of a force acting on a point without *moment arm* with respect to *O*.

Point O is not on the Line of Action of the Force.

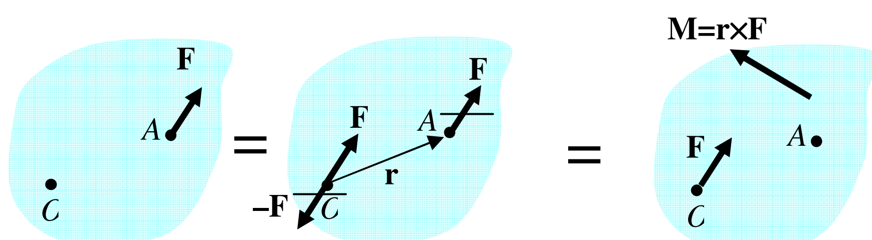


Figure 4.13. Reduction of a force acting on a point with *moment arm* with respect to *O*.

4.8 RESULTANT OF A FORCE AND COUPLE SYSTEM

When a rigid body is subjected to a system of forces and couple moments, it is usually more convenient to study the external effects on the body by replacing the system by an equivalent single resultant force acting at a specified point *O* and a resultant couple moment. Thus if we have a rigid body subjected to a system of forces and moments as shown in figure 4.14, the system of forces and moments can be reduced to a one resultant force and one resultant moment as

$$\vec{F}_R = \sum \vec{F}$$
$$\sum \vec{M}_{Ro} = \sum \vec{M}_C + \sum \vec{M}_O$$

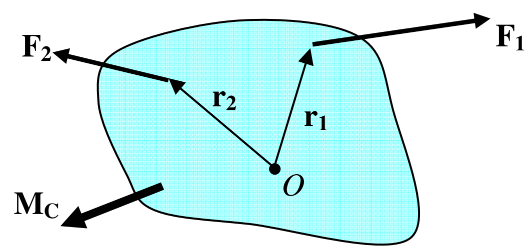


Figure 4.14. A system of forces and moments applied to a rigid body can be replaced by one resultant force and one resultant moment.

Homework No. 4.5:
4–109, 4–110, 4–113, 4–115.