# Calculus with Vectors

*Additional Problems*

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**1)** Section 1.1

1) You are $5\,km$ north, $30\,km$ east and $1\,km$ of the origin and a second person is $3\,km$ south, $3\,km$ east and $1\,km$ above you. How far is the second person from the origin?

2) Points $A$ in rectangular coordinates is $(3, -5)$ and point $B$ in polar coordinates is $\left(5,\ \frac{\pi}{4}\right)$. What is the distance between $A$ and $B$?

3) Two points, in polar coordinates, are $\left(3,\ \frac{\pi}{3}\right)$ and $\left(2,\ -\frac{\pi}{4}\right)$. What is the distance between the two points?

4) Two points, in polar coordinates, are $\left(-3,\ \frac{\pi}{6}\right)$ and $\left(4,\ \frac{5\pi}{6}\right)$. What is the distance between the two points?

5) The midpoint of the line segment from $\mathbf{A} = (a_1, a_2, a_3)$ to $\mathbf{B} = (b_1, b_2, b_3)$ is the point on the segment with endpoints $\mathbf{A}$ to $\mathbf{B}$ which is equidistant from $\mathbf{A}$ and $\mathbf{B}$.

(a) Show that the point $\left(\frac{a_1+b_1}{2},\ \frac{a_2+b_2}{2},\ \frac{a_3+b_3}{2}\right)$ is equidistant from $\mathbf{A}$ and $\mathbf{B}$.

(b) Find the midpoint of the segment from $(3, 0, 2)$ to $(-5, 0, 2)$.

(c) Find the midpoint of the segment from $(0, -7, -3)$ to $(0, -2, 1)$.

(d) Find the midpoint of the segment from $(0, 1, 1)$ to $(0, -2, -1)$.

(e) Find the midpoint of the segment from $(5, 4, 0)$ to $(3, 4, 0)$.

(f) Find the midpoint of the segment from $(2, 0, 4)$ to $(-1, 0, 5)$.

**2)** Section 1.2

1) Three forces $\alpha(1, 2, -1)$, $(-4, 3, 5)$ and $(-12, -6, 9)$ are acting on a mass. What additional force is required to keep the mass stationary?

2) Do following triples of vectors form right hand systems?

(a) $(1, 2, 0)$, $(-1, 4, 0)$, and $\hat{k}$

(b) $(3, 0, 2)$, $(-5, 0, 2)$, and $\hat{\jmath}$

(c) $(0, -7, -3)$, $(0, -2, 1)$, and $-\hat{\imath}$

(d) $(0, 1, 1)$, $(0, -2, -1)$, and $\hat{\imath}$

(e) $(5, 4, 0)$, $(3, 4, 0)$, and $\hat{k}$

(f) $(2, 0, 4)$, $(-1, 0, 5)$, and $-\hat{\jmath}$

3) Two forces $(2, -1)\,N$ and $(5\beta, 6)\,N$ are acting on a mass. What is the magnitude (length) of the force required to keep the mass stationary?

4) Three forces $\alpha(3, 1, -2)\,N$, $(-2, 1, 5)\,N$ and $(-6, 7, 6)\,N$ are acting on a mass. What is the magnitude (length) of the force required to keep the mass stationary?

5) Prove part $(ii)$ of Theorem 1.

6) Prove part $(vi)$ of Theorem 1.

7) All vectors in this problem are in polar coordinates. If $r > 0$, find a unit vector in the same direction as $(r, \theta)$.

**3)** Section 1.3

1) Find the cosines of the angles between the following pairs of vectors.
   (a) $(1, 2)$ and $(-1, 4)$
   (b) $(-3, 2)$ and $(5, 2)$
   (c) $(-7, -3)$ and $(-2, 1)$
   (d) $(1, 1, 1)$ and $(2, -2, -1)$
   (e) $(5, 4, 3)$ and $(3, 4, 5)$
   (f) $(-2, 4, 0)$ and $(1, 3, 5)$

2) Use the dot product to find the sine of the angle between the vectors $\mathbf{y} = (3, -1, 7)$ and $\mathbf{z} = (-3, 4, 2)$.

3) Use the dot product to find the sine of the angle between the vectors $\mathbf{y} = (3, -1, 7)$ and $\mathbf{z} = (-3, 4, 2)$.

4) A $100\,kg$ block is on a frictionless incline going from $(0, 0)\,m$ to $(30, 15)\,m$ at the earth's surface. A person is pushing against the block with a force in the direction of $(20, 8)$. What force, in Newton's, must the person exert to keep the block stationary?

5) In each of the following a mass of $M$ kilograms is on an incline without friction that goes from $A$ to $B$. Assume that the mass is under the influence of gravity at the surface of the earth. Find the force pushing the mass against the incline.

(a) $M = 1$, $A = (0, 0)$, and $B = (4, 3)$.
(b) $M = 5$, $A = (0, 3)$, and $B = (1, 4)$.
(c) $M = 2$, $A = (1, 1)$, and $B = (7, 2)$.
(d) $M = 1$, $A = (0, 5)$, and $B = (4, 5)$.

6) In each of the following a mass of $M$ kilograms is restricted to traveling on the line containing the points $A$ and $B$. If the forces $F_1$ and $F_2$ are acting on the mass, what is the acceleration of the mass? (Remember that $\mathbf{F} = M\mathbf{a}$.)

(a) $M = 1\,kg$, $A = (0, 0)\,m$, $B = (4, 3)\,m$, $\mathbf{F}_1 = (-2, 3)\,N$, and $\mathbf{F}_2 = (-3, -2)$.
(b) $M = 7\,kg$, $A = (7, 2)\,m$, $B = (-1, 5)\,m$, $\mathbf{F}_1 = (4, 1)\,N$, and $\mathbf{F}_2 = (-2, 6)$.
(c) $M = 4\,kg$, $A = (1, 2, 2)\,m$, $B = (4, 5, 4)\,m$, $\mathbf{F}_1 = (-1, 4, 1)\,N$, and $\mathbf{F}_2 = (3, -1, 6)$.
(d) $M = 5\,kg$, $A = (2, -2, 5)\,m$, $B = (4, 7, -1)\,m$, $\mathbf{F}_1 = (1, 1, 1)\,N$, and $\mathbf{F}_2 = (3, -2, 2)$.

7) Use the dot product to find two unit vectors in $\mathbb{R}^3$ perpendicular to both $(2, 3, -1)$ and $(2, 0, 1)$.

**4)** Section 1.4

1) Find the unit vector in the same direction as the given $\mathbf{a}$ and find the unit vector in the opposite direction of the given $\mathbf{a}$.

(a) $\mathbf{a} = (2, 1, 2, 1)$

(b) $\mathbf{a} = (-4, 2, 3, 5)$

(c) $\mathbf{a} = (3, 2, 1, 2, 1, 3)$

(d) $\mathbf{a} = (-1, -1, 1, 1, -1, 1)$

2) Explain why there are more than two unit vectors orthogonal to the vectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ in $\mathbb{R}^4$.

3) Assume we have a sequence of data collected in an experiment $(x_1, y_1)$, $(x_2, y_2), \ldots,$ $(x_n, y_n)$ such that $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n y_i = 0$ and not all of the $x_i$'s and not all of the $y_i$'s are 0. In this case the correlation coefficient is defined by

$$\frac{\sum_{i=1}^n x_i\, y_i}{\left(\sum_{i=1}^n x_i^2\right)\,\left(\sum_{i=1}^n y_i^2\right)}\ .$$

What are the maximum and minimum possible value of the correlation coefficient? Give a geometric interpretation of this correlation coefficient.

**5)** Section 1.5

1) Assume a planet has a circular orbit around its sun with the sun at the center. The force of gravity from the sun acting on the planet always points toward the sun. Show that the dot product of position vector of the planet relative to its sun and the force vector is constant.

2) A bead is restricted to traveling on a rod parametrized by $\ell(s) = (t-1, 2t-3, 3t+2)\ m$ force $-2 \leq s \leq 2$. If the bead is at $(-2, -4, 1)\, m$ and a force $\mathbf{F} = (3, 1, 7)\, N$ is acting on the bead, what is the force pushing the bead against, perpendicular to, the rod at that point?

3) A 10 kilogram mass is initially stationary at $(2, 10, -2)\, km$ and will always have a force $\mathbf{F}_1(t) = \left(\frac{10}{1+t^2}, 3 + 2\sin(t), 3 - e^{-t^2}\right)\ N$ acting on it. What force must you apply to the mass to keep it accelerating at the constant acceleration $(4, 1, 2)$. This force is a vector valued function.

4) Show that no parabola of the form $x = ay^2 + by + c$ in the $xy$-plane where $a \neq 0$ can be parametrized as $\mathbf{r}(t) = \mathbf{v} + t\,\mathbf{z}$.

5) Find parametrizations for the graphs of the following functions.

(a) $g(x) = x\, \sin(x)$

(b) $h(y) = \exp(y^2 + 10)$

(c) $\mathbf{r}(s) = \left(s^2,\, \ln(s^2+1)\right)$

(d) $\mathbf{u}(t) = (-\sin(t),\, -\cos(t))$

(e) $\mathbf{w}(x) = \left(x^3,\, x^3,\, x^4 - x^2\right)$

(f) $\mathbf{z}(s) = \left(s - \sqrt{s},\, \dfrac{s^2-1}{s^2+1},\, \sin(s),\, \exp(s)\right)$

6) Explain why the graph of a function from $\mathbb{R}$ to $\mathbb{R}^n$ is in $\mathbb{R}^{n+1}$.

7) Give two examples of parametrized curves that are not graphs of functions.

8) Give three distinct parametrizations of the line through each of the following pairs of points.

(a) $(0,0,0)$ and $(1,1,1)$
(b) $(2,1,3)$ and $(2,2,3)$
(c) $(4,-5,4)$ and $(3,4,3)$
(d) $(1,1,2)$ and $(-1,-1,-2)$
(e) $(-3,-1,0)$ and $(-1,-2,3)$
(f) $(-7,1,-3)$ and $(-5,2,-2)$
(g) $(1,2,-2,3)$ and $(4,-4,1,2)$
(h) $(5,-5,-7,1,4)$ and $(6,2,2,6,4)$

**6)** Section 2.1

1) Some sequences can be defined recursively by $a_{n+1} = f(a_n)$. Consider an arithmetic progression of the form $a_{n+1} = C\,a_n + D$ where $C$ and $D$ are real numbers.
(a) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges if $a_1 = 1$, $C = 2$ and $D = 2$.
(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges if $C \geq 1$, $D > 0$ and $a_1 = 1$.

2) When does the sequence $\mathbf{b}_n = (-1)^n\,\mathbf{v}$ converge? When does it diverge?

3) Assume the sequence $\{\mathbf{a}_n\}_{n=1}^{\infty}$ converges. Prove, without using any theorem from this section, that every sequence of the form $\{\mathbf{a}_n + \mathbf{v}\}_{n=1}^{\infty}$ converges when $\mathbf{v}$ is a fixed vector.

**7)** Section 2.2

1) Show that the function

$$h(w) = \begin{cases} w^2 - 4w & \text{if w} \leq 2 \\ \sin\left(\frac{\pi x}{2}\right) & \text{if w} > 2 \end{cases}$$

is not continuous at $w = 2$.

2) Where is the function

$$h(z) = \begin{cases} \lfloor \frac{1}{3z} \rfloor & \text{if z} \neq 0 \\ 0 & \text{if z} = 0 \end{cases}$$

continuous?

3) Find two functions $f(x)$ and $g(x)$ that are defined on $\mathbb{R}$ such that both have an infinite number of discontinuities but the sum has only one discontinuity. In addition, make certain that the discontinuity is not removable.

4) Assume that $\mathbf{c}$ is a constant vector. Show that the places where the functions $\mathbf{f}(x)$ and $\mathbf{g}(x) = \mathbf{f}(x) + \mathbf{c}$ are continuous and are discontinuous are the same. (You may not use any theorems from this section.)

5) Which of the discontinuities of $\mathbf{f}(t) = \left(\dfrac{1}{t^2}, \dfrac{t}{t^2+t}, \dfrac{t+1}{t^2-1}\right)$ are removable?

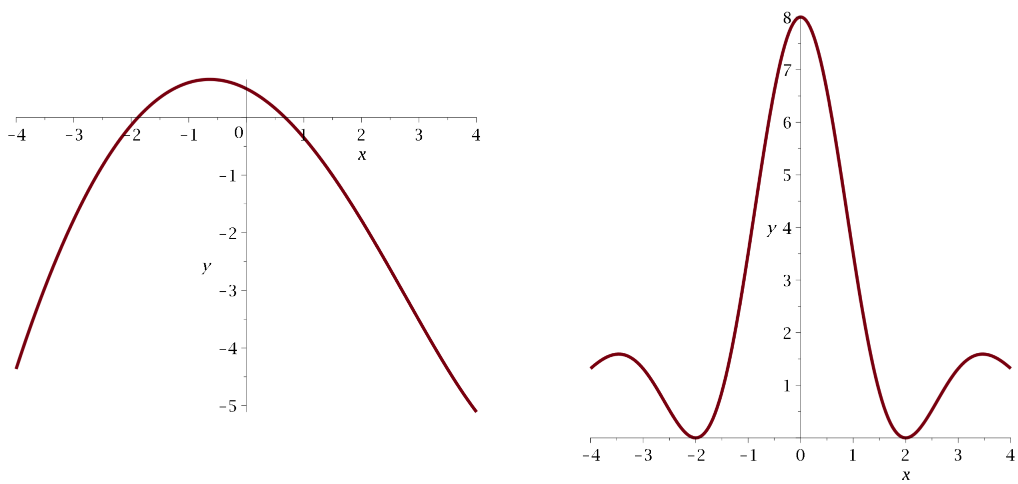
**8)** Section 2.3

1) If the derivative of $\mathbf{c}(s)$ at $s = -2$ is $\mathbf{c}'(-2) = (2,3,-10)$ and $\mathbf{c}(-2) = (3,-4,0)$, find a parametrization of the tangent line to the image of $\mathbf{c}(s)$ at $\mathbf{c}(-2)$.

2) Give an example showing that a function from $\mathbb{R}$ to $\mathbb{R}^3$ can be continuous at a point but not differentiable at that same point.

3) Figure 1(a) is the graph of $f(x)$. At approximately what $x$ values does $f(x)$, as in Figure 1(a), have derivatives 0, 1, $-1$, and 2?

4) Figure 1(b) is the graph of $g(x)$. At approximately what $x$ values does $g(x)$, as in Figure 1(b), have derivatives 0, 1, $-1$, 2, and $-6.5$?




(a) The graph of $f(x)$.

(b) The graph of $g(x)$.

FIGURE 0.0.1.

5) Consider a function of the form $f(a\,x)$ where $a$ is a nonzero constant. Then $\lim_{x\to b} a\,x = a\,b$ and, if $h \neq 0$

$$\frac{f(a\,(b+h)) - f(a\,b)}{h} = \frac{f(a\,(b+h)) - f(a\,b)}{a\,(b+h) - a\,b}\ \frac{a\,(b+h) - a\,b}{h}$$

$$= \frac{f(a\,b + a\,h) - f(a\,b)}{a\,h}\ a\ . \tag{Eq 1}$$

(a) Assume that $f(x)$ is a differentiable function at $a\,x$. Use Eq 1, the definition of the derivative, the fact that if $h \neq 0$ then $a\,h \neq 0$ and the fact that if $h \to 0$ then $a\,h \to 0$ to show that

$$\frac{d}{dx} f(a\,x) = a\,f'(a\,x)\ . \tag{0.0.1}$$

(b) Use Equation 0.0.1 to show that for vector valued function $\mathbf{r}(t)$ that is differentiable at $a\,t_0$ with $a \neq 0$,

$$\frac{d}{dt}\mathbf{r}(a\,t_0) = a\,\mathbf{r}'(a\,t_0)\ . \tag{0.0.2}$$

6) Find the derivatives of the following functions, if they exist, at the given points $a$. If the derivative does not exist at $a$, explain why it does not exist.

(a) $f(x) = \lfloor x \rfloor$ at $a = \frac{3}{2}$.

(b) $h(z) = \lfloor z \rfloor$ at $a = 5$.

(c) $g(y) = \begin{cases} y^2 + 1 & y < 0 \\ y + 1 & y \geq 0 \end{cases}$ at $a = 0$.

(d) $f(x) = \begin{cases} x + 2 & x < 1 \\ x^2 - 2x + 4 & x \geq 1 \end{cases}$ at $a = 1$.

7) Assume that the even function $f(z)$ is differentiable at $z = 0$. Use the definition of the derivative to show that $f'(0) = 0$.

8) Assume that an even function $f(z)$ is differentiable on an interval $I = (-a, a)$ with $a > 0$. Show that $f'(z)$ is an odd function of $z$. (Hint: Consider the difference quotients for $f$ at $z$ with $h > 0$ and for $f$ at $-z$ with $-h$.)

9) Assume that an odd function $f(z)$ is differentiable on an interval $I = (-a, a)$ with $a > 0$. Show that $f'(z)$ is an even function of $z$. (Hint: Consider the difference quotients for $f$ at $z$ with $h > 0$ and for $f$ at $-z$ with $-h$.)

**9)** Section 2.4

1) Refer to Example 82 for this problem. Assume that a population of bacteria satisfies the condition that the rate of growth of the population is equal to the population.

(a) If the population at $t = 0$ is $100,000$ bacteria, is there a function of the form $P(t) = Ce^t$ that satisfies this condition? What is the function, if it exists?

(b) If the population at $t = 0$ is $N$ bacteria, is there a function of the form $P(t) = Ce^t$ that satisfies this condition? What is the function, if it exists?

(c) Let $t_0$ be any real number. If the population at $t = t_0$ is $N$ bacteria, is there a function of the form $P(t) = Ce^t$ that satisfies this condition? What is the function, if it exists?

(d) What does this tell you about populations that have a rate of growth equal to the population?

2) Use trigonometric identities to find the derivatives of the following functions.

(a) $\cos\left(x + \frac{\pi}{4}\right)$ (b) $\sin\left(x - \frac{2\pi}{3}\right)$

**10)** Section 2.5

1) Use a linear approximation for the volume function of a cube to approximate the volume of a cube with side length $4.91\,cm$ without using a calculator.

**11)** Section 3.1

1) Find the left and right sided limits of the following functions at the given point. Are the functions continuous at the point?

(a) $\mathbf{r}(t) = \left(\sin\left(t^2+t\right), \cos(t^2+t)\right)$, $t=3$

(b) $\mathbf{r}(t) = \left(\exp\left(\left\lfloor \frac{t^2+t}{2} \right\rfloor\right), \tan\left(\left\lfloor \pi\frac{t^2+t}{4} \right\rfloor\right)\right)$, $t=3$

(c) $\mathbf{r}(t) = \left(\exp\left(\left\lfloor \frac{t^2+t}{8} \right\rfloor\right), \sin\left(\left\lfloor \pi\frac{t^2+t}{8} \right\rfloor\right)\right)$, $t=3$

(d) $\mathbf{r}(t) = \left(\sin\left(\lfloor t^2-6t+9\rfloor\right), \exp\left(\lfloor t^2-6t+9\rfloor\right), \cos\left(\lfloor t^2-6t+9\rfloor\right)\right)$, $t=3$

2) Find both one-sided limits as $x \to 2$ of $f(x) = \lfloor x\rfloor + (x-\lfloor x\rfloor)^2$. What does this say about $\lim_{x\to 2} f(x)$? Is $f(x)$ continuous at $x=2$?

3) In this problem we consider the differentiability of piecewise defined functions. The situation we consider is where

$$\mathbf{f}(x) = \begin{cases} \mathbf{g}(x) & x<a \\ \mathbf{h}(x) & x\geq a \end{cases}.$$

We assume that $\mathbf{g}(a)=\mathbf{h}(a)$ and both $\mathbf{g}(x)$ and $\mathbf{h}(x)$ are differentiable at $a$. Then

$$\lim_{t\to 0^-}\frac{\mathbf{f}(a+t)-\mathbf{f}(a)}{t} = \lim_{t\to 0^-}\frac{\mathbf{g}(a+t)-\mathbf{g}(a)}{t} = \mathbf{g}\,'(a)$$

and

$$\lim_{t\to 0^+}\frac{\mathbf{f}(a+t)-\mathbf{f}(a)}{t} = \lim_{t\to 0^+}\frac{\mathbf{h}(a+t)-\mathbf{h}(a)}{t} = \mathbf{h}\,'(a)\ .$$

From this we conclude that $\mathbf{f}(x)$ is differentiable at $a$ if and only if $\mathbf{g}\,'(a)=\mathbf{h}\,'(a)$. We state this as follows.

THEOREM 1. *Assume that*

$$\mathbf{f}(x) = \begin{cases} \mathbf{g}(x) & x<a \\ \mathbf{h}(x) & x\geq a \end{cases}.$$

*has* $\mathbf{g}(a)=\mathbf{h}(a)$ *and that both* $\mathbf{g}(x)$ *and* $\mathbf{h}(x)$ *are differentiable at* $a$*. Then* $\mathbf{f}(x)$ *is differentiable at* $a$ *if and only if* $\mathbf{g}\,'(a)=\mathbf{h}\,'(a)$*.*

Use the theorem above to decide if the following functions are differentiable at the given $a$. If the theorem does not apply at $x=a$, explain why the theorem does not apply. If a function is not differentiable at $x=a$, explain why.

(a) $f(x) = \begin{cases} x^2 & x\leq 0 \\ x^4 & x>0 \end{cases}$ at $a=0$.

(b) $g(y) = \begin{cases} x^2-1 & x<1 \\ 1-x^4 & x\geq 1 \end{cases}$ at $a=1$.

(c) $h(z) = \begin{cases} e^z - z & z \le 0 \\ \cos(2z) & z > 0 \end{cases}$ at $a = 0$.

(d) $f(x) = \begin{cases} \lfloor \frac{x}{2} \rfloor & x \le -1 \\ x + 1 & x > -1 \end{cases}$ at $a = 0$.

(e) $\mathbf{h}(w) = \begin{cases} \left(w^2,\, -1,\, \sin(\pi w)\right) & w \le -1 \\ \left(1,\, 3 - 2w^2,\, \cos(\pi w)\right) & w > -1 \end{cases}$ at $a = -1$.

**12)** Section 3.2

1) Find the following limits if they exist.

(a) $\displaystyle\lim_{z\to\infty} \cos\left(z^2 - e^z\right)$

(b) $\displaystyle\lim_{y\to 1} \cos\left(\pi\,\frac{\sin(x-1)}{x-1}\right)$

(c) $\displaystyle\lim_{t\to -\infty} \tan\left(\frac{t^3 - t}{\exp(-t)}\right)$

(d) $\displaystyle\lim_{x\to 4^+} \ln\left(\frac{x^2 - 16}{x^2 - x - 12}\right)$

(e) $\displaystyle\lim_{a\to -2^-} \frac{|a+2|}{6a^2 + 30a + 36}$

2) Evaluate the following limits if they exist. What is the relationship between the two limits?

(a) $\displaystyle\lim_{n\to\infty} \frac{\left(n^2 - 3n^3 + 5\right)\sin(n\pi)}{n^3 + 2n - 6}$. Here the $n$'s are integers.

(b) $\displaystyle\lim_{x\to\infty} \frac{\left(4x^2 - 3x^3 + 5\right)\sin(\pi\, x)}{x^3 + 2x - 6}$

**13)** Section 4.1

1) Find the derivatives of the following functions using the product rule. You will probably need trigonometric identities to rewrite most of the problems.

(a) $f(x) = \cos(2x)$

(b) $f(y) = \sin(2y)$

(c) $g(\theta) = \sin\left(4\theta - \frac{3\pi}{4}\right)$

(d) $h(x) = \cos\left(x^2\right)$

(e) $w(\phi) = \cos\left(\phi + \theta\right)$

(f) $v(\theta) = \sin\left(\phi + \theta\right)$

(g) $f(y) = \sin\left(y^2 + \frac{\pi}{3}\right)$

(h) $h(x) = \cos\left(4x^2\right)$

(i) $g(x) = \exp\left(4 + x\right)$

(j) $w(z) = \exp\left(2z\right)$

2) The *hyperbolic sin* and *hyperbolic cosine* are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}\ .$$

(a) Show that $\cosh(x)$ is an even function and show that $\sinh(x)$ is an odd function.

(b) Find the derivatives of $\cosh(x)$ and $\sinh(x)$.

(c) Show that $\cosh(x)\sinh(x) = \sinh(2x)/2$.

(d) Show that $\cosh^2(x) - \sinh^2(x) = 1$.

(e) Find the derivative of $h(y) = \cosh(x)\sinh(x)$.

(f) Find the derivative of $g(w) = \sinh^2(x)$.

3) Let $\mathbf{x}(t)$ be the position vector of a mass in $\mathbb{R}^2$ or $\mathbb{R}^3$ that is confined to a circle centered at the origin or a sphere centered at $\mathbf{0}$.

(a) Show that $\|\mathbf{x}(t)\|$ and $\|\mathbf{x}(t)\|^2$ are constant functions of $t$.

(b) Using the product rule for dot products, find an expression in vector format for the derivative of $\|\mathbf{x}(t)\|^2$.

(c) If a vector valued function $\mathbf{x}(t)$ is confined to a circle centered at the origin or a sphere centered at $\mathbf{0}$, what can you say about the relationship between $\mathbf{x}(t)$ and $\mathbf{x}'(t)$?

**14)** Section 4.2

1) Find the derivatives of the following functions.

(a) $f(x) = \dfrac{x^2+16}{x+4}$

(b) $g(y) = \tan(y)\ \sec(y)$

(c) $h(z) = \dfrac{\sin(z)+\cos(z)}{\sin(z)-\cos(z)}$

(d) $r(w) = \dfrac{w^2+\exp(w)}{w^2-\exp(w)}$

(e) $f(x) = \dfrac{5\sec(x)}{\cos(x)}$

(f) $g(y) = \dfrac{\ln(y^4)+\ln(y)}{y^4+y}$

(g) $h(z) = \dfrac{\exp(z)+\exp(-z)}{\exp(z)+\exp(-z)}$

(h) $r(w) = \dfrac{w^3+6w^2+7w-2}{\sqrt{w}+1}$

(i) $y(z) = \tanh(z) = \dfrac{\sinh(z)}{\cosh(z)}$

(j) $h(s) = \operatorname{csch}(s) = \dfrac{1}{\sinh(s)}$

(k) $\mathbf{r}(t) = \dfrac{\big(t^2,\ \cos(t),\ \ln(t)\big)}{\sqrt{t^2+1}}$

2) Find the derivatives of the following functions.

(a) $f(x) = \dfrac{x^2+a^2}{x+a}$

(b) $g(y) = \dfrac{4x}{ax^2+b}$

(c) $h(z) = \dfrac{c\sin(z)+a\cos(z)}{c\sin(z)-a\cos(z)}$

(d) $r(w) = \dfrac{r\,w^2+\exp(w)}{w^2-r\ \exp(w)}$

(e) $f(x) = \dfrac{5}{\cos(x) + s}$

(f) $g(y) = \dfrac{\ln(s\,y^4) + \ln(y)}{s\,y^4 + y}$

(g) $h(z) = \dfrac{\exp(z)}{a + \exp(-z)}$

(h) $r(w) = \dfrac{w^3 + bw^2 + cw - d}{\sqrt{w} + a}$

(i) $\mathbf{s}(t) = \dfrac{\left(at^2 + 6,\ 3t^3 - bt,\ ce^{-t}\right)}{t + e^t}$

**15)** Section 4.3

1) Find the derivatives of the following functions.

(a) $f(x) = \sin\left(a\,x + b\right)$

(b) $g(y) = \tan\left(a\,y^2 + 4\right)$

(c) $h(z) = e^{c\,z^2 + 15z}$

(d) $r(w) = \exp\left(A\,w^4 + B\,\sin(w)\right)$

(e) $f(t) = A\sin\left(\omega\,t + \phi\right)$

(f) $g(y) = \ln\left(r\sin(y) + \cos(r\,y)\right)$

(g) $\mathbf{h}(z) = \Big(\exp\left(z^2 + Az\right),\ B\,\cos\left(z^2 + Az\right),\ C\,\sec\left(z^2 + Az\right)\Big)$

(h) $\mathbf{r}(w) = \Big(\cos\left(\csc\left(A\,w + w^2\right)\right),\ \ln\left(\csc\left(A\,w + w^2\right)\right),\ \tan\left(\csc\left(A\,w + w^2\right)\right)\Big)$

2) Find the derivatives of the following functions.

(a) $f(x) = \sin\left(\dfrac{x^2 + 16}{x + 4}\right)$

(b) $g(y) = \tan\left(\sec^2\left(y - y^2\right)\right)$

(c) $h(z) = \dfrac{\sin(4z^2) + \cos(4z^2)}{\sin(4z^2) - \cos(4z^2)}$

(d) $r(w) = \exp\left(w^4 + \exp(w^4)\right)$

(e) $f(x) = \cos\left(\dfrac{5\sec(2x)}{\cos(2x)}\right)$

(f) $g(y) = \dfrac{\ln(y^4) + \ln(y)}{\cos\left(\sqrt{y^2 + 1}\right)}$

(g) $h(z) = \exp\left(z^2 + \dfrac{1}{1 + z^2}\right)$

(h) $r(w) = \cos\left(w^2 - w^{-2} + \csc(w)\right)$

(i) $g(z) = x^2\,\ln\left(3x^2 + 4\right)$

(j) $h(y) = \cos(4y)\,\sin(4y)$

(k) $r(x) = \cos^2\left(6x\right) - \sin^2\left(6x\right)$

(l) $r(w) = \cos\left(w^2 - w^{-2} + \csc(w)\right)$

(m) $f(x) = \sqrt{\dfrac{x + 1}{x - 1}}$

(n) $g(y) = \dfrac{1}{\sqrt{1 - \sin\left(y^2/4\right)}}$

(o) $g(y) = \cosh\left(y^2 - a^2\right)$

(p) $h(z) = \tanh\left(C\sin(z)\right)$

3) Find the derivatives of the following functions.

(a) $f(x) = \sqrt{4x + 2}\,\cos(3x)\,\exp\left(x^2 + 2x + 1\right)$

(b) $s(w) = \cos^2\left(aw + \exp(bw)\right) - \sin^3\left(aw^2 - b\right)$

(c) $\mathbf{s}(t) = \left(t^{-\frac{3}{2}}\exp^{\frac{3t}{2}},\ t^{-\frac{1}{3}}\,\ln\left(\dfrac{2t}{3}\right),\ t^{-\frac{5}{4}}\,\tan\left(a^2t^2\right)\right)$

4) If $f(x) = \cos(ax) + bx^2$, find an equation for the tangent line to the graph of $f(x)$ at $x = \pi$.

5) Let $\mathbf{g}(z) = \big(\exp(az), \cos(cz), \sin(bz)\big)$. Find a parametrization for the tangent line to the image of $\mathbf{g}(z)$ at $z = 1$.

6) Find the equation of the secant line to the graph of $h(y) = ay^2 + b\exp(y)$ between $\big(0, h(0)\big)$ and $\big(2, h(2)\big)$.

**16)** Section 4.4

1) Assume that $y^2x + \cos(x+y) - 2x = 0$ defines $y$ as a function of $x$ around the point $(1,-1)$. Find the slope intercept form of the tangent line to the graph of the function $y(x)$ at $(1,-1)$.

2) Find a parametrization of the tangent line to the curve parametrized by $\mathbf{x}(t) = \big(\cos(t), \sin(2t), t^2\big)$ at the point $\mathbf{x} = \left(\frac{\sqrt{2}}{2}, -1, \frac{\pi^2}{16}\right)$.

3) An object is moving along the path satisfying

$$y^2\exp(x\,y) - x\cos(y) = 1$$

with a speed of $3\,m/sec$ at the point $(0,-1)$.

(a) Find $\frac{dy}{dx}$ for the path at $x \in (0,-1)$.

(b) What are the possible velocities for the object at the point $x \in (0,-1)$?

4) In the following the equation defines $y$ as a function of $x$. Find $\dfrac{dy}{dx}$ at $\mathbf{a}$ using implicit differentiation.

(a) $\mathrm{e}^{xy} - \cos\left(\pi\,\left(x^2+y^2\right)\right) = 2$, $\mathbf{a} = (1,0)$

(b) $x^2 + y^2 - \mathrm{e}^{\frac{x}{y}} = 0$, $\mathbf{a} = (0,-1)$

(c) $\cos\left(\pi\,x^2\right) - \sin\left(\pi\,y^2\right) - 2\,\dfrac{y}{x} = 1$, $\mathbf{a} = (-1,1)$

(d) $\tan\left(\dfrac{\pi\,x}{4\,y}\right) - \dfrac{y^2}{2x} = 0$, $\mathbf{a} = (2,2)$

(e) $\cosh^2(x) - \sinh^2(y) = 1$, $\mathbf{a} = \left(\dfrac{5}{4}, -\dfrac{3}{4}\right)$

5) Consider the following equation.

$$a\,x^2\,y^2 - a^2\,x + y = 2\ .$$

(a) If this equation defines $y$ as a function of $x$, find $\dfrac{dy}{dx}$.

(b) If $y = 2$, for what values of $a$ does your equation give a value for $\dfrac{dy}{dx}$.

(c) If this equation defines $y$ as a function of $a$, find $\dfrac{dy}{da}$.

(d) If $y = 2$, for what values of $x$ does your equation give a value for $\dfrac{dy}{da}$.

6) Give an example of an equation in $x$ and $y$, besides an equation of a circle, that has a least one point satisfying the equation where you cannot do implicit differentiation.

**17)** Section 4.5

1) Find the derivatives of the following functions.

(a) $h(y) = \tan^{-1}\left(\dfrac{a\,y+5}{\sin(b\,y)}\right)$.

(b) $f(x) = \arcsin\left(b\,\sin(x) + x\right)$.

(c) $g(z) = \operatorname{arcsec}\left(\pi\,z + a^{z^2+\phi}\right)$.

(d) $\mathbf{r}(t) = \left(\cos^{-1}\left(\dfrac{a}{t}\right), \cot^{-1}\left(\dfrac{1}{at^2+1}\right), \operatorname{arccsc}\left(\dfrac{at}{t^2+5}\right)\right)$.

(e) $\mathbf{v}(w) = \left(\dfrac{w}{1+\arcsin(a\,w)}, \exp\left(\arctan\left(1+b^2w^2\right)\right)\right)$

(f) $h(y) = \operatorname{arcsec}\left(\arctan(\theta\,y)\right)$

(g) $f(\theta) = \operatorname{arcsec}\left(\arctan(\theta\,y)\right)$

(h) $h(y) = \arctan(\theta\,y)$

(i) $f(\theta) = \arctan\left(\theta\,y\right)$

(j) $f(x) = x^2\,\arcsin(x)$

(k) $g(x) = \cos^{-1}(x)\,\tan^{-1}(2x)$

(l) $h(y) = \exp(x)\,\sec^{-1}\left(x^2\right)$

(m) $r(z) = \ln\left(z^2\right)\,\operatorname{arccot}\left(\pi z\right)$

(n) $s(\alpha) = \tan\left(\alpha\,y^2\right)\sqrt{\cos^2\left(\alpha^2\right)+1}$

(o) $\mathbf{r}(t) = \left(t\,\sin^{-1}(t), \arctan\left(t\,y^4\right), \operatorname{arcsec}\left(t^2+t\right)\right)$

(p) $\mathbf{s}(\tau) = \left(\arctan\left(\tau\,y\right), \exp\left(\tau^2\right), \ln(\tau)\,\cos^{-1}(y\,\tau)\right)$

(q) $f(x) = \dfrac{\cos(x)}{\arccos(x)}$

(r) $g(y) = \dfrac{1+\tan(y)}{\operatorname{arccot}(y)}$

(s) $h(z) = \dfrac{z^3}{\sqrt{4+\tan^{-1}(z)}}$

(t) $f(x) = \dfrac{1+\sqrt{a+b\cos(x)}}{\pi+\sin^{-1}(c\,x)}$

(u) $g(y) = \dfrac{\arctan(4y}{\frac{y}{2} + \arctan(6y)}$

(v) $h(z) = \dfrac{\cos(z)\ \sin^{-1}(2z)}{az^4 + 1}$

2) Find the ranges and domains of the following functions.

(a) $h(y) = 4\arcsin(ay)$ .

(b) $f(x) = b\arccos(2x + a)$ .

(c) $h(y) = \dfrac{5}{\arccos(4y + c)}$ .

(d) $g(z) = c\cot^{-1}(2z^2 + b)$ .

**18)** Section 4.6

1) Find the first three derivatives of the following functions.

(a) $h(y) = \tan^{-1}\left(a\,y^2 + b\right)$ .

(b) $f(x) = \dfrac{c\,x}{x^2 + b}$ .

(c) $g(z) = \exp\left(Az^2\right)$ .

(d) $\mathbf{r}(t) = \left(\cos\left(\dfrac{a}{t}\right), \cot\left(b\,t + c\right), \csc\left(E\,t^2 + F\right)\right)$ .

(e) $\mathbf{v}(w) = (\sin\left(a\,w\right), \ln\left(b\,w + c\right)))$

2) Find the fourth degree Taylor polynomials for the following functions centered at $a = 0$.

(a) $f(x) = \exp(Ax)$ .

(b) $h(y) = \ln(B\,y + 1)$ .

(c) $g(z) = \cos\left(C\,z + \dfrac{\pi}{2}\right)$ .

3) Use Exercises 8 and 9 from Section 2.3 of these additional problems for the following. If $f(x)$ is an even function with $n$ derivatives at $x = 0$, show that the Taylor polynomial for $f(x)$ centered at zero of degree $n$ only uses even powers of $x$.

4) Use Exercises 8 and 9 from Section 2.3 of these additional problems for the following. If $f(x)$ is an odd function such that $f(x)$ has $n$ derivatives at $x = 0$ and $f(0) = 0$, show that the Taylor polynomial for $f(x)$ centered at zero of degree $n$ only uses odd powers of $x$.

**19)** Section 5.1

1) Assume that the functions $f(x)$ and $g(x)$ are continuous on an interval $[a, b]$. Show that if $f(a) > g(a)$ and $f(b) < g(b)$, then $f(c) = g(c)$ for some $c \in (a, b)$.

**20)** Section 5.2

1) Consider the function $g(x) = \begin{cases} -\sqrt{-x} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$.

(a) Show that for any $u \neq 0$, one Newton's step for $g(x)$ takes you from $u$ to $-u$.

(b) For which values of $x$ does Newton's method converge to a root of $g(x)$?

(c) Explain why Theorem 44 in the text does not apply to this function.

**21)** Section 5.3

1) A window consists of a rectangle that is twice as wide as it is high with a half disk on to of the rectangle. The diameter of the window matches the width of the rectangle. If the area of the window is changing at a rate of $1\,m/hr$, how fast is the radius of the half disk changing when the radius of the half disk is $\frac{3}{2}\,m$?

2) A person is sitting in their office that is $50\,ft$ above the ground. A crane that is $500\,ft$ away from the office is lifting a beam from the ground at $30\,ft/min$. How fast is the angle from horizontal to the persons the line of sight to the beam changing when the beam is $10\,ft$ above the ground?

3) The ideal gas law states that $PV = nRT$ where $P$ is the pressure of the gas, $V$ is the volume of the gas, $n$ is the amount of substance of gas (also known as the number of moles), $R$ is the ideal gas constant, and $T$ is the absolute temperature of the gas in degrees Kelvin. Assume that at $280^o\,K$ you have a gas occupying $100\,\ell$ with a pressure of $50\,N/m^2$. Also assume that the amount of gas is constant. If $V$ is increasing at $0.2\,m^3/sec$ and $T$ is increasing at $0.02^o\,K/sec$, what is the rate of change of $P$?

**22)** Section 5.4

1) Find the maximum and minimum values of $g(\theta) = 2\sin(\theta) + \cos(\theta)$ on $[0, \pi]$.

2) Find the local and global maximizing and minimizing points for the following functions.

(a) $f(x) = \exp\left(A^2 x^2 - 4\right)$ .

(b) $h(y) = \ln\left(B^2 y^2 + 1\right)$ .

(c) $g(z) = \cos\left(a\, y^3\right)$ .

(d) $f(w) = \sin\left(c\, w^2\right)$ .

(e) $h(x) = x^3 - A\, x$ .

(f) $g(y) = y^4 - b\, y^2$ .

3) The *value function* of a minimization problem that has an unknown parameter $a$ is the function

$$V(a) = \min_x f(a, x).$$

Find the value function for the optimization problem

$$\min_x x^4 - a\, x^2 \ .$$

4) Show that the minimum distance from a point $(x_1, y_1)$ to a line $Ax + By = C$, $A$ or $B$ is not zero, is achieved at a point $(x_0, y_0)$ on the line where $(x_0 - x_1, y_0 - y_1) \cdot (A, B) = 0$.

**23)** Section 5.5

1) Find the domain and the range of $h(z) = \arcsin(z) + \arccos(z)$. Justify your answer.

2) Let $\mathbf{x}(t)$ be the position vector of a mass in $\mathbb{R}^2$ or $\mathbb{R}^3$ that is always differentiable. Assume that $\mathbf{x}(t) \cdot \mathbf{x}'(t) = 0$ on an interval $(a, b)$.

(a) What does this say about the derivative of $\|\mathbf{x}(t)\|$? What does this say about $\|\mathbf{x}(t)\|$? (Hint: Think about $\|\mathbf{x}(t)\|$ in terms of a dot product.)

(b) What does this tell you about the motion of mass whose position vector is always orthogonal to its velocity?

3) Use four steps of Newton's method to approximate a point guaranteed by the Mean Value Theorem for $h(x) = \cosh(x) - x^2$ on $[0, 4]$ starting at the midpoint of the interval. (This requires several different steps. Get as far as you can.)

4) If $f(4) = 17$ and $|f'(x)| \leq 3$ when $x \in [4, 6]$, what can you say about the values of $f(6)$?

**24)** Section 5.6

1) Where is $g(w) = \sin(w) - \frac{w}{2}$ concave up? Where is $g(w)$ concave down?

**25)** Section 5.7

1) Find all appropriate information for plotting $f(x) = \dfrac{x^2 - 4}{x^4 - 1}$. Then plot the function.

**26)** Section 5.8

1) You need to make a rectangular poster with an area for text that is $\frac{1}{4}\,m^2$. The area for text will have a border that is $\frac{1}{20}\,m$ on all sides. The border cost $60\,\$/m^2$ for the top and bottom borders, including the areas above and below the side borders, and $40\,\$/m^2$ for the side borders. The text area costs $20\,\$/m^2$. What dimensions of the text area will minimize the cost of the poster materials?

**27)** Section 6.1

1) Use algebra or trigonometric identities to simplify and then find antiderivatives for the following functions.

(a) $f(x) = \dfrac{x^3 + \sqrt{x}}{\sqrt[3]{x}}$

(b) $h(\theta) = \sin(\theta + \pi/4)$

(c) $r(t) = \dfrac{t^2 + 2t + 1}{t + 1}$

(d) $s(\theta) = \cos(\theta + \pi)$

(e) $f(x) = \dfrac{x^2 + 4}{x^2 + 1}$

(f) $h(w) = \cos^2\left(\dfrac{w}{2}\right)$

(g) $r(t) = \sin^2\left(\dfrac{t}{2}\right)$

**28)** Section 6.2

1) Let $f(x)$ be a function that is continuous on $[a, b]$ for some finite $a < b$. All Riemann sums approximations in the problem are assumed to use $n$ intervals with equal lengths. The division points of the intervals are $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ where $x_i = x_0 + i\,\frac{b-a}{n}$. Take $h = \frac{b-a}{n}$.

(a) Let $y_i = \max_{x \in [x_{i-1}, x_1]} f(x)$ for $i = 1, \ldots, n$, the maximum value of $f(x)$ in the $i$th interval. Explain why, for any points $\xi_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} f(\xi_i)\, h \leq \sum_{i=1}^{n} y_i\, h\ .$$

(b) Let $z_i = \min_{x \in [x_{i-1}, x_1]} f(x)$ for $i = 1, \ldots, n$, the minimum value of $f(x)$ in the $i$th interval. Explain why, for any points $\xi_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} f(\xi_i)\, h \geq \sum_{i=1}^{n} z_i\, h\ .$$

(c) Assume that $f(x)$ is increasing on $[a, b]$. Explain why, for any points $\xi_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} f(\xi_i)\, h \leq \sum_{i=1}^{n} f(x_i)\, h\ .$$

(d) Assume that $f(x)$ is increasing on $[a, b]$. Explain why, for any points $\xi_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} f(\xi_i)\, h \geq \sum_{i=1}^{n} f(x_{i-1})\, h\ .$$

(e) Assume that $f(x)$ is decreasing on $[a, b]$. Explain why, for any points $\xi_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} f(\xi_i)\, h \geq \sum_{i=1}^{n} f(x_i)\, h\ .$$

(f) Assume that $f(x)$ is decreasing on $[a,b]$. Explain why, for any points $\xi_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} f(\xi_i)\, h \leq \sum_{i=1}^{n} f(x_{i-1})\, h \ .$$

2) Consider the function

$$s(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases} \ .$$

on the interval $[0,1]$. Show that the Riemann sums for $s(x)$ on $[0,1]$ do not converge using either all rational points or all irrational points for evaluating $s(x)$.

**29)** Section 7.1

1) Each part of this problem has two vector valued functions and a point where the curves parametrized by the functions intersect. Find a vector orthogonal to both of the the tangent lines to the curves at the point of intersection of the curves.

(a) $\mathbf{f}(t) = (3+t,\, 1-2t,\, 4-t)$, $\mathbf{g}(s) = (4-2s,\, s-1,\, 2s+3)$, $\mathbf{a} = (4,\, -1,\, 3)$

(b) $\mathbf{f}(t) = (\sin(t),\, \cos(t),\, 4)$, $\mathbf{g}(s) = \left(4-s^2,\, \exp(s-2),\, s^3-4\right)$, $\mathbf{a} = (0,\, 1,\, 4)$

(c) $\mathbf{f}(t) = \left(\dfrac{\arctan(t)}{\pi},\, \ln\left(t^2\right),\, t^2-3\right)$, $\mathbf{g}(s) = \left(\dfrac{1}{s^2-5},\, \sin\left(s^2-9\right),\, 7-s^2\right)$, $\mathbf{a} = \left(\dfrac{1}{4},\, 0,\, -2\right)$

**30)** Section 7.2

1) Consider a problem where we have forces $\mathbf{F}_i$ at points $\mathbf{r}_i$ for $i = 1, 2, \ldots, k$ acting on a stationary plate in the $xy$-plane that can rotate around the origin. We want to know the $\alpha$ such that a force $\alpha\mathbf{F}_0$ applied at $\mathbf{r}_0$ will keep the plate from rotating. Explain why using the **incorrect** expression $\mathbf{T} = \mathbf{F} \times \mathbf{r}$ for torque will give the correct value of $\alpha$?

**31)** Section 7.3

1) If two planes in $\mathbb{R}^3$ are not parallel, the intersection of the planes is a line. Since the line is both planes, the direction of the line is perpendicular to the normals to the planes. In each of the following, find a direction of the intersection of the two planes.

(a) $4x - 3y + 6z = 3$, $x - 2y + z = 0$

(b) $-3x - y + 2z = 5$, $3x - 2y + z = 5$

(c) $2x + 2y - 3z = 7$, $5x - 2y - 4z = 8$

**32)** Section 7.4

1) Evaluate the following limits

(a) $\lim_{x \to -\infty} x\, e^x$

(b) $\lim_{x \to 0} \dfrac{x}{\tan(x)}$

(c) $\lim_{w \to \frac{\pi}{4}} \dfrac{\cos^2(w) - \sin^2(w)}{w - \frac{\pi}{4}}$

(d) $\lim_{z \to \pi} \dfrac{\cos(z)}{e^{z-\pi} - 1}$

(e) $\lim_{w \to \infty} \dfrac{\exp(4w)}{w^2 + \cos(w)}$

(f) $\lim_{z \to 0} \dfrac{1}{z^2 \ln^2(z)}$

(g) $\lim_{x \to 0} \dfrac{1 - \cosh(x)}{1 - \cos(x)}$

(h) $\lim_{s \to \infty} \dfrac{\ln\left(e^{3s} + s\right)}{s}$

**33)** Section 8.1

1) Evaluate the integrals

(a) $\displaystyle\int 2^z\, dz$ .

(b) $\displaystyle\int \frac{4}{(1+\delta)^t}\, dt$ .

(c) $\displaystyle\int \frac{1}{(1+t)\sqrt{t}}\, dt$ .

(d) $\displaystyle\int \tau \Big( \sin\left(\pi\tau^2\right), \cos\left(\pi\tau^2\right) \Big)\, d\tau$ .

(e) $\displaystyle\int_0^{\sqrt{\frac{3\pi}{4}}} y \cos\left(y^2\right) \sin\left(y^2\right)\, dy$ .

(f) $\displaystyle\int_2^3 \frac{(x-2)\, \arctan\left((x-2)^2\right)}{(x-2)^4 + 1}\, dx$ .

(g) $\displaystyle\int_{-1}^3 \frac{\sec^2\left(\ln\left(x^2 + 2x + 1\right)\right)}{x+1}\, dx$ .

**34)** Section 8.2

1) Evaluate the integrals

(a) $\displaystyle\int \arctan(4s)\, ds$

(b) $\displaystyle\int \cos(x) \exp(4x+2)\, dx$

(c) $\displaystyle\int (y+1)\, \cos(2y+2)\, dy$

(d) $\displaystyle\int \sin(z) \cos(4z+2)\, dz$

(e) $\displaystyle\int x^5 \sin(2x^2+3)\, dx$

(f) $\displaystyle\int \sin(a\, y) \sin(4y)\, dy$ where $a \neq 0$.

(g) $\displaystyle\int \cos(b\, z)\, e^{3z}\, dz$

(h) $\displaystyle\int x \cos(x) \sin(3x)\, dx$

(i) $\displaystyle\int y^2 \ln\left(4y^2\right)\, dy$

(j) $\displaystyle\int s^2 e^s \sin(3s)\, ds$

(k) $\displaystyle\int w \arctan(w)\, dw$

(l) $\displaystyle\int \ln^2(a)\, da$

**35)** Section 8.3

1) Consider the goal of using the logistic equation as opposed to the exponential growth equation to be taking into consideration the effects of limited resources. A possible way to limit growth is to assume that the reproduction rate decreases as a function of time.

(a) Why does replacing the positive reproductive rate $k$ in the differential equation

$$\frac{d\,P}{dt} = k\,P$$

with a reproductive rate function $\frac{r}{(1+\delta)^t}$ with $\delta > 0$ force the reproduction rate toward 0 as $t$ gets large?

(b) Solve the differential equation

$$\frac{d\,P}{dt} = \frac{r}{(1+\delta)^t}\,P$$

assuming $r$ and $\delta$ are positive.

(c) Plot the solution to the differential equation and compare your results with the solutions to the logistic equation. What do you conclude about the solutions to part (b) of this problem and how well those solutions model populations with limited resources?

2) Integrate the following:

(a) $\displaystyle\int \frac{4\,x^3 - 2\,x^2 - 5\,x + 4}{(x-2)\,(x+1)}\,dx$

(b) $\displaystyle\int \frac{4\,x^2 - 5\,x + 19}{x^2 - 2\,x + 5}\,dx$

(c) $\displaystyle\int -\frac{2\,w^3 - 3\,w^2 + 10\,w - 9}{(w^2 - 2\,w + 5)\,(w+1)}\,dw$

(d) $\displaystyle\int -\frac{z^4 - 26\,z^2 - 17\,z + 364}{(z^2 + 8\,z + 20)\,(-4+z)}\,dz$

(e) $\displaystyle\int \frac{2\,y^3 + 7\,y^2 - 24\,y - 17}{(y+1)\,(y+5)}\,dy$

(f) $\displaystyle\int \frac{w^3 - 6\,w^2 + 36\,w - 1}{w^2 - 6\,w + 34}\,dw$

(g) $\displaystyle\int \frac{z^5 - 2\,z^4 - 3\,z^3 + 15\,z^2 - 19\,z + 27}{(z+2)\,(z^2 - 4\,z + 5)}\,dz$

(h) $\displaystyle\int \frac{y^4 + 3\,y^3 + 5\,y^2 - 29\,y - 88}{(y+2)\,(y^2 + 4\,y + 13)}\,dy$

(i) $\displaystyle\int \frac{4\,x^3 + 6\,x^2 - 11\,x + 13}{(x^2 - 2\,x + 1)\,(x+2)}\,dx$

(j) $\displaystyle\int \frac{4\,x^3 + 6\,x^2 - 11\,x + 13}{(x^2 - 2\,x + 1)\,(x+2)}\,dx$

(k) $\displaystyle\int \frac{x + a}{(x-2)\,(x+2)}\,dx$

(l) $\displaystyle\int \frac{x^2 - a^2}{(x^2 - 2\,x + 1)}\,dx$

(m) $\displaystyle\int \frac{a\,x^2 - x}{(x+1)\,(x+2)}\,dx$

(n) $\displaystyle\int \frac{1}{(x-a)\,(x-b)}\,dx$

**36)** Section 8.4

1) Evaluate the following integrals.

(a) $\displaystyle\int \cos(3z)\,\sin^2(z)\,dz$ .

(b) $\displaystyle\int_{\pi/4}^{\pi/2} \cot^4(\theta)\, d\theta$

(c) $\displaystyle\int \sin^2(2z)\cos^3(2z)\, dz$

(d) $\displaystyle\int \sin(2x)\cos\left(2x+\frac{\pi}{4}\right)\, dx$

(e) $\displaystyle\int \sin\left(3z-\frac{\pi}{6}\right)\cos(3z)\, dz$

(f) $\displaystyle\int \sin(2x)\sin\left(2x+\frac{\pi}{3}\right)\, dx$

(g) $\displaystyle\int \cos\left(3z-\frac{\pi}{6}\right)\cos^2(3z)\, dz$

(h) $\displaystyle\int \left(\cos^2(3z),\, \sin^2(3z),\, \sin^3(3z)\right)\, dz$

(i) $\displaystyle\int \left(\tan^2(w),\, \sec^4(w),\, \csc^4(w)\right)\, dw$

**37)** Section 8.5

1) Here we consider a term that one gets from a partial fractions decomposition that were not considered earlier. We concentrate on the form $\displaystyle\int \frac{Az+B}{(1+z^2)^2}\, dz$ .

(a) Rewrite the integral $\displaystyle\int \frac{1}{(1+z^2)^2}\, dz$ using the substitution $z=\tan(\theta)$.

(b) Evaluate the trigonometric integral from part (a).

(c) Rewrite the answer from part (b) in terms of $z$.

(d) Use this to evaluate the integral $\displaystyle\int \frac{2z^2+4z-1}{(1+z^2)^2}\, dz$ .

2) Evaluate the following integrals.

(a) $\displaystyle\int \frac{5}{(4+x^2)^{3/2}}\, dx$ .

(b) $\displaystyle\int \left(\frac{x-5}{x^2-2x-3},\, \frac{3x-1}{x^2-2x+5}\right)\, dz$ .

**38)** Section 8.7

1) There are cases when we can decide if an integral of the form $\int_a^\infty f(x)\, dx$ exists without begin able to evaluate it exactly. The ideas are very simple. First, if our function $f(x)$ is

nonnegative and there is a function $g(x)$ with $g(x) \geq f(x)$ for all $x \geq a$ and $\int_a^\infty g(x)\,dx = M$, then, for any $b \in (a, \infty)$,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \leq M\ .$$

This says that $F(b) = \int_a^b f(x)\,dx$ is an increasing function that is always less than or equal to $M$. Using an analog to Theorem 82 on page 345, we have that any increasing function $F(x)$ that is always less than or equal to some $M \in \mathbb{R}$ has a limit $L$ as $x$ goes to $\infty$. In our case, the integral

$$\int_a^\infty f(x)\,dx$$

converges to some $L$ with $L \leq M$.

Second, if our function $f(x)$ is nonnegative and and there is a function $g(x)$ with $g(x) \leq f(x)$ for all $x \geq a$ and $\int_a^\infty g(x)\,dx = \infty$, then, for any $b \in (a, \infty)$,

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx\ .$$

Since $\int_a^\infty g(x)\,dx = \infty$, for any $M \in \mathbb{R}$ there is a $b > a$ such that $\int_a^b g(x)\,dx > M$. This says that $F(b) = \int_a^b f(x)\,dx$ is an increasing function that is eventually greater than any $M$. In our case, the integral

$$\int_a^\infty f(x)\,dx \geq \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx > M\ .$$

This means that $\int_a^\infty f(x)\,dx = \infty$.

The same type of results hold for $\int_{-\infty}^a f(x)\,dx$ and for decreasing $f(x)$. The statements of these cases are left to the reader.

Are the values of the following integrals finite or infinite? Justify your answers.

(a) $\displaystyle\int_2^\infty \frac{2+\cos(z)}{z^2}\,dz$

(b) $\displaystyle\int_1^\infty \frac{2w+\cos(w)}{w^2}\,dw$

(c) $\displaystyle\int_0^\infty \frac{1}{x^3+5}\,dx$

(d) $\displaystyle\int_7^\infty \frac{3}{\sqrt{y}-1}\,dy$

(e) $\displaystyle\int_2^\infty \frac{1}{x+\cos(x)}\,dx$

(f) $\displaystyle\int_4^\infty \frac{1}{\sqrt{r^3}-r\sin(r)}\,dr$

2) Evaluate the following integrals.

(a) $\displaystyle\int_2^{\infty} \frac{2w}{w^4+2w^2+5}\,dw$

(b) $\displaystyle\int_2^{\infty} \frac{y^3+y}{\sqrt{y^4+2y^2+5}}\,dy$

(c) $\displaystyle\int_2^{\infty} \frac{2y}{\sqrt{y^4+2y^2+5}}\,dy$

(d) $\displaystyle\int_{-\infty}^{0} \frac{w^3+w}{(w^4+2w^2+5)^5}\,dw$

**39)** Section 8.8

1) Evaluate the integral $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$ .

2) Evaluate the integral $\displaystyle\int \left(\sec^2(\theta), \sec^4(\theta)\right) d\theta$ .

3) Evaluate the integral $\displaystyle\int x \left(e^{x^2}, \frac{2}{\sqrt{1+x^2}}\right) dx$ .

4) Evaluate the integral $\displaystyle\int_{-1}^{0} \frac{x-1}{\sin^2\left(x^2-2x+1\right)}\,dx$ .

**40)** Section 8.9

1) Use the trapezoid rule to approximate the integral of $f(x)$ from $-2$ to 2 using the following table of data.

| $x$ | $-2$ | $-1.5$ | $-1.1$ | $-0.5$ | $-0.1$ | 0.1 | 0.4 | 0.9 | 1.5 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 1. | 0.35 | $-0.2$ | $-0.3$ | $-0.2$ | 0 | 0.2 | 0.1 | $-0.2$ | 1.0 |

2) Consider the integral $\displaystyle\int_1^4 e^{-z}\cos(z)\,dz$.

(a) Approximate the integral using both the right endpoint and midpoint rules with 17 intervals of equal length.

(b) Are the errors as you expect they should be? The integral has the exact value $\frac{1}{2} - \frac{1}{2}\,\mathrm{e}^{-4}\cos(4) + \frac{1}{2}\,\mathrm{e}^{-4}\sin(4)$ .

**41)** Section 9.3

1) Find the volume of the in $\mathbb{R}^3$ inside the cylinder $x^2+y^2=16$, above the plane $z=-10$, and below the plane $z=2x-1$.

2) Find the volume of the region whose base is the region in the first quadrant of the $xy$-plane with boundaries in the curves $x = 0$, $y = 0$ and $y = \cos(x)$ and whose cross sections with planes parallel to the $xz$-plane are equilateral triangles with one side in the $xy$-plane.

3) The intersection of a volume with the $xy$-plane is the region in the first quadrant of the $xy$-plane whose boundaries are the curves $y = 0$, $y = \sqrt[3]{x}$, and $x = 8$. The cross sections of the volume parallel to the $yz$-plane are circles with a diameter in the $xy$-plane. Find the volume of the region.

4) The base of a volume is the region in the first quadrant of the $xy$-plane boundaries are the curves $y = 0$, $y = \sqrt[3]{x}$, and $x = 8$. The cross sections of the volume parallel to the $yz$-plane are parabolas of the form $z = a\,x - x^2$. Find the volume of the region.

5) Find the volume of the region whose intersection with the $xy$-plane is the region in the first quadrant of the $xy$-plane whose boundaries are the curves $y = 0$, $y = x^2$ and $x = 4$ and whose cross sections with planes parallel to the $yz$-plane are squares with one diagonal in the $xy$-plane.

**42)** Section 9.4

1) Find the volume of the solid generated by rotating the region with $0 \leq x \leq 5$ and $0 \leq y \leq x\,e^{-x}$ around the $x$-axis.

2) Find the volume of the solid generated by rotating the region with $0 \leq x \leq \pi$ and $x\,\sin(x) \leq y \leq 5\sin(x)$ around the line $y = 0$.

3) Find the volume of the solid generated by rotating the region with $0 \leq x \leq \pi$ and $x\,\sin(x) \leq y \leq 5\sin(x)$ around the line $y = 6$.

4) Find the volume, as a function of $b$, of the solid generated by rotating the region(s) bounded by $x = 0$, $x = 4$, $y = 0$, and $y = x + b$ around the $x$-axis.

5) Find the volume, as a function of $b$, of the solid generated by rotating the region(s) bounded by $x = 0$, $x = 4$, $y = 0$, and $y = 2x + b$ around the $y$-axis.

**43)** Section 9.5

1) Use **both** washers/disks and shells to find the volume of the solid obtained by rotating the region in the first quadrant bounded by $y = x^2$, and $y = \sqrt[3]{x}$ around the line $y = -2$.

2) Find the volume of the solid generated by rotating the region with $0 \leq x \leq \pi$ and $x\,\sin(x) \leq y \leq 5\sin(x)$ around the line $x = -1$.

3) A volume is formed by rotating the region in the first quadrant bounded by $y = x$, $x = 0$, and $y = \frac{a+2}{2}\,x - a$ around the $x$-axis. Assume that $a > 0$. Find the volume as a function of $a$.

4) A volume is formed by rotating the region in the first quadrant bounded by $y = x$, $x = 0$, and $y = \frac{a+2}{2}\,x - a$ around the $y$-axis. Assume that $a > 0$. Find the volume as a function of $a$.

**44)** Section 9.7

1) Use Euler's method to approximate $y(3)$ if $y'(t) = y(t)\,\sin(t)$ and $y(1) = -5$. Use 11 equal steps.

**45)** Section 10.1

1) Does the sequence $a_n = \dfrac{(n^2 - 10)\,\cos(n)}{15 - n^2}$ converge? Justify your answer.

2) Give an example, besides those in the text, of a function $f(x)$ such that $\lim_{n\to\infty} f(n)$ exists, but $\lim_{x\to\infty} f(x)$ does not exist.

**46)** Section 10.2

1) In Example 346 it was shown that

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$$

converges. What can you say about the convergence of the series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \begin{cases} 1 & \text{if } n = m^2 \text{ for some integer } m \\ \frac{1}{n\,(n+1)} & \text{otherwise} \end{cases} \quad ?$$

**47)** Section 10.3

1) Do the following series converge or diverge?

(a) $\displaystyle\sum_{m=1}^{\infty} \frac{m+1}{m^2 + 2m + 2}$ .

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+1}{k^3 + k - 5}$ .

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{(n^2 + 4n + 5)^3}$ .

(d) $\displaystyle\sum_{m=4}^{\infty} \frac{\ln(m)}{m^2}$ .

2) Why do we need the hypothesis that $f(x)$ is decreasing in the integral test for the convergence of series?

**48)** Section 10.4

1) Determine if the following series converge or diverge?

(a) $$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n} \ .$$

(b) $$\sum_{n=1}^{\infty} \frac{3^n + 5^n}{4^n} \ .$$

(c) $$\sum_{n=0}^{\infty} \frac{4^n + 8^n}{4^{2n}} \ .$$

(d) $$\sum_{n=1}^{\infty} \frac{n^3 + 3^n}{4^n} \ .$$

(e) $$\sum_{m=10}^{\infty} \frac{4}{m^{3/2}\ \ln(m)} \ .$$