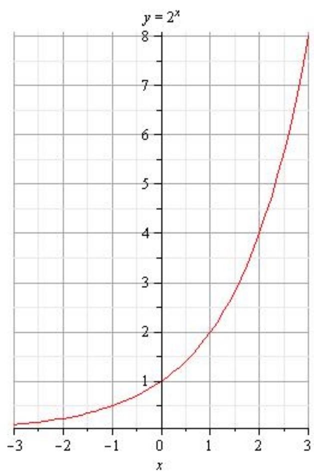
Exponential and Logaritmic Functions

In this section, we consider the *exponential* and *logarithmic functions* and their derivatives. These elementary functions have many important applications. They are used in fields such as econommics, biology, physics, and engineering. We will not prove every theorem in this section, that will be done later, but we want to use these functions for the rest of this course.

First we consider exponential functions of the form $f(x) = b^x$, $b > 1$. (Later we will examine exponential functions with $0 < b < 1$.) The constant b is called the *base* of the exponential function. The graph below shows an example with $b = 2$. Recall that such exponential functions are always positive, always increasing, and one-to-one on their domains, which consist of all real numbers. The graph of $f(x) = 2^x$ indicates the slope is always positive and appears to increase as x increases.



Since exponential functions with a positive base are one-to-one, they have *inverse functions*, generally denoted by $f^{-1}(x) = \log_b(x)$. Thus we have the following:

$$f^{-1}(f(x)) = \log_b(b^x) = x$$

for all real x (which is the domain of f), and

$$f(f^{-1}(x)) = b^{\log_b(x)} = x$$

for all $x > 0$ (which is the domain of f^{-1}). We summarize here some important properties about exponential and logarithmic expressions (sometimes called "laws of exponents" and "laws of logarithmns").

exponential properties	logarithmic properties
$b^r b^s = b^{r+s}$	$\log_b(uv) = \log_b(u) + \log_b(v)$
$\frac{1}{b^s} = b^{-s}$	$\log_b\left(\frac{1}{v}\right) = -\log_b(v)$
$\frac{b^r}{b^s} = b^{r-s}$	$\log_b\left(\frac{u}{v}\right) = \log_b(u) - \log_b(v)$
$(b^s)^r$	$\log_b(u^r) = r\ \log_b(u)$

Now consider the derivative of $f(x) = b^x$, with base $b > 1$.

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{b^{x+h}-b^x}{h}\\ &= \lim_{h\to 0}\frac{b^x b^h - b^x}{h}\\ &= \lim_{h\to 0}\frac{b^x(b^h-1)}{h}\\ &= b^x\left(\lim_{h\to 0}\frac{b^h-1}{h}\right)\end{aligned}$$

Assuming that this last limit exists (which is exactly what we will not prove rigorously now), it will be $f'(0)$. We will denote this by

$$m(b) = \lim_{h\to 0}\frac{b^h-1}{h}.$$

Although numerical work can never "prove" a limit exists, if the limit exists one can get a guess for the limit through computation. We use the following table to get approximate values for $m(2)$ and for $m(3)$.

h	$\frac{2^h-1}{h}$		h	$\frac{3^h-1}{h}$
-0.1	0.66967		-0.6	0.804530
-0.002	0.69267		-0.005	1.095600
-0.00003	0.693140		-0.00004	1.098588
0.00004	0.693157		0.00003	1.098630
0.005	0.694350		0.002	1.099820
0.6	0.859528		0.1	1.161232

From this numerical work (assuming these limits exist), we can estimate that $m(2) \approx 0.693$ and that $m(3) \approx 1.099$. In general, for $f(x) = b^x$, we get that $f'(x) = m(b)\, b^x$. There is a base, denoted e, between 2 and 3 where $m(e) = 1$. The function $f(x) = e^x$ is called the *natural exponential function*, and its inverse $f^{-1}(x) = \ln x$ is called the *natural logarithm*. Among all exponential and logarithmic functions, the "natural" thing about the base $b = e$ is that the derivatives of these natural functions take a very simple form. In particular, we have the following.

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

Most calculators provide keystrokes for the natural exponential and the natural logarithm. Computer applications and languages generally have you type `exp`(x) and `ln`(x). You can see an approximate value for e by evaluating e^1 on a calculator or computer. For example, on a TI-89 we get the following.

$$e \approx 2.71828182846$$

There is a simple *change-of-base* formula that allows us to convert from one exponential base to another. Since $e^{\ln b} = b$,

$$b^x = \left(e^{(\ln b)}\right)^x = e^{(\ln b)\,x}.$$

If we let $u = b^x$, then we can also derive the *change-of-base* formula for logarithms. In one case we have

$$\log_b u = \log_b\left(b^x\right) = x$$

and in a different way

$$\ln u = \ln\left(e^{(\ln b)\,x}\right) = (\ln b)\,x.$$

Solving for x in both lines above and setting the results equal, we get the following

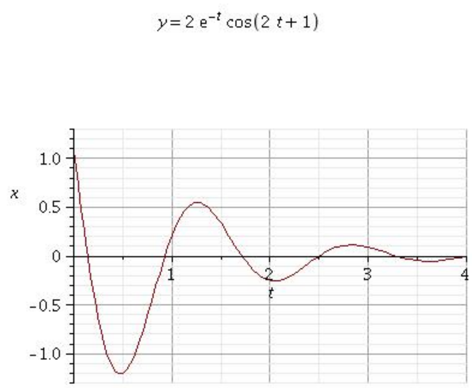
$$\log_b u = \frac{\ln u}{\ln b}.$$

Because these conversions are so simple and the calulus formulas for the natural functions are simple, most applications using exponential and logarithmic functions involving calculus use the natural functions.

Example 0.1. *The function that is often used to describe damped harmonic motion, e.g. a pendulum with some friction, is*

$$x(t) = e^{-kt}\cos(\omega t + \phi)$$

with $k > 0$. The graph of this function with $k = 1$, $\omega = 2$ and $\phi = 1$ shows the general behavior of these functions.



The function

Example 0.2. (Calculator) *Most graphing calculators provide an exponential regression. Consider the following early U.S. Census Data.*

Year x	*Population y*		*Year x*	*Population y*
1790	*3,929,214*		*1810*	*7,239,881*
1800	*5,308,483*		*1820*	*9,638,453*

Performing an ***`ExpReg`*** *on a TI-89 with this data gives* **`y = a*b^x`** *with $a \approx 1.80290695 \times 10^{-17}$ and $b \approx 1.030477846$. To make calculus operations easier, we would prefer a result of the form $y = a\,e^{kx}$. The change of base formula for exponents makes this easy using $a \approx 1.80290695 \times 10^{-17}$ and $k \approx \ln(1.03048) \approx 0.030022623$. By the way, this model does a reasonable job of predicting the U.S. population at the next census [where the actual count in 1830 was 12,866,020 and the value of $y = a\,e^{k(1830)} \approx 13,083,388$]. The model does not do so well at predicting the U.S. population much later [where the actual count in 2000 was 281,421,906 and the value of $y = a\,e^{k(2000)} \approx 2,154,231,253$].*

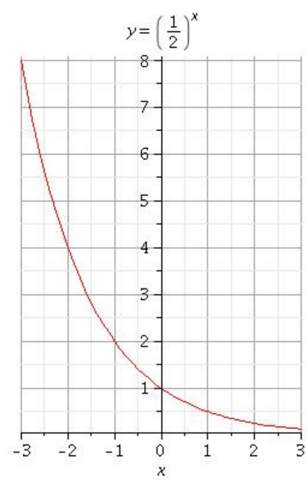
Assuming $\frac{d}{dx}(e^x) = e^x$, we can use the change-of-base formula for exponentials and the chain rule for derivatives to find the value of $m(b)$ for any b.

$$\begin{aligned}\frac{d}{dx}(b^x) &= \frac{d}{dx}\left(e^{(\ln b)\,x}\right) \\ &= \left(e^{(\ln b)\,x}\right)\frac{d}{dx}((\ln b)\,x) \\ &= \left(e^{(\ln b)\,x}\right)(\ln b) \\ &= (\ln b)\,b^x\end{aligned}$$

Thus the constant $m(b)$ that we approximated earlier for few values of b will be $\ln b$. Again evaluating on a TI-89, we get

$$\ln(2) \approx 0.69314718056 \quad \text{and} \quad \ln(3) \approx 1.09861228867$$

An exponential function $f(x) = b^x$ with $0 < b < 1$ can also be handled now. The figure below shows $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$.



Note that exponential functions with $0 < b < 1$ are one-to-one and decreasing, thus there is a well-defined logarithm function as the inverse. We can use the chain rule for derivatives to evaluate the derivative of $f(x) = 2^{-x}$. We have an expression for the derivative of 2^x,

$$\frac{d}{dx}(2^x) = (\ln 2)\,2^x.$$

Using this one has

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}\right)^x &= \frac{d}{dx}\left(2^{-x}\right) \\ &= (\ln 2)\,2^{-x}\frac{d}{dx}(-x) \\ &= -(\ln 2)\,2^{-x} \\ &= \ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x.\end{aligned}$$

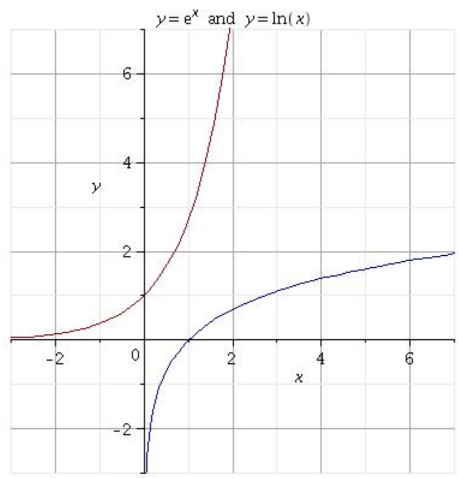
In general, the formula $f'(x) = (\ln b)\,b^x$ remains valid for $0 < b < 1$. Note that $\ln(b)$ will be negative for $0 < b < 1$. This substantiates the claim that an exponential funciton with base between 0 and 1 is decreasing everywhere. We have avoided the case $b = 1$ because we do not label the function $f(x) = 1^x = 1$ as an exponential function. (It is *not* one-to-one and has no inverse function.) However, because $\ln 1 = 0$, the formula $f'(x) = (\ln b)\, b^x$ remains valid even for the case when $b = 1$.

Remark 0.1. Considering the case when $b < 0$ would require working with complex numbers, which we generally avoid i n a first calculus course. As an illustration, let $b = -4$ and $x = \frac{1}{2}$, then

$$(-4)^{\frac{1}{2}} = \sqrt{-4} = 2i.$$

Exponential and logarithmic functions involving complex numbers are very important in engineering and scientific applications and are considered in later courses.

Now we look at the inverse functions of exponentials, namely the logarithms, and seek a derivative rule for them. First we consider the natural logarithm $f(x) = \ln x$. This is plotted below together with the natural exponential functions.



One easy way to get the derivative of an inverse function is to apply the chain rule to the right-hand-side of $f\left(f^{-1}(x)\right) = x$.

$$\begin{aligned}\frac{d}{dx}\left\{f\left(f^{-1}(x)\right)\right\} &= f'\left(f^{-1}(x)\right)\frac{d}{dx}\left(f^{-1}(x)\right)\\ &= \frac{d}{dx}(x)\\ &= 1.\end{aligned}$$

If we know a formula for the derivative of the original function, we can solve the above equation for the derivative of the inverse. We get the following.

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

In the case where $f(x) = e^x$ and $f^{-1}(x) = \ln x$, this reduces to:

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \frac{1}{e^{\ln x}}\\ &= \frac{1}{x}.\end{aligned}$$

Thus the natural logarithm also has a particularly simple derivative formula. Applying the change-of-base formula for logarithms and the derivative rules, we can then differentiate any logarithm $\log_b x$ for $0 < b < 1$ or $1 < b$.

$$\begin{aligned}\frac{d}{dx}\left(\log_b x\right) &= \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right)\\ &= \frac{1}{\ln b}\,\frac{d}{dx}\left(\ln x\right)\\ &= \frac{1}{\ln b}\,\frac{1}{x}\end{aligned}$$

We sumarize all of the derivatives derived in this section (subject to our assumption that the limits defining $m(b)$ exist). The special case when $b = 10$ is also listed since *common logarithms* with this base are frequently used.

$b = e$		$0 < b$ and $b \neq 1$		$b = 10$
$\frac{d}{dx}\left(e^x\right) = e^x$		$\frac{d}{dx}\left(b^x\right) = (\ln b)\, b^x$		$\frac{d}{dx}\left({}^{x}\right) = (\ln 10)\,{}^{x} \approx (2.303)\, 10^x$
$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$		$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{\ln b}\,\frac{1}{x}$		$\frac{d}{dx}\left(\log x\right) = \frac{1}{\ln 10}\,\frac{1}{x} \approx (0.4343)\,\frac{1}{x}$

Example 0.3. *Find the derivative of* $\frac{d}{dx}\left(e^{5x}\sin(2x)\right)$.

$$\begin{aligned}\frac{d}{dx}\left(e^{5x}\sin(2x)\right) &= \sin(2x)\,\frac{d}{dx}\left(e^{5x}\right) + e^{5x}\,\frac{d}{dx}\sin(2x)\\ &= 5e^{5x}\sin(2x) + 2e^{5x}\cos(2x)\end{aligned}$$

Example 0.4. *Find the derivative of* $\frac{d}{dx}\left(\ln\left(x^2 + 2x + 3\right)\right)$.

$$\begin{aligned}\frac{d}{dx}\left(\ln\left(x^2 + 2x + 3\right)\right) &= \frac{1}{(x^2 + 2x + 3)}\,\frac{d}{dx}\left(x^2 + 2x + 3\right)\\ &= \frac{2x + 2}{(x^2 + 2x + 3)}\end{aligned}$$

Exercises.

Find the derivatives of the following functions.

(1) 6^x

(2) $\left(\frac{1}{6}\right)^x$

(3) $\log_6(x)$

(4) $\log_{\frac{1}{6}}(x)$

(5) x^4

(6) $\left(\frac{1}{4}\right)^x$

(7) $\log_4(x)$

(8) $\log_{\frac{1}{4}}(x)$

(9) e^{7x}

(10) $e^{7x}\cos(4x)$

(11) $\frac{e^{7x}+x^2}{e^{2x}+1}$

(12) e^{x^2}

(13) e^{-x^2}

(14) $\ln\left(x^2\right)$

(15) $\frac{e^x+e^{-x}}{2}$

(16) $\frac{e^x-e^{-x}}{2}$

(17) $\ln(5x)$

(18) $\ln(10x)$

(19) $\ln(x+1)$

(20) $\ln\left(3x^2+5x+11\right)$

(21) $\ln\left(3x^2+5x+11\right)$

(22) $\ln(4x+6)$

(23) $\ln\left(\frac{3x^2+5x+11}{4x+6}\right)$

(24) $e^{2x}+5x$

(25) $\ln\left(e^{2x}+5x\right)$

(26) $\frac{\ln(e^{2x}+5x)}{\sin(3x)}$

(27) $\ln\left(\sqrt{7x+5}\right)$

(28) $\frac{1}{2}\ln(7x+5)$

(29) $\sqrt{\ln(7x+5)}$

(1) Numericaly approximate the values for $m(2.7)$ and $m(2.8)$ using the limit definition. Compare your approximations with $\ln(2.7)$ and $\ln(2.8)$.

(2) Plot The functions $f(x)=2.5^x$ and $g(x)=2.9^x$ together. Indicate where f is larger than g.

(3) Plot The functions $f(x)=1.6^x$ and its derivative together. Indicate where f is larger than g.

(4) Plot The functions $g(x)=\left(\frac{1}{2}\right)^x$ and $g(x)=log_{\frac{1}{2}}(x)$ together. Numerically approximate the intersection of these two functions.

(5) Plot The function $f(x)=e^x$ and the tangent line to its graph at $x=0.6$ together.

(6) Plot The function $f(x)=\ln(x)$ and the tangent line to its graph at $x=2$ together.