Math 1710 Exam 3

Name ______________________

Show all work! November 21, 2008

1) Evaluate the following.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^2 - e^{-x}}{e^x - 4x}$

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2 - e^{-x}}{e^x - 4x} &= \lim_{t\to\infty} \frac{2x + e^{-x}}{e^x - 4} \qquad \text{L'Hôpitol's rule}\\ &= \lim_{t\to\infty} \frac{2 + e^{-x}}{e^x} \qquad \text{L'Hôpitol's rule}\\ &= 0. \qquad \text{Since the form is 2 over } \infty.\end{aligned}$$

(b) $\displaystyle\int_0^\infty w\, e^{-w}\, dw$

First, integrate by parts:

$$\begin{aligned}\int w\, e^{-w}\, dw &= -we^{-w} + \int e{-}w\, dw\\ &= -we^{-w} - e^{-w} + C.\end{aligned}$$

Then do the improper integral.

$$\begin{aligned}\int_0^\infty w\, e^{-w}\, dw &= \lim_{a\to\infty} \int_0^a w\, e^{-w}\, dw\\ &= \lim_{a\to\infty} -(w+1)\, e^{-w}\Big|_0^a\\ &= \lim_{a\to\infty} -(a+1)e^{-a} + 1\\ &= 1 \qquad \text{(After applying L'Hôpital's rule.)}\end{aligned}$$

2) Do the following series converge or diverge? *Justify your answers.*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{n^2}$

One can compare this to $\sum_{n=1}^{\infty} \frac{1}{n}$ as follows. Note that

$$\frac{n+1}{n^2} = \frac{n}{n^2} + \frac{1}{n^2} > \frac{1}{n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=0}^{\infty} \frac{n+1}{n^2}$ diverges.

(b) $\displaystyle\sum_{m=1}^{\infty} \frac{4^m}{5^m}$

Note that $\frac{4^m}{5^m} = \left(\frac{4}{5}\right)^m$. This means that the series is a geometric series with $r = \frac{4}{5} < 1$. The series converges.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2^k + k^3}$

Here the limit ratio test will work.

$$\begin{aligned}
\lim_{k\to\infty} \frac{\frac{(k+1)^3}{2^{k+1}+(k+1)^3}}{\frac{k^3}{2^k+k^3}} &= \lim_{k\to\infty} \frac{(k+1)^3\,(2^k+k^3)}{k^3(2^{k+1}+(k+1)^3)} \\
&= \lim_{k\to\infty} \frac{(k+1)^3}{k^3}\,\frac{1+\frac{k^3}{2^k}}{2+\frac{(k+1)^3}{2^k}} \\
&= \frac{1}{2} < 1.
\end{aligned}$$

This means the series converges.

3) Explain why the logistic equation is often a better model for the real world than the exponential growth model.

The logistic equation is often a better model since there is a finite limiting population. The reflect that fact that there are always limits on space and food. For small populations, it behaves like the exponential growth model. For small populations space and food do not limit the populations.

4) Find the volume generated by rotating the region in the first quadrant bounded by $y = 1 - x^2$ and $y = 4 - x/2$ around the x-axis.

Note that this can be viewed as taking the volume generated by rotating the region bounded by $y = 1 - x^2$ in the first quadrant around the x-axis out of a right circular cone with base radius 4 and height 8. One can compute this as follows:

$$\begin{aligned}
\text{Volume} &= \int_0^8 \pi\left(4 - \frac{x}{2}\right)^2 dx - \int_0^1 \pi\left(1 - x^2\right)^2 dx \\
&= \frac{2\pi}{3}\left(4 - \frac{x}{2}\right)^3 \Big|_{x=0}^{8} - \pi\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right) \\
&= \frac{128\pi}{3} - \frac{8\pi}{15} \\
&= \frac{632\pi}{15}.
\end{aligned}$$

5) The temperature of a object follows Newton's Law of Cooling. If the initial temperature of the object is 50^o, the ambient temperature is 10^o, and the temperature of the object after 20 minutes is 40^o, what is the temperature of the object after 50 minutes?

The differential equation for Newton's law of cooling is

$$\frac{dT}{dt} = k\,(T - T_A).$$

The solution of this differential equation is

$$T(t) = T_A + T_1\,e^{-kt}.$$

In our case we have, at $t = 0$,

$$50 = 10 + T_1\,e^0.$$

This gives $T_1 = 40$. Plugging in $T(20) = 40$ gives

$$40 = 10 + 40\,e^{-kt}.$$

Solving for k gives

$$k = \frac{\ln\left(\frac{4}{3}\right)}{20}.$$

Putting this back into the solution and setting $t = 50$ gives

$$T(50) = 10 + 40\,e^{-\frac{50\ln\left(\frac{4}{3}\right)}{20}} \approx 29.49.$$