FE 5204 Stochastic Differential Equations

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Lecture 3: Stochastic processes and Itô integrations

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We focus this lecture on three topics:

- Continuous stochastic processes: they can be regarded as the limits of its discrete counterpart.
- Brownian motions: a fundamental example of continuous martingale, and
- the Itô integration which can be viewed as a continuous version of the martingale transform.
- Our main reference for this lecture is Chapter 2 and 3 of the textbook.

A continuous stochastic process

Stochastic process

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $[0, T]$ be an interval. $(X_t), t \in [0, T]$ is a stochastic process if for every t, X_t is a random variable on $(\Omega, \mathcal{F}, P)$.

Path

For each fixed $\omega \in \Omega$,

$$t \longrightarrow X_t(\omega)$$

is called a *path* of the stochastic process (X_t).

As a two variable function

A stochastic process (X_t) can also be viewed as a two variable function

$$(t, \omega) \to X_t(\omega) = X(t, \omega).$$

Usually we will need a certain properties of this function. We often assume every path is continuous which ensures the joint measurability of $X(t, \omega)$.

General filtration

Definition of filtration

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $[0, T]$ be an interval. $(\mathcal{F}_t), t \in [0, T]$ is a filtration if for every t, $\mathcal{F}_t \subset \mathcal{F}$ is a σ-algebra and, for any $s < t$,

$$\mathcal{F}_s \subset \mathcal{F}_t.$$

Again, $\mathcal{F}_t$ represents information available up to time t.

Adapted stochastic process

Adapted stochastic process

Let $(\mathcal{F}_t), t \in [0, T]$ is a filtration on probability space $(\Omega, \mathcal{F}, P)$. We say a stochastic process (X_t) is $(\mathcal{F}_t)$-adapted provided that, for every t, X_t is $\mathcal{F}_t$ measurable.

Note: there is no corresponding concept for predictable.

Filtration generated by a stochastic process

Filtration generated by a stochastic process

Let $(X_t), t \in [0, T]$ be a stochastic process on probability space $(\Omega, \mathcal{F}, P)$. Define $\mathcal{F}_t = \sigma(\{X_s^{-1}(F) : s \in [0, t], F \in \mathcal{B}\})$, where $\mathcal{B}$ is all the Boral sets in R^n. Then $(\mathcal{F}_t)$ is a filtration and (X_t) is $\mathcal{F}_t$-adapted.

Brownian motion: history

1. Named after Scottish botanist Robert Brown who in 1828 observed this motion from pollen suspended in liquid.
2. Louis Bachelier use it to model stock market in 1900, wrongly for allowing negative stock price.
3. Paul Samuelson gave the widely used geometric Brownian motion model for stock price movements in 1965, which do not allow any price jump.
4. "All models are wrong. Some are wronger than others."

Definition of 1-dimensional Brownian motion (Wiener)

A stochastic process $\{B_t : t \in [0, T)\}$ is called a Brownian motion starting from x if

1. $B_0 = x$,
2. for $0 \le t_1 < t_2 < \ldots < t_k \le T$, the random variables
$$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_k} - B_{t_{k-1}}$$
are independent,
3. for $0 \le s \le t \le T$, $B_t - B_s$ has Gaussian distribution with mean 0 and variance $t - s$,
4. for ω in a set of probability one, the path $B_t(\omega)$ is continuous.

Definition of multi-dimensional Brownian motion

Multi-dimensional Brownian motion

A vector stochastic process $\{B_t : t \in [0, T]\}$ in R^n is called a Brownian motion starting from $x = (x^1, x^2, \ldots, x^n)$ if $B_t = (B_t^1, B_t^2, \ldots, B_t^n)$ where $B_t^i, i = 1, 2, \ldots, n$ are independent 1-dimensional Brownian motion starting from x^i.

If $x = 0$, we say (B_t) is a standard Brownian motion. A Brownian motion starting from x can be viewed as a standard Brownian motion shifted by x.

Existence

1. Existence of a Brownian motion needs to be justified.
2. By and large, there are two ways to do it:
3. by construction (pioneered by Wiener, see e.g. Steele's book), or
4. by Kolmogorov's extension theorem (if time permits, other wise see e.g. [T]).

Uniqueness

- Brownian motions are not uniquely defined,
- but their effects are equivalent.
- We usually pick a 'convenient' version.

Gaussian distribution

Let $v = [v^1, \dots, v^d]^\top$ be a d-dimensional random vector. We define the mean vector and the covariance matrix by $\mu = \mathbf{E}[v] = [\mu_1, \dots, \mu_d]^\top$, and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{bmatrix},$$

where $\sigma_{ij} = \mathrm{cov}(v_i, v_j)$.

Gaussian distribution

Gaussion distribution

A d-dimensional random vector v is Gaussian with mean μ and covariance matrix Σ if the density of v is given by

$$f(x) = (2\pi)^{-d/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(x-\mu)^\top\Sigma^{-1}(x-\mu)\right).$$

Two simple and important facts of Gaussian distribution

- Gaussian random vectors are completely determined by its mean and covariance.
- The components of a Gaussian random vector are independent if and only if Σ is diagonal.

Characteristic functions

Let v be a d-dimensional random vector. Then

$$\phi(\theta) = \mathbf{E}[\exp(i\theta^\top v)]$$

is the characteristic function of v which uniquely determines the distribution of a random vector. By direct computation we have

Characteristic function of a Gaussion vector

Let v be a d-dimensional Gaussian random vector v with mean μ and covariance matrix Σ. Then

$$\phi(\theta) = \mathbf{E}[\exp(i\theta^\top v)] = \exp\left(i\theta^\top \mu - \frac{1}{2}\theta^\top \Sigma \theta\right).$$

Gaussian characterization

Gaussian characterization

Let v be a d-dimensional random vector. Then v is Gaussian if and only if for any $\theta \in R^d$, $\theta^\top v$ is a univariate Gaussian random variable.

Gaussian process

Gaussian process

Let X_t be a stochastic process. If, for any $0 \le t_1 < t_2 < \ldots < t_k$, $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ is a k-dimensional Gaussian random vector then we say (X_t) is a Gaussian process.

A Gaussian process is completely determined by the mean and covariance functions $\mu(t) = \mathbf{E}[X_t]$ and $f(s,t) = \mathrm{cov}(X_s, X_t)$.

Covariance of a Brownian motion

A standard 1-dimensional Brownian motion (B_t) is a well behaved Gaussian process. For $s < t$, its covariance is $\mathrm{cov}(B_s, B_t) = \mathbf{E}[B_t - B_s + B_s, B_s] = \mathbf{E}[B_s^2] = s$. In general,

Covariance of a Brownian motion

$\mathrm{cov}(B_s, B_t) = s \wedge t$.

Covariance of a Brownian motion

The converse is also true:

Covariance of a Brownian motion

Let (X_t) be a Gaussian process with continuous paths and $\mathbf{E}[X_t] = 0$ for all $t \in [0, T]$. Then (X_t) is a standard Brownian motion if and only if $\text{cov}(X_s, X_t) = s \wedge t$.

Proof. We need to show that $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent. This can be done by expending, for $i < j$, $\mathbf{E}[(X_{t_i} - X_{t_{i-1}})(X_{t_j} - X_{t_{j-1}})]$ to find

$$\mathbf{E}[X_{t_i} X_{t_j}] - \mathbf{E}[X_{t_i} X_{t_{j-1}}] - \mathbf{E}[X_{t_{i-1}} X_{t_j}] + \mathbf{E}[X_{t_{i-1}} X_{t_{j-1}}]$$

$$= t_i - t_i - t_{i-1} + t_{i-1} = 0.$$

Higher moments

For a standard 1-dimensional Brownian motion (B_t), we can also use its density function to directly calculate higher moments

$$\mathbf{E}[(B_t - B_s)^3] = 0,$$

and

$$\mathbf{E}[(B_t - B_s)^4] = 3(t-s)^2.$$

Alternatively, one can also use the method in Exercise 2.8.

Multi-dimensional generalizations

Let (B_t) be an n-dimensional standard Brownian motion. Then we have

Covariance of a Brownian motion

$\mathrm{cov}(B_s, B_t) = n(s \wedge t)$. In particular, $\mathbf{E}[|B_t - B_s|^2] = n(t - s)$.

and

Higher moments

$\mathbf{E}[|B_t - B_s|^4] = n(n+2)(t-s)^2$.

Itô Integration

- the Itô integration can be viewed as a continuous version of the martingale transform.
- This is one of the key concepts in the stochastic analysis.
- Its role in solving SDE is similar to that of integration in solving ODE.
- We focus on the idea rather than technical details.

Integrable function space $\mathcal{V}$

- Let $B(t)$ be a one dimensional Brownian motion.
- Let $\mathcal{F}_t := \sigma\{B^{-1}(s) : s \leq t\}$ be the standard Brownian filtration.
- We define Itô integration for the class of functions $\mathcal{V}$ from $[0, \infty) \times \Omega \longrightarrow R$ satisfying:
  - (i) $f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable;
  - (ii) for fixed t, $f(t, \omega)$ is $\mathcal{F}_t$ adapted;
  - (iii) $\mathbf{E}[\int_S^T f(t, \omega)^2 dt] < \infty$.

Itô integral for elementary functions

- Again we use the idea of approximation using Itô integration for elementary functions.
- They are of the form

$$\phi(t,\omega) = \sum_j e_j(\omega)\chi_{[t_j,t_{j+1})}(t),$$

- where e_j is $\mathcal{F}_{t_j}$ adapted and $\chi_{[t_j,t_{j+1})}(t) = 1, t \in [t_j, t_{j+1})$ and 0 otherwise.
- For an elementary function ϕ we define its Itô integration with respect to $B(t)$ as

$$\int_S^T \phi(t,\omega)dB(t) = \sum_j e_j(\omega)(B(t_{j+1}) - B(t_j)).$$

- This is a martingale transform.

Approximating $\mathcal{V}$

- From the measure and integration theory we know that for any function $f \in \mathcal{V}$ there exists a sequence of elementary functions ϕ_n such that

$$\mathbf{E}[\int_S^T [f(t,\omega) - \phi_n(t,\omega)]^2 dt] \to 0, \text{ as } n \to \infty. \qquad (1)$$

- The Itô integration for f in $\mathcal{V}$ is defined by

$$\int_S^T f(t,\omega)dB(t) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB(t), a.s.$$

where ϕ_n that satisfies (1).

- Will different approximating sequences yield different limits?
- The answer is no by virtue of the following important lemma.

Itô isometry

Itô isometry

Let $\phi \in \mathcal{V}$ be an elementary function. Then

$$\mathbf{E}[\int_S^T \phi^2(t,\omega)dB(t)] = \mathbf{E}[\int_S^T \phi^2(t,\omega)dt].$$

Linearity and additivity

- $\int_S^T f dB(t) = \int_S^U f dB(t) + \int_U^T f dB(t)$.
- $\int_S^T (af + bg) dB(t) = a \int_S^T f dB(t) + b \int_S^T g dB(t)$.

Measurability and mean

- $\int_S^T f dB(t)$ is $\mathcal{F}_T$ measurable.
- $\mathbf{E}[\int_S^T f dB(t)] = 0$.

Itô integration and continuous martingale

- The most important feature of the Itô integration is that it produces continuous martingales.
- This is not surprising given that the Itô integrals can be viewed as a continuous version of the martingale transforms.
- However, we need to pay attention to a number of technical details.
- We start with the definition of a continuous martingale.

Definition of continuous martingale

Definition of continuous martingale

A stochastic process M_t is a martingale with respect to the filtration $\mathcal{F}_t$ on probability space $(\Omega, \mathcal{F}, P)$ if

(i) M_t is $\mathcal{F}_t$ measurable,

(ii) $\mathbf{E}[|M_t|] < \infty$, and

(iii) $\mathbf{E}[M_s|\mathcal{F}_t] = M_t$, for $s > t$.

Here the expectation is with respect to probability P.

We emphasize that martingale is relative to a filtration and a probability measure. We will revisit this topic in Girsanov Theory.

Examples

Brownian motion is a martingale

Brownian motion $B(t)$ on R is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(\{B(s) : s \leq t\})$.

Justification

- (i) follows from the definition of $\mathcal{F}_t$.
- (ii) can be derived from Jensen's inequality:

$$\mathbf{E}[|B(t)|]^2 \leq \mathbf{E}[B(t)^2] \leq t < T < \infty.$$

- (iii) is due to the independence of the increment of $B(t) - B(s)$ and $B(s) = B(s) - B(0)$, which implies that

$$\mathbf{E}[B(t)|\mathcal{F}_s] = \mathbf{E}[B(t) - B(s) + B(s)|\mathcal{F}_s]$$

$$= \mathbf{E}[B(t) - B(s)] + \mathbf{E}[B(s)|\mathcal{F}_s] = B(s)$$

Itô integration generates martingales

'Brownian motion is a martingale' is a special case of the following more general result:

Itô integration generates martingales

For $f \in \mathcal{V}$, $M_t = \int_0^t f(s,\omega)dB(s)$ is a martingale.

Remarks

- The notation $\int_0^t f(s,\omega)dB(s)$ signifies a unique continuous stochastic process whose existence need technical verification.
- The proof of $M_t = \int_0^t f(s,\omega)dB(s)$ is a martingale depend on discrete approximation.
- Intuitively, this is the limit of a discrete martingale derived from a martingale transform of a discrete version of $B(t)$.

Remark

The definition and properties of Itô integration also applies to functions adapted to a filtration $\mathcal{H}_t$ such that $B(t)$ is a martingale with respect to $\mathcal{H}_t$ and $\mathcal{F}_t \subset \mathcal{H}_t$.

Itô integrations for multi-dimensional Brownian motions

Let $B = (B_1, B_2, \dots, B_n)$ be an n-dimensional Brownian motion and $v = (v_{ij})$ be an $m \times n$ matrix function. A vector form of the Itô integration $\int_S^T v dB(t)$ is formally defined componentwise: the ith component is

$$\sum_{j=1}^{n} \int_S^T v_{ij}(t,\omega) dB_j(t,\omega).$$

The technical part

- Let $\mathcal{F}_t^n$ be the σ-algebra generated by $(B_1(s_1,\cdot),\dots,B_n(s_n,\cdot))$, $s_i \le t$.
- The matrix function v should be adapted to a filtration that contains $\mathcal{F}_t^n$.
- under this condition each of the Itô integral in the sum is well defined.
- Itô integrations for multi-dimensional Brownian motions also produce a continuous martingale

$$M_t = \int_0^t v dB_s$$

for v satisfying the above technical conditions (in short hand $v \in \mathcal{V}^n$).

Exercises

Exercises: [T] 3.1, 3.4,3.5, 3.8(a),3.13(a),(b),3.16,3.17.