**Answers to Worksheet 5, Math 272**

1. Calculate the directional derivative of the function $f(x,y,z) = x\cos y \sin z$ at the point $\vec{a} = (1, \pi/4, 5\pi/6)$ in the direction $\vec{u} = (3, 0, -1)$.

   **Solution:** The gradient of $f$ at $\vec{a}$ is

$$\nabla f(\vec{a}) = (\cos y \sin z, -x \sin y \sin z, x \cos y \cos z)_{(1,\pi/4,5\pi/6)} = \left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{\sqrt{6}}{4}\right)$$

$$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{\sqrt{6}}{4}\right) \cdot \frac{(3,0,-1)}{\sqrt{10}} = \frac{3\sqrt{2}+\sqrt{6}}{4\sqrt{10}}$$

2. Find the equation for the line tangent to the curve $x^3+y^3 = -\frac{7}{2}xy$ at the point $\vec{a} = (2, -1)$.

   **Solution:** Let

$$f(x,y) = x^3 + y^3 + \frac{7}{2}xy$$

   The gradient vector of $f$ at $\vec{a}$ is

$$\nabla f(\vec{a}) = (3x^2 + \frac{7}{2}y, 3y^2 + \frac{7}{2}x)_{(2,-1)} = (\frac{17}{2}, 10)$$

   Hence the equation for the line tangent to the level set $f(x,y) = 0$ at the point $\vec{a} = (2, -1)$ is

$$((x,y) - (2,-1)) \cdot (\frac{17}{2}, 10) = 0$$

   That is

$$\frac{17}{2}(x-2) + 10(y+1) = 0$$

3. Find the point on the paraboloid $2x - z^2 - 3y^2 = 0$ at which the tangent plane is parallel to the plane $5x + y + z = 1$.

   **Solution:** Let $f(x,y,z) = 2x - z^2 - 3y^2$. The gradient vector of $f$ is

$$\nabla f = (2, -6y, -2z)$$

   Hence, there exists a constant $k$ such that

$$(2, -6y, -2z) = k(5, 1, 1).$$

   This yields

$$k = \frac{2}{5}, \quad y = -\frac{1}{15} \quad z = -\frac{1}{5}$$

   Finally, we have

$$2x - (-\frac{1}{5})^2 - 3(-\frac{1}{15})^2 = 0$$

   This gives us $x = \frac{2}{75}$. So the point is

$$(\frac{2}{75}, -\frac{1}{15}, -\frac{1}{5})$$

4. Calculate the divergence and curl of the function

$$\vec{F}(x,y,z) = \left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$$

**Solution:**

$$\text{div}\vec{F}(x,y,z) = \frac{1}{y} + \frac{1}{z} + \frac{1}{x}$$

and

$$\text{curl}\vec{F}(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{y} & \frac{y}{z} & \frac{z}{x} \end{vmatrix} = \left(\frac{y}{z^2}, \frac{z}{x^2}, \frac{x}{y^2}\right)$$

5. Find the second-degree Taylor polynomial of the function $f(x,y,z) = x\cos(yz)$ at the point $\vec{a} = (-4, \pi/4, 1)$

**Solution:** We have

$$f(-4, \pi/4, 1) = -2\sqrt{2}.$$

Further

$$\nabla f(x,y,z) = (\cos(yz), -xz\sin(yz), -xy\sin(yz))$$

and

$$Hf(x,y,z) = \begin{bmatrix} 0 & -z\sin(yz) & -y\sin(yz) \\ -z\sin(yz) & -xz^2\cos(yz) & -x\sin(yz) - xyz\cos(yz) \\ -y\sin(yz) & -x\sin(yz) - xyz\cos(yz) & -xy^2\cos(yz) \end{bmatrix}$$

Hence,

$$\nabla f(-4, \pi/4, 1) = (\frac{\sqrt{2}}{2}, 2\sqrt{2}, \frac{\sqrt{2}\pi}{2})$$

and

$$Hf(-4, \pi/4, 1) = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}\pi}{8} \\ -\frac{\sqrt{2}}{2} & 2\sqrt{2} & (4+\pi)\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}\pi}{8} & (4+\pi)\frac{\sqrt{2}}{2} & \frac{\sqrt{2}\pi^2}{8} \end{bmatrix}$$

The second-degree Taylor polynomial at $\vec{a} = (-4, \pi/4, 1)$ is

$$\begin{aligned} p(x,y,z) &= -2\sqrt{2} + (\frac{\sqrt{2}}{2}, 2\sqrt{2}, \frac{\sqrt{2}\pi}{2}) \cdot (x+4, y - \frac{\pi}{4}, z-1) \\ &\quad + \frac{1}{2} \begin{bmatrix} x+4 & y-\frac{\pi}{4} & z-1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}\pi}{8} \\ -\frac{\sqrt{2}}{2} & 2\sqrt{2} & (4+\pi)\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}\pi}{8} & (4+\pi)\frac{\sqrt{2}}{2} & \frac{\sqrt{2}\pi^2}{8} \end{bmatrix} \begin{bmatrix} x+4 \\ y-\frac{\pi}{4} \\ z-1 \end{bmatrix} \end{aligned}$$

6. Categorize the following quadratic forms as positive definite, negative definite, indefinite, or none of these

(a) $f(x,y,z) = x^2 + y^2 - z^2 + 5xy - 3yz$

**Solution:** The symmetric matrix is

$$A = \begin{bmatrix} 1 & 5/2 & 0 \\ 5/2 & 1 & -3/2 \\ -3/2 & 0 & -1 \end{bmatrix}.$$

Because $a_{11} = 1 > 0$,

$$\det\left(\begin{bmatrix} 1 & 5/2 & 0 \\ 5/2 & 1 & -3/2 \\ -3/2 & 0 & -1 \end{bmatrix}\right) < 0, \quad \text{and} \quad \det(A) > 0,$$

By Sylvester Theorem, this quadratic form is **indefinite**.

(b) $f(x,y) = 3x^2 - 8xy + 6y^2$

**Solution:** The symmetric matrix is

$$A = \begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix}.$$

Because $a_{11} = 3 > 0$ and $\det(A) > 0$, By Sylvester Theorem, this quadratic form is **positive definite**.

(c) $f(x,y,z) = 12x^2 + 10xy + 8xz + 3y^2 + 6yz + 4z^2$

**Solution:** The symmetric matrix is

$$A = \begin{bmatrix} 12 & 5 & 4 \\ 5 & 3 & 3 \\ 4 & 3 & 4 \end{bmatrix}.$$

Because $a_{11} = 12 > 0$,

$$\det\left(\begin{bmatrix} 12 & 5 \\ 5 & 3 \end{bmatrix}\right) > 0 \quad \text{and} \quad \det(A) > 0$$

By Sylvester Theorem, this quadratic form is **positive definite**.

7. Show that if $a$, $b$, and $c$ are all nonzero and

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix},$$

then $p(\vec{x}) = \vec{x}^T A\vec{x}$ is indefinite.

**Solution:** It is easy to know

$$p(\vec{x}) = 2(axy + bxz + cyz)$$

Let's take two nonzero vectors $\vec{x} = (x, x, 0)$ and $\vec{y} = (x, -x, 0)$ for nonzero $x$, we have

$$p(\vec{x}) = 2ax^2 \quad \text{and} \quad p(\vec{y}) = -2ax^2.$$

Since $a$ is nonzero, we know that $p(\vec{x})$ and $p(\vec{y})$ have opposite signs, by the definition, $p(\vec{x}) = \vec{x}^T A\vec{x}$ is **indefinite**.

8. Sketch the level curves of the function $f(x, y) = \cos y$ for the indicated values of $c = 0, 1/2, 1/\sqrt{2}$.

   **Solution:** We skip the plot of the level curves. We only give the solutions of $f(x, y) = \cos y = c$.

   - If $f(x, y) = \cos y = 0$, then $y = \pm\frac{\pi}{2} + k\pi$.
   - If $f(x, y) = \cos y = 1/2$, then $y = \pm\frac{\pi}{3} + 2k\pi$.
   - If $f(x, y) = \cos y = 1/\sqrt{2}$, then $y = \pm\frac{\pi}{4} + 2k\pi$.

9. For the following functions, find all the critical points and use the second derivative tests to identify the local extrema.

   - $f(x, y) = x^2 + 3y^2 - 2x + 12y + 15$.

     **Solution:**
     Solving equation
     $$0 = \nabla f(x, y) = (2x - 2, 6y + 12)$$
     we get one critical point $(1, -2)$. The Hessian of $f$ at the critical point,
     $$Hf(1, -2) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$
     is positive definite. Thus, $(1, -2)$ is a local minimum.

   - $f(x, y) = x + y - \ln(xy)$.

     **Solution:**
     Solving equation
     $$0 = \nabla f(x, y) = (1 - \frac{1}{x}, 1 - \frac{1}{y})$$
     we get one critical point $(1, 1)$. The Hessian of $f$ at the critical point,
     $$Hf(1, 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
     is positive definite. Thus, $(1, 1)$ is a local minimum.

10. Find a point on the plane $x + y + z = 1$ that is closest to the point (1,1,1).

**Solution:** We need to find point $(x, y, z)$ on the given plane that minimizes the distance $d = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$. Equivalently we can minimize $d^2$. Thus, we can formulate the problem as

$$\begin{aligned} &\text{minimize} && f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2 \\ &\text{subject to} && g(x, y, z) = x + y + z - 1 = 0. \end{aligned}$$

Using the Lagrange multiplier rule we know a potential minimizing point must satisfy

$$\nabla f(\bar{x}, \bar{y}, \bar{z}) = \lambda \nabla g(\bar{x}, \bar{y}, \bar{z}),$$

that is,

$$2(x-1, y-1, z-1) = \lambda(1, 1, 1).$$

Clearly, $\bar{x} = \bar{y} = \bar{z}$. Since $g(\bar{x}, \bar{y}, \bar{z}) = \bar{x} + \bar{y} + \bar{z} - 1 = 0$ we obtain $\bar{x} = \bar{y} = \bar{z} = 1/3$. That is $(1/3, 1/3, 1/3)$ is the only candidate and, therefore, $(1/3, 1/3, 1/3)$ is the closest point to $(1, 1, 1)$ on the plane $x + y + z = 1$.